Pergamon

Computers & Geosciences Vol. 23, No. 2, pp. 195–202, 1997

0098-3004/97 \$17.00 + 0.00

PII: S0098-3004(96)00090-8

AN EFFICIENT ALGORITHM FOR EVALUATING COUPLED PROCESSES IN RADIAL FLUID FLOW

MAO BAI

Rock Mechanics Institute, The University of Oklahoma, Norman, OK 73019-1014
(*e-mail*: mao@rmg.uoknor.edu)

(*Received 15 December 1995; revised 21 September 1996*)

Abstract—An algorithm for calculating the solution of coupled processes such as fluid flow through fractured porous media using dual-porosity conceptualization is presented. In the Laplace domain, instead of solving the coupled system of equations using special functions as practiced by most analytical methodologies, this algorithm offers a direct solution technique using the method of differential operators. In modeling fluid flow through fractured porous media, this method maintains the rigor of fluid flow within the matrix blocks without the complications in assessing the values of special functions. This efficient algorithm is general and can be applied to any pertinent modeling of coupled processes. © 1997 Elsevier Science Ltd

Key Words: Fluid flow, Coupled process, Differential operator, Fractured porous media.

INTRODUCTION

Modeling of fluid flow and contaminant transport through fractured porous media has been a subject of intensive interest in numerous earth science disciplines (Bai, Elsworth, and Roegiers, 1993; Bai, Roegiers, and Inyang, 1996). Because of significant differences in the characteristics of flow and transport between fractures and rock matrix, the flow and transport processes predicted by dual-porosity models are substantially different from conventional models using single-porosity models or homogeneous approaches (Bai, Elsworth, and Roegiers, 1993).

As an initial approximation or in some simplified circumstances, analytical solutions become convenient tools to achieve certain qualitative results. When studying fluid flow through fractured porous media, typical analytical solutions were obtained via decoupling technique by neglecting the fluid flow within the matrix blocks (Warren and Root, 1963).

The influence of matrix flow can be maintained in some rigorous solutions through solving eigenvalue problems using special functions for radial or Cartesian geometries (Bai, Ma, and Roegiers, 1994; Bai, Roegiers, and Elsworth, 1994). Although these solutions are analytical, evaluation of the special functions with open series frequently becomes so tedious that their practical utilization is discouraged.

To avoid the inconvenience and complication without sacrificing the accurate representation of realistic physical processes, this paper presents an efficient algorithm to solve the system of equations within the Laplace domain using the method of differential operators. The closed form analytical solutions are obtained in relatively few steps without invoking any special functions. The computer implementation of this algorithm is straightforward. This algorithm is presented in a complete example of modeling fluid flow through a naturally fractured reservoir where the fluid flow within the dual-porosity media is fully coupled.

DUAL-POROSITY FORMULATION

In the literature, a typical dual-porosity formulation is usually written as (Barenblatt, Zheltov, and Kochina, 1960):

$$\frac{k_1}{\mu}\nabla^2 p_1^* = c_1 n_1 \frac{\partial p_1^*}{\partial t^*} + \frac{\Gamma}{\mu}(p_1^* - p_2^*), \tag{1}$$

$$\frac{k_2}{\mu}\nabla^2 p_2^* = c_2 n_2 \frac{\partial p_2^*}{\partial t^*} - \frac{\Gamma}{\mu}(p_1^* - p_2^*), \tag{2}$$

where subscripts 1 and 2 represent matrix and fractures, respectively; p^* is the fluid pressure, k is the permeability, μ is the dynamic viscosity, n is the porosity, c is the compressibility, t^* is the time, $\Gamma = \alpha k_1$, and α is the geometric leakage factor indexing the interporosity flow between the fractures and the matrix blocks. It is noted from Equations (1) and (2) that the fluid flow is retained in both matrix blocks and fractures.

For the one-dimensional radial flow example, Equations (1) and (2) can be written as:

$$\frac{\partial^2 p_1^*}{\partial (r^*)^2}+\frac{1}{r^*}\frac{\partial p_1^*}{\partial r^*}=\frac{\mu c_1 n_1}{k_1}\frac{\partial p_1^*}{\partial t^*}+\alpha(p_1^*-p_2^*), \quad (3)$$

$$\frac{\partial^2 p_2^*}{\partial (r^*)^2}+\frac{1}{r^*}\frac{\partial p_2^*}{\partial r^*}=\frac{\mu c_2 n_2}{k_2}\frac{\partial p_2^*}{\partial t^*}-\alpha\frac{k_1}{k_2}(p_1^*-p_2^*), \quad (4)$$

where r^* is the radial distance from the well.

Conventional application frequently invokes a dimensionless formulation, as described by the following terms:

$$\begin{cases} p_i=\dfrac{2\pi k_2 h(p_0-p_i^*)}{q\mu} \quad (i=1,2), \\ r=\dfrac{r^*}{r_w}, \; \lambda=\alpha r_w^2, \; R_k=\dfrac{k_1}{k_2}, \\ t=\dfrac{(k_1+k_2)t^*}{\mu r_w^2(c_1n_1+c_2n_2)}, \; \omega=\dfrac{c_2n_2}{c_1n_1+c_2n_2}, \end{cases} \quad (5)$$

where h is the reservoir thickness, p_0 is the initial reservoir pressure, q is the flow rate at the well, and r_w is the wellbore radius. Although the rate of flow at the well from the matrix may be proportional in any amount to that from the fractures, it is assumed that the fluid flows at the well only through the fractures in view of significantly higher fracture permeability, and for simplicity.

The dimensionless form for fluid flow through a fractured porous formation can be written as

$$\frac{\partial^2 p_1}{\partial r^2}+\frac{1}{r}\frac{\partial p_1}{\partial r}=\left(1+\frac{1}{R_k}\right)(1-\omega)\frac{\partial p_1}{\partial t}+\lambda(p_1-p_2), \quad (6)$$

$$\frac{\partial^2 p_2}{\partial r^2}+\frac{1}{r}\frac{\partial p_2}{\partial r}=(1+R_k)\omega\frac{\partial p_2}{\partial t}-\lambda R_k(p_1-p_2), \quad (7)$$

where it is indicated that the fluid flow is characterized by the reservoir equivalent storage ratio for the fractures ω, the permeability ratio R_k and the inter-porosity coefficient λ.

For constant flow rate at the well ($r^*=r_w$) and constant pressure at the reservoir boundary ($r^*=r_e^*$), the boundary and initial conditions in the dimensionless form are expressed as

$$\left.\frac{\partial p_1}{\partial r}\right|_{r=1}=0, \; \left.\frac{\partial p_2}{\partial r}\right|_{r=1}=-1, \; p_1=p_2|_{r=r_e}$$
$$=0, \; p_1=p_2|_{t=0}=0, \quad (8)$$

where r_e is the dimensionless reservoir radius.

Using the Laplace transform to Equations (6) and (7) results in the following ordinary differential equations:

$$\frac{d^2\bar{p}_1}{dr^2}+\frac{1}{r}\frac{d\bar{p}_1}{dr}=\left(1+\frac{1}{R_k}\right)(1-\omega)s\bar{p}_1+\lambda(\bar{p}_1-\bar{p}_2), \quad (9)$$

$$\frac{d^2\bar{p}_2}{dr^2}+\frac{1}{r}\frac{d\bar{p}_2}{dr}=(1+R_k)\omega s\bar{p}_2-\lambda R_k(\bar{p}_1-\bar{p}_2), \quad (10)$$

where s is a Laplace parameter.

Boundary conditions are changed to

$$\left.\frac{d\bar{p}_1}{dr}\right|_{r=1}=0, \; \left.\frac{d\bar{p}_2}{dr}\right|_{r=1}=-\frac{1}{s}, \; \bar{p}_1=\bar{p}_2|_{r=r_e}=0. \quad (11)$$

SOLUTION TECHNIQUE

It is noted that Equations (9) and (10) are coupled differential equations, which can be solved by the method of differential operators (Mathematical Handbook, 1979). The differential operators, D^n, is applied as

$$D^n f(x_i)=\frac{d^n f(x_i)}{dx_i^n}, \quad (12)$$

where i indexes an arbitrary variable, and n is the order of the differential equations.

Applying the differential operators to Equations (9) and (10) gives

$$\left\{D^2+\frac{D}{r}-\left[\lambda+\left(1+\frac{1}{R_k}\right)(1-\omega)s\right]\right\}\bar{p}_1+\lambda\bar{p}_2=0, \quad (13)$$

$$\left\{D^2+\frac{D}{r}-[\lambda R_k+(1+R_k)\omega s]\right\}\bar{p}_2+\lambda R_k\bar{p}_1=0. \quad (14)$$

The following relationship is derived from Equation (14):

$$\bar{p}_1=\frac{-1}{\lambda R_k}\left\{D^2+\frac{D}{r}-[\lambda R_k+(1+R_k)\omega s]\right\}\bar{p}_2. \quad (15)$$

Substituting Equation (15) into Equation (13) results in

$$(D^4+B_1D^3+B_2D^2+B_3D+B_4)\bar{p}_2=0, \quad (16)$$

where

$$\begin{cases} B^*=\left(1+\dfrac{1}{R_k}\right)\{\lambda R_k+s[1-\omega(1-R_k)]\}, \\ B_1=\dfrac{2}{r}, \; B_2=\dfrac{1}{r^2}-B^*, \; B_3=-\dfrac{B^*}{r}, \\ B_4=s\left(1+\dfrac{1}{R_k}\right)[\lambda R_k+\omega s(1+R_k)(1-\omega)]. \end{cases} \quad (17)$$

Equation (16) has four roots from the following equations (Mathematical Handbook, 1979):

$$D^2 + \xi_1 D + \phi_1 = 0, \tag{18}$$

$$D^2 + \xi_2 D + \phi_2 = 0, \tag{19}$$

where

$$\begin{cases} \Delta_1 = 8z + B_1^2 - 4B_2 \\ \xi_1 = 0.5(B_1 + \sqrt{\Delta_1}), \quad \xi_2 = 0.5(B_1 - \sqrt{\Delta_1}), \\ \phi_1 = z + \dfrac{B_1 z - B_3}{\sqrt{\Delta_1}}, \quad \phi_2 = z - \dfrac{B_1 z - B_3}{\sqrt{\Delta_1}}, \end{cases} \tag{20}$$

where z is any real root of the following equation:

$$8z^3 + 4E_1 z^2 + E_2 z + E_3 = 0, \tag{21}$$

and where

$$\begin{cases} E_1 = -B_2, \\ E_2 = 2B_1 B_3 - 8B_4, \\ E_3 = B_4(4B_2 - B_1^2) - B_3^2. \end{cases} \tag{22}$$

Furthermore, assume

$$z = u - \frac{E_1}{6}, \tag{23}$$

where the parameter u can be determined from the third-order equation

$$u^3 + \tau_1 u + \tau_2 = 0, \tag{24}$$

and where

$$\begin{cases} \tau_1 = 0.125\left(E_2 - \dfrac{2E_1^2}{3}\right), \\ \tau_2 = 0.125\left(\dfrac{2E_1^3}{27} - \dfrac{E_1 E_2}{6} + E_3\right). \end{cases} \tag{25}$$

The three roots of u may be described as

$$\begin{cases} u_1 = F_1 + F_2, \\ u_2 = \theta_1 F_1 + \theta_2 F_2, \\ u_3 = \theta_2 F_1 + \theta_1 F_2, \end{cases} \tag{26}$$

where

$$\begin{cases} \Delta^* = \left(\dfrac{\tau_2}{2}\right)^2 + \left(\dfrac{\tau_1}{3}\right)^3, \\ F_1 = \left(-\dfrac{\tau_2}{2} + \sqrt{\Delta^*}\right)^{\frac{1}{3}}, \quad F_2 = \left(-\dfrac{\tau_2}{2} - \sqrt{\Delta^*}\right)^{\frac{1}{3}}, \\ \theta_1 = -0.5 + \dfrac{\sqrt{3}}{2} i, \quad \theta_2 = -0.5 - \dfrac{\sqrt{3}}{2} i. \end{cases} \tag{27}$$

Three possible solutions of the real root, z, exist depending upon the signs of Δ^* in Equation (27), as indicated in the following. When $\Delta^* > 0$, the real root is

$$z_1 = F_1 + F_2 - \frac{E_1}{6}. \tag{28}$$

If $\Delta^* = 0$, then the real root becomes

$$z_1 = -\sqrt[3]{4\tau_2} - \frac{E_1}{6}. \tag{29}$$

However, if $\Delta^* < 0$, the real root is recovered in trigonometric form as

$$z_1 = 2\sqrt[3]{\bar{r}} \cos\left(\frac{\alpha}{3}\right) - \frac{E_1}{6}, \tag{30}$$

where

$$\begin{cases} \bar{r} = \sqrt{-\left(\dfrac{\tau_1}{3}\right)^3}, \\ \alpha = \arccos\left(-\dfrac{\tau_2}{2\bar{r}}\right). \end{cases} \tag{31}$$

Once the real root, z, is determined for Equation (21), the four roots of Equations (18) and (19) can then be expressed as

$$\begin{cases} \Delta_2 = \dfrac{\xi_1^2}{4} - \phi_1, \quad \Delta_3 = \dfrac{\xi_2^2}{4} - \phi_2, \\ \psi_1 = -\dfrac{\xi_1}{2} + \sqrt{\Delta_2}, \quad \psi_2 = -\dfrac{\xi_1}{2} - \sqrt{\Delta_2}, \\ \psi_3 = -\dfrac{\xi_2}{2} + \sqrt{\Delta_3}, \quad \psi_4 = -\dfrac{\xi_2}{2} - \sqrt{\Delta_3}. \end{cases} \tag{32}$$

The solutions in the Laplace domain may become complicated as a result of uncertainty in the signs of Δ_1, Δ_2, and Δ_3. However, for a choice of limiting physical parameters: $r(1 \to \infty)$, $s(0 \to \infty)$, $\lambda(0 \to \infty)$, $\omega(0 \to 1)$, $R_k(0 \to 1)$, Δ_1 is predominantly positive. Although Δ_2 and Δ_3 are also most positive for the range of representative parameters in this paper, the complete solutions including those under simultaneously or alternately negative Δ_2 and Δ_3, along with the solution procedure, are described in Appendix A. The solutions in the real-time space may be recovered by using a numerical inversion technique.

A SIMPLE SOLUTION

For the reservoir with relatively distant boundary, the solutions can be substantially simplified as a result of: (a) Δ_1, Δ_2, and Δ_3 are all positive values within the changes of any physical parameters; and (b) some solution parameters are naturally determined.

In the Laplace domain, the pressure in fractures can be derived from Equation (16) as

$$\bar{p}_2 = \sum_{i=1}^{4} g_i e^{\psi_i r} \tag{33}$$

where constants g_i are determined by satisfying the boundary conditions. For the situations with finite fluid pressure and as a result of the fact that $\psi_3 > 0$ and $\psi_4 > 0$, it is readily known from the outer boundary conditions in Equation (11) that $g_3 = g_4 = 0$. As a result,

$$\bar{p}_2 = \sum_{i=1}^{2} g_i e^{\psi_i r}. \quad (34)$$

The fluid pressure in the matrix in the Laplace domain can be determined from Equation (15) as

$$\bar{p}_1 = -\frac{1}{\lambda R_k}\sum_{i=1}^{2} g_i \eta_i e^{\psi_i r}, \quad (35)$$

where

$$\eta_i = \psi_i^2 + \frac{\psi_i}{r} - [\lambda R_k + \omega s(1 + R_k)]. \quad (36)$$

The constants g_1 and g_2 can be determined from the inner boundary conditions in Equation (11), which yields

$$\begin{bmatrix} A_1^* & A_2^* \\ A_1 & A_2 \end{bmatrix}\begin{bmatrix} g_1 \\ g_2 \end{bmatrix} = \begin{bmatrix} 0 \\ \delta \end{bmatrix}, \quad (37)$$

where

$$A_i = \psi_i e^{\psi_i}, A_i^* = \eta_i A_i, \delta = -\frac{1}{s}. \quad (38)$$

Solving Equation (37) gives

$$g_1 = -\frac{\delta A_2^*}{A^*}, \; g_2 = \frac{\delta A_1^*}{A^*}, \quad (39)$$

where

Table 1. Selected modeling parameters

Figure	R_k	ω	λ
Figure 1	0.1	0.01–0.5	1
Figure 2	0.1	0–1	1
Figure 3	0.05–0.5	0.5	1
Figure 4	0.1	0.5	1–20

$$A^* = A_1^* A_2 - A_1 A_2^*. \quad (40)$$

The utility of the simple solution can be demonstrated in an illustrative example. In order to compare the solution with those of a finite reservoir boundary, the pumping rate at the well is modified to consider the steady-state discharge [proportional to $\ln(r_e/r_w)$] from the reservoir with a finite boundary. The selected parameters are listed in Table 1.

The comparison between this model and that by Bai, Ma, and Roegiers (1994) is shown in Figure 1. As a result of the difference in the assumed flow paths at the well, using identical parameters still leads to a time lag in the pressure development for the latter model (curve 2). However, this difference is reduced as the dimensionless storativity ω decreases (curves 3 and 4). The latter curves depict pressure slope changes at later time, evidence of the dual-porosity behavior of the fractured reservoir.

The sensitivities of three major parameters of this model are tested. The pressure variations for a few limiting examples of ω are indicated in Figure 2.

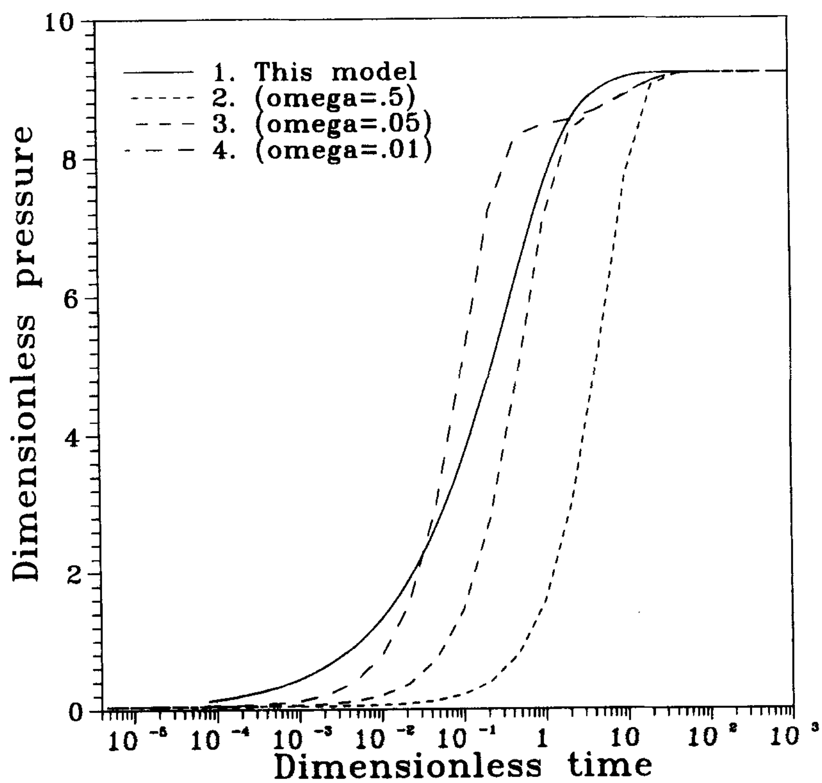

Figure 1. Comparison of temporal pressures.

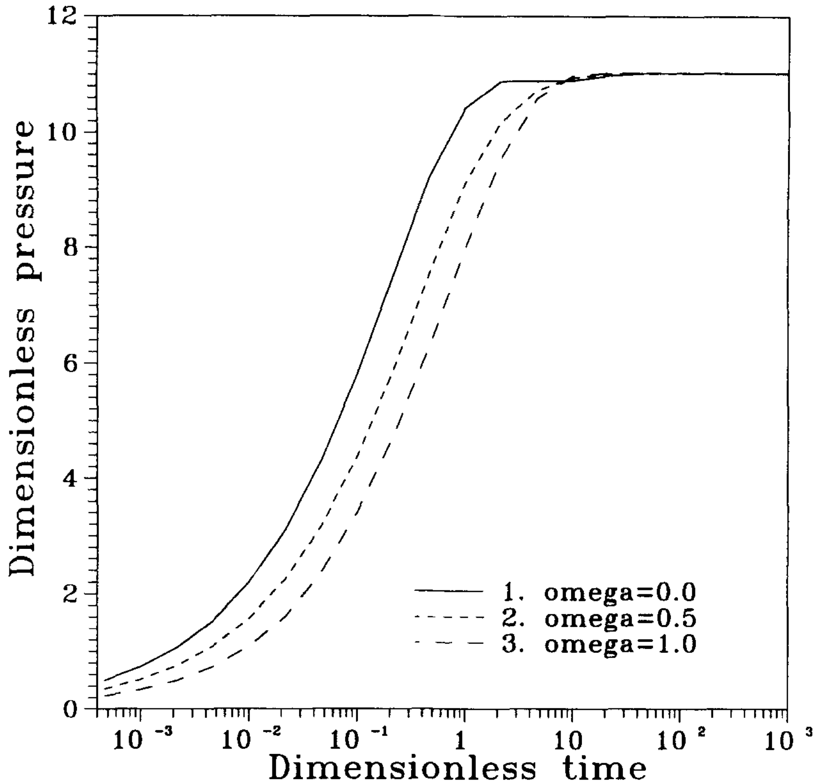

Figure 2. Temporal pressure for various ω.

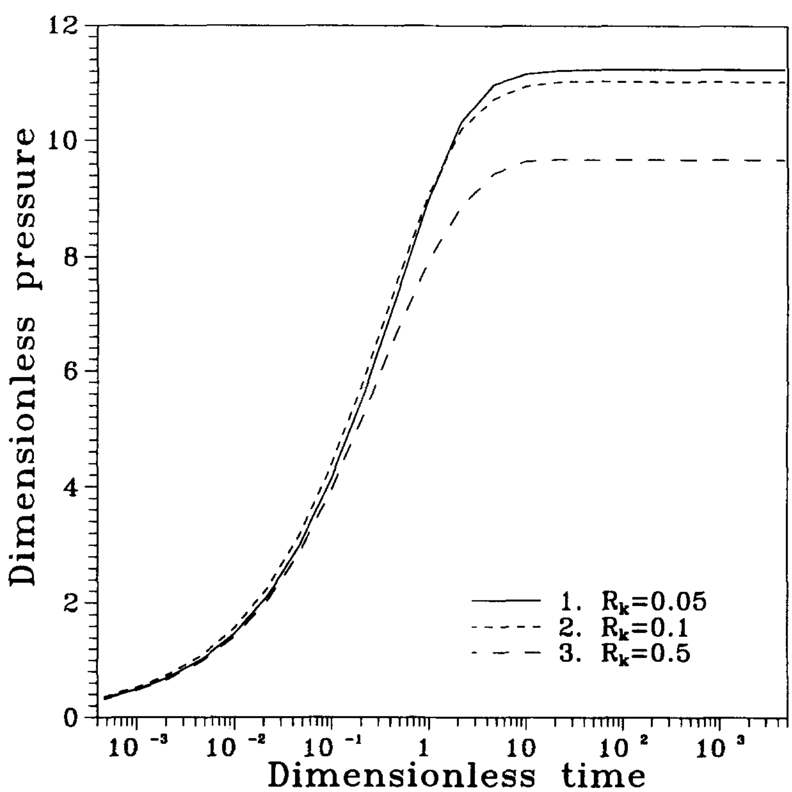

Figure 3. Temporal pressure for various R_k.

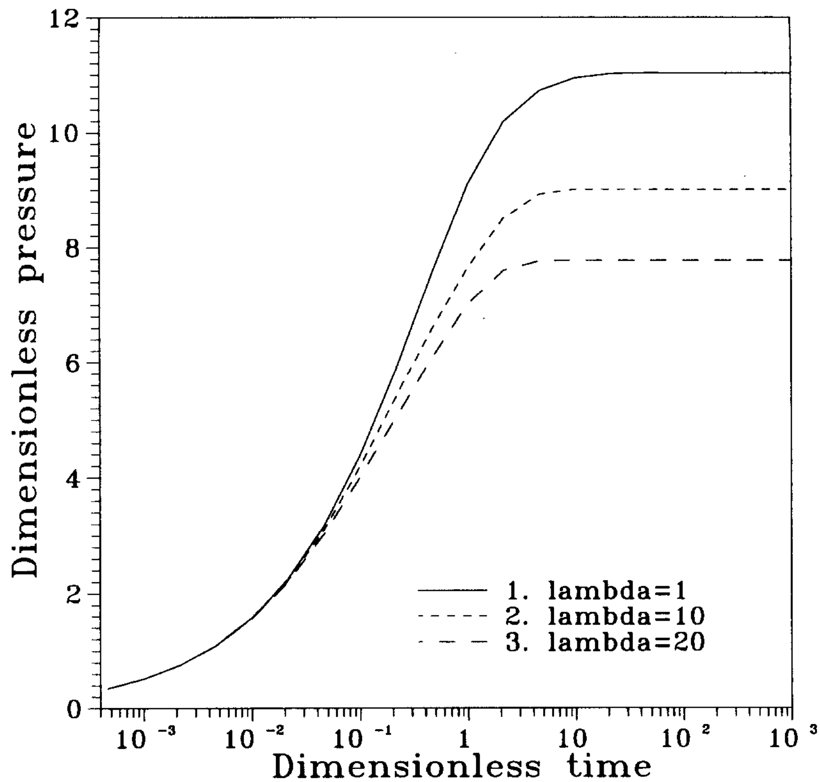

Figure 4. Temporal pressure for various λ.

Smaller equivalent fracture storativity appears to result in earlier pressure changes. However, the time lag seems to be less significant for the instances of larger ω. The changes of permeability ratio R_k create the differences in the pressure magnitudes, as shown in Figure 3. A smaller ratio represents a reduced matrix permeability, which leads to higher fracture pressure changes as a result of more dominant flow within the fractures. This effect diminishes as the ratio becomes larger. Similarly, the apparent variation of pressure magnitude is as a result of the changes of the interporosity coefficient λ, as depicted in Figure 4. The larger λ represents more manifest fluid exchanges between the matrix blocks and the fractures, which shows a lower pressure magnitude, an indication of reduced flow in the fractures.

CONCLUSIONS

An efficient algorithm for evaluating the coupled processes which involve solving a simultaneous system of equations is presented. The application of this algorithm is given in the alternative modeling of fluid flow through fractured reservoirs conceptualized as the dual porosity media. The most significant advantage of using this method rests on its ability to analyze any coupled equations (e.g. solute transport in dual-porosity media) in a straightforward fashion without resorting to complicated special functions. This feature is especially useful to practicing engineers whose knowledge in mathematics is not far beyond fundamental calculus. Despite the simplicity of the algorithm along with its computer implementation, the utility using this method to study complex physical phenomena without excessive approximation has been demonstrated.

Acknowledgments—Support of the National Science Foundation under contract EEC-9209619, along with the support and encouragement of Dr J.-C. Roegiers, are gratefully acknowledged.

REFERENCES

Bai, M., Elsworth, D., and Roegiers, J.-C., 1993, Multi-porosity/multi-permeability approach to the simulation of naturally fractured reservoirs: Water Resources Res., v. 29, no. 6, p. 1621–1633.

Bai, M., Ma, Q., and Roegiers, J.-C., 1994, Dual-porosity behavior of naturally fractured reservoirs: Int. Jour. Numer. Anal. Meth. Geomech., v. 18, no. 6, p. 359–376.

Bai, M., Roegiers, J.-C., and Elsworth, D., 1994, An alternative model of fractured reservoir simulation: Proc. 8th Int. Conference Computer Methods Advanced Geomechanics, Morgantown, West Virginia, p. 2031–2036.

Bai, M., Roegiers, J.-C., and Inyang, H. I., 1996, Contaminant transport in nonisothermal fractured por-

ous media: Jour. Environmental Engineering, v. 122, no. 5, p. 416–423.
Barenblatt, G. E., Zheltov, I. P., and Kochina, I. N., 1960, Basic concept in the theory of homogeneous liquids in fractured rocks: Jour. Appl. Math. Mech., v. 24, no. 5, p. 1286–1303.
Mathematical Handbook, 1979, People's Education Publishing Company, 1398 p. (in Chinese).
Warren, J. E., and Root, P. J., 1963, Behavior of naturally fractured reservoirs: Soc. Petrology Engineering Jour., v. 228, p. 245–255.

APPENDIX A

Solution Procedure and Solutions

General Procedure

The following system of equations can be derived by satisfying boundary conditions as expounded in Equation (11):

$$\begin{bmatrix} \beta_1^* & \beta_2^* & \beta_3^* & \beta_4^* \\ \beta_1 & \beta_2 & \beta_3 & \beta_4 \\ \gamma_1^* & \gamma_2^* & \gamma_3^* & \gamma_4^* \\ \gamma_1 & \gamma_2 & \gamma_3 & \gamma_4 \end{bmatrix} \begin{bmatrix} g_1 \\ g_2 \\ g_3 \\ g_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \delta \end{bmatrix}, \tag{A1}$$

where δ is given in Equation (38).

The solutions of Equation (A1) can be expressed as

$$g_i = \frac{I_i}{J} \; (i = 1, 2, 3, 4), \tag{A2}$$

where

$$J = \sum_{i=1}^{4} (-1)^{i-1} \beta_i^* J_i, \tag{A3}$$

and where J_i can be described as

$$J_i = \beta_j(\gamma_k^* \gamma_l - \gamma_k \gamma_l^*) - \beta_k(\gamma_j^* \gamma_l - \gamma_j \gamma_l^*) + \beta_l(\gamma_j^* \gamma_k - \gamma_j \gamma_k^*), \tag{A4}$$

and it follows the rotation rule, that is, $i \neq j \neq k \neq l$. More specifically,

$$\begin{cases} i = 1, \; j = 2, \; k = 3, \; l = 4; \\ i = 2, \; j = 1, \; k = 3, \; l = 4; \\ i = 3, \; j = 1, \; k = 2, \; l = 4; \\ i = 4, \; j = 1, \; k = 2, \; l = 3. \end{cases} \tag{A5}$$

Similarly,

$$I_i = (-1)^i \delta[\beta_j^* (\beta_k \gamma_l^* - \gamma_k^* \beta_l) - \beta_k^* (\beta_j \gamma_l^* - \gamma_j^* \beta_l) + \beta_l^* (\beta_j \gamma_k^* - \gamma_j^* \beta_k)], \tag{A6}$$

where the rotation rule also applies.

Analytical Solutions

The solutions in the Laplace domain can be divided into four groups depending on the signs of Δ_2 and Δ_3 while $\Delta_1 > 0$.

Case 1. $\Delta_2 \geq 0$ and $\Delta_3 \geq 0$

The solutions have been partially given previously, and are rewritten below:

$$\bar{p}_2 = \sum_{i=1}^{4} g_i e^{\psi_i r}, \tag{A7}$$

$$\bar{p}_1 = -\frac{1}{\lambda R_k} \sum_{i=1}^{4} g_i \eta_i e^{\psi_i r}, \tag{A8}$$

where η_i is expressed in Equation (36).

The parameters required in Equation (A1) are

$$\beta_i = e^{\psi_i r_e}, \; \beta_i^* = \eta_i \beta_i, \; \gamma_i = \psi_i e^{\psi_i}, \; \gamma_i^* = \eta_i \gamma_i, \tag{A9}$$

where, again, $i = 1, 2, 3, 4$.

Case 2. $\Delta_2 \geq 0$ and $\Delta_3 < 0$

$$\bar{p}_2 = \sum_{i=1}^{2} g_i e^{\psi_i r} + e^{W_3 r}[g_3 \cos(W_4 r) + g_4 \sin(W_4 r)], \tag{A10}$$

where

$$W_3 = -\frac{\xi_2}{2}, \; W_4 = \sqrt{-\Delta_3}, \tag{A11}$$

$$\bar{p}_1 = -\frac{1}{\lambda R_k} \Big\{ \sum_{i=1}^{2} g_i \eta_i e^{\psi_i r} + e^{W_3 r} \{ g_3[\lambda_1^* \cos(W_4 r) - \lambda_2^* \sin(W_4 r)] + g_4[\lambda_2^* \cos(W_4 r) + \lambda_1^* \sin(W_4 r)]\} \Big\}, \tag{A12}$$

where

$$\begin{cases} \lambda_1^* = W_3^2 - W_4^2 + \Omega_1 W_3 + \Omega_2, \\ \lambda_2^* = W_4(2W_3 + \Omega_1), \\ \Omega_1 = \frac{1}{r}, \; \Omega_2 = -[\lambda R_k + (1 + R_k)\omega s]. \end{cases} \tag{A13}$$

For $i = 1, 2$, β_i, β_i^*, γ_i and γ_i^* are identical to those in case 1. However, for $i = 3, 4$:

$$\begin{cases} \eta_3 = \lambda_1^* \cos(W_4 r_e) - \lambda_2^* \sin(W_4 r_e), \\ \eta_4 = \lambda_2^* \cos(W_4 r_e) + \lambda_1^* \sin(W_4 r_e), \\ \beta_3 = e^{W_3 r_e} \cos(W_4 r_e), \; \beta_4 = e^{W_3 r_e} \sin(W_4 r_e), \\ \beta_3^* = \eta_3 e^{W_3 r_e}, \; \beta_4^* = \eta_4 e^{W_3 r_e}, \\ \gamma_3 = e^{W_3}[W_3 \cos(W_4) - W_4 \sin(W_4)], \\ \gamma_4 = e^{W_3}[W_3 \sin(W_4) + W_4 \cos(W_4)], \\ \gamma_3^* = e^{W_3}[\lambda_3^* \cos(W_4) - \lambda_4^* \sin(W_4)], \\ \gamma_4^* = e^{W_3}[\lambda_4^* \cos(W_4) + \lambda_3^* \sin(W_4)], \\ \lambda_3^* = W_3 \lambda_1^* - W_4 \lambda_2^*, \\ \lambda_4^* = W_3 \lambda_2^* + W_4 \lambda_1^*. \end{cases} \tag{A14}$$

Case 3. $\Delta_2 < 0$ and $\Delta_3 \geq 0$

$$\bar{p}_2 = e^{W_1 r}[g_1 \cos(W_2 r) + g_2 \sin(W_2 r)] + \sum_{i=3}^{4} g_i e^{\psi_i r}, \tag{A15}$$

where

$$W_1 = -\frac{\xi_1}{2}, \; W_2 = \sqrt{-\Delta_2}, \tag{A16}$$

$$\bar{p}_1 = -\frac{1}{\lambda R_k} \Big\{ e^{W_1 r} \{ g_1[\lambda_1 \cos(W_2 r) - \lambda_2 \sin(W_2 r)] + g_2[\lambda_2 \cos(W_2 r) + \lambda_1 \sin(W_2 r)]\} + \sum_{i=3}^{4} g_i \eta_i e^{\psi_i r} \Big\}, \tag{A17}$$

and where

$$\begin{cases} \eta_1 = \lambda_1 \cos(W_2 r_e) - \lambda_2 \sin(W_2 r_e), \\ \eta_2 = \lambda_2 \cos(W_2 r_e) + \lambda_1 \sin(W_2 r_e), \\ \lambda_1 = W_1^2 - W_2^2 + \Omega_1 W_1 + \Omega_2, \\ \lambda_2 = W_2(2W_1 + \Omega_1). \end{cases} \tag{A18}$$

For $i = 3, 4$, β_i, β_i^*, γ_i and γ_i^* are identical to those in case 1. However, for $i = 1, 2$:

$$\begin{cases} \beta_1 = e^{W_1 r_e}\cos(W_2 r_e),\ \beta_2 = e^{W_1 r_e}\sin(W_2 r_e), \\ \beta_1^* = \eta_1 e^{W_1 r_e},\ \beta_2^* = \eta_2 e^{W_2 r_e}, \\ \gamma_1 = e^{W_1}[W_1 \cos(W_2) - W_2 \sin(W_2)], \\ \gamma_2 = e^{W_1}[W_1 \sin(W_2) + W_2 \cos(W_2)], \\ \gamma_1^* = e^{W_1}[\lambda_3 \cos(W_2) - \lambda_4 \sin(W_2)], \\ \gamma_2^* = e^{W_1}[\lambda_4 \cos(W_2) + \lambda_3 \sin(W_2)], \\ \lambda_3 = W_1\lambda_1 - W_2\lambda_2, \\ \lambda_4 = W_1\lambda_2 + W_2\lambda_1. \end{cases} \tag{A19}$$

Case 4. $\Delta_2 < 0$ and $\Delta_3 < 0$

$$\bar{p}_2 = e^{W_1 r}[g_1 \cos(W_2 r) + g_2 \sin(W_2 r)] + e^{W_3 r}[g_3 \cos(W_4 r) + g_4 \sin(W_4 r)], \tag{A20}$$

$$\bar{p}_1 = -\frac{1}{\lambda R_k}\left[\sum_{i=1}^{4} G_i\right], \tag{A21}$$

where

$$\begin{cases} G_1 = g_1 e^{W_1 r}[\lambda_1 \cos(W_2 r) - \lambda_2 \sin(W_2 r)], \\ G_2 = g_2 e^{W_1 r}[\lambda_2 \cos(W_2 r) + \lambda_1 \sin(W_2 r)], \\ G_3 = g_3 e^{W_3 r}[\lambda_1^* \cos(W_4 r) - \lambda_2^* \sin(W_4 r)], \\ G_4 = g_4 e^{W_3 r}[\lambda_2^* \cos(W_4 r) + \lambda_1^* \sin(W_4 r)], \end{cases} \tag{A22}$$

and where

$$\begin{cases} \eta_1 = \lambda_1 \cos(W_2 r_e) - \lambda_2 \sin(W_2 r_e), \\ \eta_2 = \lambda_2 \cos(W_2 r_e) + \lambda_1 \sin(W_2 r_e), \\ \eta_3 = \lambda_1^* \cos(W_4 r_e) - \lambda_2^* \sin(W_4 r_e), \\ \eta_4 = \lambda_2^* \cos(W_4 r_e) + \lambda_1^* \sin(W_4 r_e), \\ \lambda_1 = W_1^2 - W_2^2 + \Omega_1 W_1 + \Omega_2, \\ \lambda_2 = W_2(2W_1 + \Omega_1), \\ \lambda_1^* = W_3^2 - W_4^2 + \Omega_1 W_3 + \Omega_2, \\ \lambda_2^* = W_4(2W_3 + \Omega_1), \\ \beta_1 = e^{W_1 r_e}\cos(W_2 r_e),\ \beta_2 = e^{W_1 r_e}\sin(W_2 r_e), \\ \beta_3 = e^{W_3 r_e}\cos(W_4 r_e),\ \beta_4 = e^{W_3 r_e}\sin(W_4 r_e), \\ \beta_1^* = \eta_1 e^{W_1 r_e},\ \beta_2^* = \eta_2 e^{W_2 r_e}, \\ \beta_3^* = \eta_3 e^{W_3 r_e},\ \beta_4^* = \eta_4 e^{W_3 r_e}, \\ \gamma_1 = e^{W_1}[W_1 \cos(W_2) - W_2 \sin(W_2)], \\ \gamma_2 = e^{W_1}[W_1 \sin(W_2) + W_2 \cos(W_2)], \\ \gamma_3 = e^{W_3}[W_3 \cos(W_4) - W_4 \sin(W_4)], \\ \gamma_4 = e^{W_3}[W_3 \sin(W_4) + W_4 \cos(W_4)], \\ \gamma_1^* = e^{W_1}[\lambda_3 \cos(W_2) - \lambda_4 \sin(W_2)], \\ \gamma_2^* = e^{W_1}[\lambda_4 \cos(W_2) + \lambda_3 \sin(W_2)], \\ \gamma_3^* = e^{W_3}[\lambda_3^* \cos(W_4) - \lambda_4^* \sin(W_4)], \\ \gamma_4^* = e^{W_3}[\lambda_4^* \cos(W_4) + \lambda_3^* \sin(W_4)], \\ \lambda_3 = W_1\lambda_1 - W_2\lambda_2,\ \lambda_4 = W_1\lambda_2 + W_2\lambda_1, \\ \lambda_3^* = W_3\lambda_1^* - W_4\lambda_2^*,\ \lambda_4^* = W_3\lambda_2^* + W_4\lambda_1^*. \end{cases} \tag{A23}$$