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Lubrication Theory Analysis of the Permeability of Rough-walled Fractures

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Lubrication theory is used to study the permeability of rough-walled rock fractures. In this approximation, which is valid for low Reynolds numbers and under certain restrictions on the magnitude of the roughness, the Navier–Stokes equations that govern fluid flow are reduced to the more tractable Reynolds equation. An idealized model of a fracture, in which the roughness follows a sinusoidal variation, is studied in detail. This fracture is considered to consist of a random mixture of elements in which the fluid flows either parallel or transverse to the sinusoidal bumps. The overall permeability is then found by a suitable averaging procedure. The results are similar to those found by other researchers from numerical analysis of the Reynolds equation, in that the ratio of the hydraulic aperture to the mean aperture correlates well with the ratio of the mean aperture to the standard deviation of the aperture. Higher-order approximations to the Navier–Stokes equations for flow between sinusoidal walls are then studied, and it is concluded that in order for the lubrication approximation to be valid, the fracture walls must be smooth over lengths on the order of one standard deviation of the aperture, which is much less restrictive a condition than had previously been thought to apply.

INTRODUCTION

The flow of a fluid between the rough surfaces of a rock fracture is very complex, due to the tortuous paths followed by the fluid particles. Exact analytical modelling of these flows is made difficult by the irregular geometry of rock fracture surfaces, while full 3-D numerical simulations of these flows are as yet still impractical. To overcome the difficulties of working with the 3-D Navier–Stokes equations, the simpler Reynolds lubrication equation has sometimes been used to model flow in fractures [1,2]. This paper focuses on two aspects of lubrication theory. First, lubrication theory is applied to two simplified aperture profiles, sinusoidal and "saw-tooth", and analytical expressions are found for the permeabilities. These results are then compared with numerical results [2,3] obtained by solving the lubrication equation for fractures with "random" surfaces. Secondly, the validity of the lubrication equations for modelling flow in rough fractures is studied by examining higher-order perturbation solutions, as well as numerical solutions, to the Navier–Stokes equations for flow in fractures with sinusoidally-varying apertures.

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LUBRICATION THEORY MODEL FOR FRACTURE FLOW

The flow of a Newtonian fluid through a rock fracture is governed by the Navier–Stokes equations of fluid mechanics [4,5]. These are a set of three coupled, nonlinear partial differential equations, and only for very simplified cases can they be solved analytically. One such case is that of flow under a uniform pressure gradient in the channel between two parallel, smooth surfaces. For this problem only one component of the velocity vector is nonzero, and the equations simplify greatly. The result is the well-known parabolic velocity profile, with the volumetric flow rate per unit width perpendicular to the direction of flow given by $Q = d^3\Delta P/12\,\mu L$, (the so-called "cubic law"), in which d is the aperture of the channel, μ is the dynamic viscosity of the fluid, and $\Delta P/L$ is the magnitude of the pressure gradient [6]. If the aperture is not constant along the channel, the equations cannot be solved analytically, and one must resort to approximate methods of some sort.

Under certain geometric and kinematic conditions which usually are assumed to hold for rock fractures, the Navier–Stokes equations can be reduced (locally) to the simpler Reynolds equation. One requirement for the Reynolds equation to be valid is that viscous forces dominate the inertial forces. A quantitative statement of this criterion is that [4, p. 109] the "reduced Reynolds number", Re*, be very small, i.e.

$$\mathrm{Re}^* = \frac{\rho U d_m^2}{\mu \Lambda} \ll 1, \qquad (1)$$

where ρ is the fluid density, U is the average velocity along the fracture, d_m is the mean aperture, μ is the fluid viscosity and Λ is some characteristic length of the fracture in the direction of the flow. There are also geometric conditions which specify that, in some sense, the aperture does not change too abruptly; these geometric conditions will be examined more closely below. If these dynamical and geometric conditions hold, the flow can be described by Reynolds equation:

$$\nabla \cdot (d^3 \nabla P) = 0,$$

i.e.

$$\frac{\partial}{\partial x}\left(d^3(x,y)\frac{\partial P}{\partial x}\right) + \frac{\partial}{\partial y}\left(d^3(x,y)\frac{\partial P}{\partial y}\right) = 0, \qquad (2)$$

where (x, y) are orthogonal coordinates in the plane of the fracture and $d(x, y)$ is the local aperture of the fracture. Equation (2) is a single, linear partial differential equation that describes the pressure field in the fracture plane. The volumetric flow of liquid is then related to the pressure by:

$$\mathbf{Q} = \frac{-d^3(x,y)}{12\mu}\nabla P. \qquad (3)$$

Milne-Thomson [7] derived the Reynolds equation from the Navier–Stokes equations through an order-of-magnitude analysis to allow the elimination of certain terms. Walsh [1] derived (2) by merely assuming that the cubic law holds locally at each point in the fracture, and then invoking the principle of conservation of mass. Since the conditions under which equation (2) is valid usually hold for "lubrication" flows, this is sometimes known as the lubrication approximation.

EFFECT OF APERTURE VARIATIONS

Brown [2, 8] solved (2) numerically for a fracture with a randomly-generated fractal aperture distribution, and plotted the fracture permeability as a function of the ratio of the mean aperture d_m to its standard deviation σ. The permeability was quantified in terms of the hydraulic aperture d_h, which is that value of d that would allow the flow rate to exactly satisfy the cubic law. The permeability was found to decrease as the standard deviation of the aperture increased (for fixed mean aperture), and was also found to be insensitive to the fractal dimension of the fracture surface. These results are remarkably similar to those of Patir and Cheng [3], who performed a similar analysis of lubrication flow between surfaces whose profiles obeyed a Gaussian distribution with linearly-decreasing autocorrelation functions. Figure 1 shows the normalized permeabilities, in the form of $(d_h/d_m)^3$, computed by Brown [8] for a surface fractal dimension of $D = 2.5$, along with the calculations of

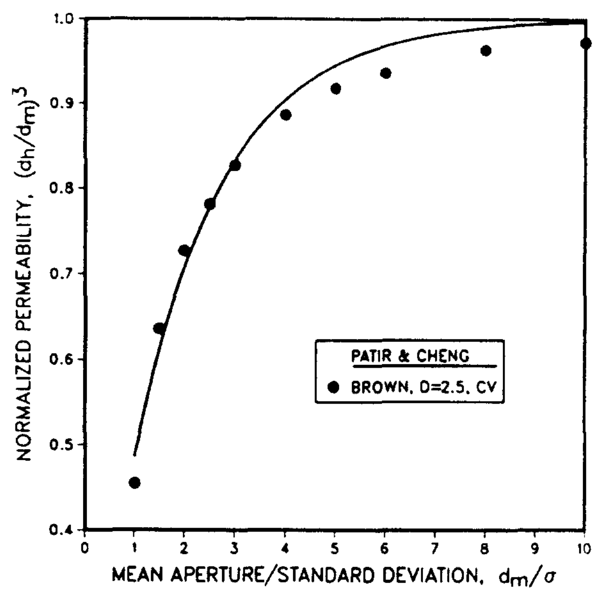

Fig. 1. Plot showing the effect of roughness on permeability. The hydraulic aperture is d_h, the mean aperture is d_m and the standard deviation of the aperture is σ. The curve labelled "Patir and Cheng" is from (4), which was fit by them to their numerical results, while the data points labelled "Brown" each represent the mean of 10 different realizations of surfaces with fractal dimensions of 2.5.

Patir and Cheng [3]. Each one of Brown's data points represents the mean of 10 different realizations. The solid curve, which was found by Patir and Cheng to provide a reasonable fit to their numerical results, can be expressed by:

$$\left(\frac{d_h}{d_m}\right)^3 = 1 - 0.90\mathrm{e}^{-0.56 d_m/\sigma}. \qquad (4)$$

The similarity between the results of Brown [8] and Patir and Cheng [3] suggests some universal (approximate) validity of the correlation (4) between the dimensionless parameters d_h/d_m and d_m/σ that is otherwise independent of the statistics of the aperture distribution. (The results found by Brown [8] for different fractal dimensions very nearly coincided. The data shown in Fig. 1 were for the case in which "conservation of volume" was imposed on the fracture during its deformation. The details of this constraint are not relevant to the present discussion, since its imposition had little influence on the calculated permeability.) In order to lend further support to this hypothesis, we will study the Reynolds equation for a fracture geometry that is simple enough to allow analytical treatment, but which still captures some of the flavour of "rough-walled" fractures. To accomplish this, we first restrict equation (2) to one dimension. Although this is an approximation, note that the flow will always be locally 1-D (see Fig. 5 of [8]); furthermore, the 2-D character of the flow field will be accounted for by an averaging procedure. If the x axis is chosen so as to coincide with the macroscopic pressure gradient, then the 1-D version of (2) is simply:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(d^3(x)\frac{\mathrm{d}P}{\mathrm{d}x}\right) = 0. \qquad (5)$$

A single integration of (5) yields:

$$d^3(x)\frac{\mathrm{d}P}{\mathrm{d}x} = C, \tag{6}$$

where C is a constant of integration. Comparison of (3) and (6) shows that the constant of integration is merely $12\mu|Q|$, where Q is the volumetric flow rate. A second integration yields:

$$P_2 - P_1 = 12\mu|Q|\int_{x_1}^{x_2}\frac{\mathrm{d}x}{d^3(x)}. \tag{7}$$

This result expresses the pressure drop between two points x_1 and x_2 in terms of a certain integral of the aperture function $d(x)$. If (7) is expressed in terms of the hydraulic aperture d_h, the result is:

$$Q = \frac{d_h^3}{12\mu}\cdot\frac{\Delta P}{L},$$

where

$$d_h = \left[\frac{1}{L}\int_{x_1}^{x_2}\frac{\mathrm{d}x}{d^3(x)}\right]^{-1/3}, \tag{8}$$

and $L = x_2 - x_1$. Using brackets to denote the "mean value", (8) can be expressed as $d_h = \langle d^{-3}\rangle^{-1/3}$ (cf. [9]). It is worth noting that although (8) was derived as a solution of the approximate Reynolds equation, it can also be derived (see [10, 11]) as a rigorous "first approximation" to the full Navier–Stokes equations.

SINUSOIDAL APERTURE VARIATION MODEL

One of the simplest aperture profile functions that captures some of the geometrical properties of a "rough-walled" fracture is a constant aperture with a sinusoidal perturbation (Fig. 2):

$$d(x) = d_m[1 + \delta\sin(2\pi x/\lambda)], \tag{9}$$

where d_m is the mean aperture, δ is the magnitude of the "roughness" and λ is the wavelength of the aperture oscillations. For now, we imagine that all cross-sections parallel to the plane of Fig. 2 are identical. If the flow is in the direction transverse to the aperture oscillations (i.e. the x direction), then the hydraulic aperture can be found by considering (8) over one period of the oscillation:

$$d_h^{-3} = \frac{1}{\lambda}\int_0^{\lambda}\frac{\mathrm{d}x}{d_m^3(1+\delta\sin 2\pi x/\lambda)^3}. \tag{10}$$

A simple change of variables, $\zeta = 2\pi x/\lambda$, reduces (10) to:

$$d_h^{-3} = \frac{1}{2\pi}\int_0^{2\pi}\frac{\mathrm{d}\zeta}{d_m^3(1+\delta\sin\zeta)^3}, \tag{11}$$

which shows that, within the framework of the lubrication approximation, the wavelength of the roughness does not affect the hydraulic aperture. (The wavelength does enter into higher-order approximations to the Navier–Stokes equations, as might be expected; see [11] and below.) Evaluation of (11) gives [12, p. 383]:

$$d_h^3 = d_m^3\frac{(1-\delta^2)^{5/2}}{1+(\delta^2/2)}. \tag{12}$$

The above 1-D model assumes, in a sense, that the resistances due to each aperture element are in series, since the flow through each element is the same. If each aperture d is thought of as having a resistance proportional to d^{-3}, then (7) corresponds to all of the resistances being placed in series. The other "extreme" assumption would be that all of the resistances are in parallel. This would correspond to flow "into the page" of Fig. 2, and would be equivalent (cf. [13]) to hydraulic aperture given by $d_h^3 = \langle d^3\rangle$, i.e.

$$\begin{aligned} d_h^3 &= \frac{1}{\lambda}\int_0^{\lambda} d^3(x)\,\mathrm{d}x \\ &= \frac{1}{2\pi}\int_0^{2\pi} d_m^3(1+\delta\sin\zeta)^3\,\mathrm{d}\zeta \\ &= d_m^3[1+(3\delta^2/2)]. \end{aligned} \tag{13}$$

Note that while (12) shows that $d_h < d_m$ for flow transverse to the roughness, (13) indicates that $d_h > d_m$ for flow parallel to the roughness.

It is known from network theory [14] that the assumption that all of the resistors in a random resistor network are in series (or parallel) provides lower (or upper) bounds respectively to the actual effective conductivity. In our problem, we know that at some points the fluid will be flowing parallel to the aperture oscillations, while in some cases it will be flowing transverse to the oscillations. One simple way to arrive at an estimate of the effective conductivity of a "random mixture" of these two cases is to use the geometric mean of the conductivities given by (12) and (13):

$$d_h^3 = \sqrt{d_h^3(\text{series}) \times d_h^3(\text{parallel})}$$

$$= \sqrt{\langle d^{-3}\rangle^{-1}\langle d^3\rangle} \tag{14a}$$

$$= d_m^3\left\{\frac{[1+(3\delta^2/2)](1-\delta^2)^{5/2}}{1+(\delta^2/2)}\right\}^{1/2}. \tag{14b}$$

A somewhat more sophisticated estimation of the effective conductivity can be found by appealing to the upper and lower bounds that were derived by Hashin and

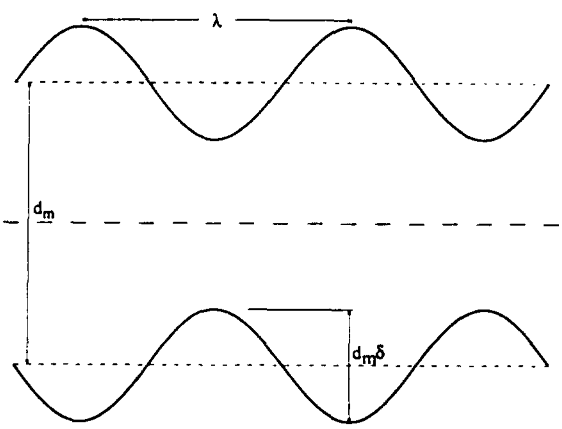

Fig. 2. Fracture with a sinusoidal variation in aperture. d_m Is the mean aperture, δ is the (relative) amplitude of the aperture variation and λ is the wavelength of the aperture variation.

Shtrikman [15] for the effective conductivity of a random mixture consisting of two "components" with different conductivities. These bounds are known to be closer together than the above-mentioned series and parallel bounds. For a 50–50 mixture of elements with conductivities G_s and G_p, where $G_s < G_p$, the Hashin–Shtrikman bounds on the overall conductance are:

$$G_s + \frac{3G_s(G_p - G_s)}{5G_s + G_p} < G < G_p - \frac{3G_p(G_p - G_s)}{5G_p + G_s}. \quad (15)$$

Aside from the factor of $1/12\,\mu$, which is common to all terms, the hydraulic conductance is equivalent to the cube of the hydraulic diameter, i.e. $G = d_h^3/12\,\mu$. Hence the bounds (15) can be applied to d_h^3, with (12) and (13) used for G_s and G_p. The geometric mean of these two bounds will provide an estimate of the effective conductivity that has an narrower possible margin of error than the geometric mean of the series and parallel bounds, as in (14).

In order to compare our semi-analytical results with those of Brown [8] or Patir and Cheng [3], we also need an expression for the standard deviation of the height distribution σ. Since $d - d_m = d_m\delta \sin(2\pi x/\lambda)$:

$$\sigma = \left[\frac{1}{2\pi}\int_0^{2\pi} d_m^2\,\delta^2 \sin^2\zeta d\zeta\right]^{1/2}$$

$$= \sqrt{d_m^2\delta^2/2} = d_m\delta/\sqrt{2}. \quad (16)$$

Hence $\delta = \sqrt{2}\sigma/d_m$ for the sinusoidal profile shown in Fig. 2, and using this fact we can plot the geometric mean from expression (14), and the mean of the Hashin–Shtrikman bounds from (15), alongside the results of Brown [8] and Patir and Cheng [3]. This is done in Fig. 3, where it is seen that all of the results are qualitatively similar, showing appreciable divergence only for small values of d_m/σ (i.e. very rough surfaces). The simple geometric mean of the series and parallel conductivities yields almost the same result as the geometric mean of the Hashin–Shtrikman upper and lower bounds. (This lends further credence to the use of the geometric mean to estimate effective permeabilities. In the somewhat different context of 2-D porous formations with stochastic permeability distributions, Dagan [16] showed that the geometric mean often yields an accurate estimate of the effective permeability.) Each of these means matches the curve of Patir and Cheng [3] closely for $d_m/\sigma > 3$, but fall below this curve for smaller values of d_m/σ. The data of Brown [8] fall close to the curve, although these data approach the asymptotic value of $d_h/d_m = 1$ more slowly as $d_m/\sigma \to \infty$.

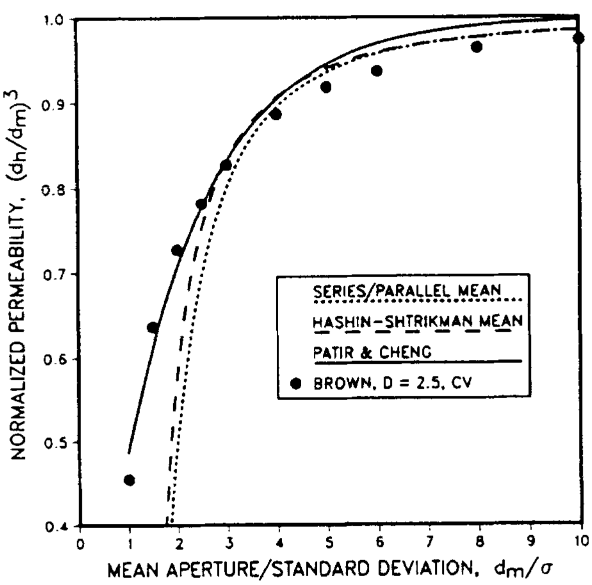

Fig. 3. Normalized permeability of a fracture as a function of the standard deviation of the roughness. Two different averaging methods, using the series/parallel bounds and the Hashin–Shtrikman bounds, have been used for the sinusoidal model.

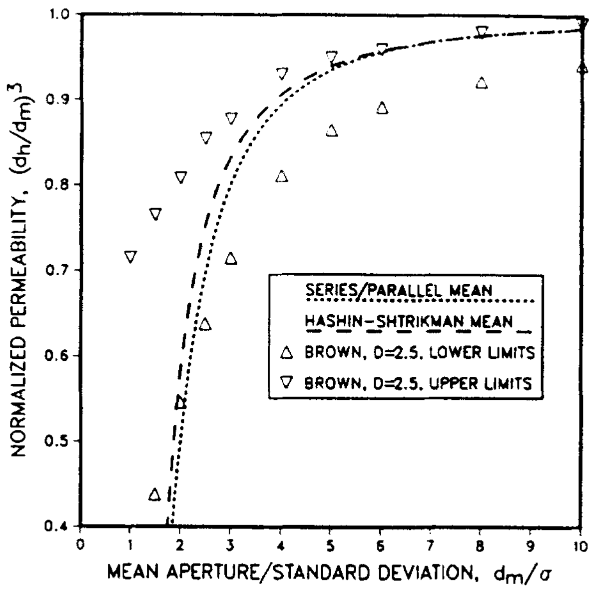

Fig. 4. Normalized permeability of a fracture as a function of the standard deviation of the roughness. The geometric mean of the series and parallel bounds for the sinusoidal model is compared to the range of values computed by Brown [2] for a fractal dimension of 2.5.

Figure 4 shows the geometric mean of the series and parallel bounds for the sinusoidal model, compared with the upper and lower values of each set of 10 realizations considered by Brown at each value of d_m/σ. This geometric mean generally lies within the range of values found by Brown; it falls near the upper range for large values of d_m/σ, and towards the lower range for small values of d_m/σ. For small values of d_m/σ, which is to say relatively rough surfaces, the sinusoidal model permeabilities drop off more rapidly than do the mean values found by either Brown [8] or Patir and Cheng [3]. This is easily understood because, for example, as $\delta \to 1$, the "series" conductivity goes to zero, and so both sets of lower bounds upon which the "geometric means" are based will vanish. Since $d_m/\sigma = \sqrt{2}/\delta$, this occurs at $d_m/\sigma = 1.414$. In reality, of course, the flow would simply flow around any local point of contact between the two faces of the fracture, and the hydraulic conductivity would not drop to zero. This possibility is not entirely accounted for by the averaging methods that we have used to find to overall effect permeability.

It is worthwhile to investigate to what extent additional roughness with higher spatial frequencies alters

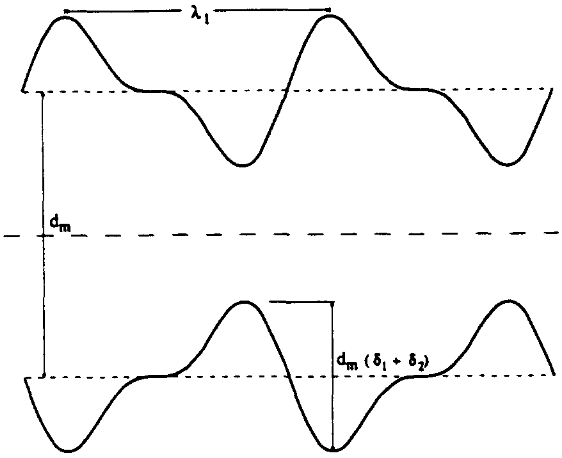

Fig. 5. Fracture whose aperture variation contains two sinusoidal components, as represented by (17), with $\lambda_1/\lambda_2 = 2$, $\delta_1/\delta_2 = 2$.

the analytical results. This can be done by using a profile that contains two sinusoidal components (see Fig. 5):

$$d(x) = d_m[1 + \delta_1 \sin(2\pi x/\lambda_1) + \delta_2 \sin(2\pi x/\lambda_2)]. \quad (17)$$

In this case an analytical evaluation of $\langle d^3 \rangle$, $\langle d^{-3} \rangle$ and σ is not practical, but can be easily carried out numerically. In general, the results show that the addition of this "smaller scale roughness" has only a minor effect on the relation between d_h/d_m and d_m/σ. As an example, consider the addition of a roughness component with a wavelength of one-half of the dominant wavelength, i.e. $\lambda_1/\lambda_2 = 2$. A reasonable value for the amplitude of this component can be found by the following considerations. The power spectral density of a rock surface is usually of a form that can be expressed as [17]:

$$\delta(\lambda) = \text{constant} \times \lambda^{3.5-D}, \quad (18)$$

where $2 < D < 3$ is the fractal dimension of the surface. Most rock surfaces seem to have fractal dimensions that lie between 2 and 2.5, since fractal dimensions near 3 correspond to an extreme degree of roughness, for which nearby apertures are completely uncorrelated [18]. Using a value of $D = 2.5$, as in those results of Brown [8] plotted in Fig. 3, we see that δ is proportional to λ, and so reasonable values for δ_2 can be found by setting $\delta_2/\delta_1 = 0.5$. Note that the purpose of this calculation is merely to find values of δ_2 that can be used as a meaningful example, and is not intended to be "exact" in any way. Figure 6 shows the geometric mean of the series and parallel conductances for this model, compared with the results for $\delta_2 = 0$. The addition of a higher-frequency roughness component is seen to have little effect on the d_h/d_m vs d_m/σ relation. We have tested this result by adding further roughness components of smaller wavelength, and this conclusion seems to hold in general.

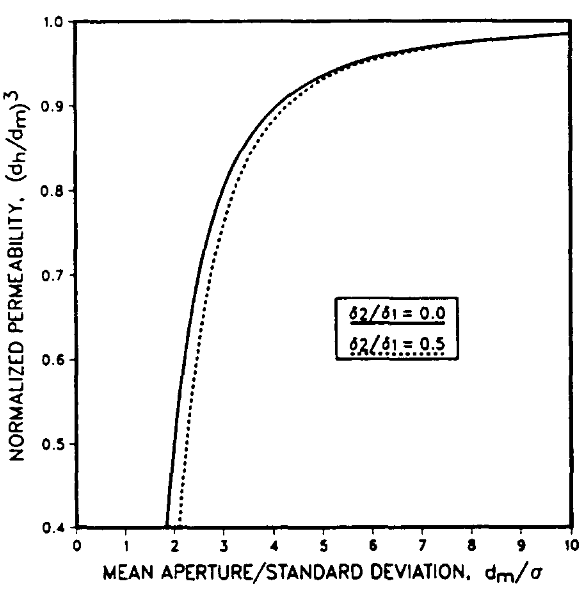

Fig. 6. Comparison of the permeabilities of fractures with one or two sinusoidal components in their aperture variation (see Figs 1 and 5), according to the geometric mean of the series and parallel values. The ratio λ_1/d_m is varied, while maintaining $\lambda_1/\lambda_2 = 2$ and $\delta_1/\delta_2 = 2$.

In order to further estimate the sensitivity of the analytical result (14b) to the shape of the aperture profile, a similar analysis has been carried out for a fracture with a "sawtooth" profile, such as that used by Elsworth and Goodman [19]. The extreme case, as far as irregularity of the aperture distribution is concerned, is when one face of the fracture is displaced from the other by one-half of a wavelength (see Fig. 11 of [19]). Within a "unit cell" consisting of one-half a wavelength, the aperture can be expressed as $d(x) = d_{min} + (d_{max} - d_{min})x/L$. The various statistical parameters can readily be found to be $d_m = (d_{max} + d_{min})/2$, $\sigma = (d_{max} - d_{min})/2\sqrt{3}$, $\langle d^3 \rangle = d_m^3 + 3d_m\sigma^2$ and $\langle d^{-3} \rangle^{-1} = (d_m^2 - 3\sigma^2)^2/d_m$. The geometric mean of the series and parallel conductances, which in general is given by (14a), takes the form:

$$d_h^3[\text{sawtooth}] = d_m^3\{[1 - 9(\sigma/d_m)^4][1 - 3(\sigma/d_m)^2]\}^{1/2}. \quad (19)$$

This is a different relation between d_h/d_m and σ/d_m than was predicted for the sinusoidal profile. However, when plotted as in Fig. 3, (19) lies very close to (14b); in fact, the two expressions agree exactly to "first order": $(d_h/d_m)^3 = 1 - 1.5(\sigma/d_m)^2$. Within the context of our analytical model, therefore, the dependence of the normalized permeability on the single parameter σ/d_m appears to be somewhat insensitive to shape.

The results discussed above, and shown in Figs 3 and 6, lead us to conclude that within the context of the lubrication approximation there is a strong correlation between the parameters d_h/d_m and d_m/σ, with the additional statistical details of the surface roughness profile providing only a small perturbation on this relation. The fact that our quasi-analytical results are similar to the numerical results found by Patir and Cheng [3] and Brown [8] lends support to this conjecture. However, all of these analyses are predicated on the use of the lubrication approximation to reduce the full Navier–Stokes equations to the Reynolds equation. We now carry out some analysis aimed at estimating the errors incurred by using the lubrication approximation, and at delineating those ranges of the fracture roughness parameters that will allow this approximation to be used.

HIGHER-ORDER CORRECTIONS TO THE LUBRICATION APPROXIMATION

One of the assumptions needed to justify the reduction of the Navier–Stokes equations to the Reynolds equation is that the velocity gradients in the plane of the fracture are much smaller than the velocity gradient in the direction perpendicular to the fracture plane. Brown [2] correctly states that this is in some sense equivalent to the condition (in the present notation) that $\sigma/\lambda \ll 1$; this can be understood by noting that rapid changes in the aperture will necessitate rapid variations in the in-plane velocity, in order to maintain conservation of mass. Interestingly enough, this condition imposes no restrictions on the ratio σ/d_m; relatively large values of σ are permissable, as long as the variation in aperture occurs slowly in the x or y directions. Brown [2] examined the velocity gradients that he computed numerically, and found that the condition on their magnitudes was not always satisfied. In fact, he concluded that the fracture walls would have to be smooth on length scales on the order of 500–5000 μm, which is certainly not usually the case for real fractures. (Note that Fig. 3 implies that a fracture can be considered "hydraulically" smooth if the amplitude of any roughness is less than about 0.1 d_m.) Furthermore, Brown's analysis does not quantify the errors that are incurred by use of the lubrication approximation for fractures with "rapidly-varying" apertures.

Strict "error estimates" for the lubrication theory approximation to the Navier–Stokes equations are difficult to derive. A more tractable approach is to focus on a specific geometry such as that shown in Fig. 2, and examine the solutions to higher-order approximations to the Navier–Stokes equations. In this way we can find the range of values for the parameter σ/λ for which the lubrication approximation is "valid". This will provide us with a rough rule-of-thumb that should apply to more general fracture aperture profiles.

Hasegawa and Izuchi [11] performed a perturbation analysis of the problem of flow between a smooth wall and a wall with sinusoidal roughness. Although this geometry is slightly different from that shown in Fig. 2, it can still serve to demonstrate the influence of the effects of roughness, wavelength, etc. on permeability. Their results can be put into a form in which the small perturbation parameters are σ/λ and Re, the Reynolds number. We will set $\mathrm{Re} = 0$ in their expansions, and concentrate on the effect of σ/λ. Note that these authors fixed the pressure gradient along the channel, and found perturbation expansions for the volumetric flow rate; their results are therefore more readily usable, for our purposes, than the related results of Van Dyke [20], who considered a fixed flow rate, and found expressions for the stream function.

When translated into the present notation, the second-order expansion found by Hasegawa and Izuchi [11] can be expressed as [see their equations (25) and (27), and Fig. 1]:

$$d_h^3 = d_m^3 \frac{(1-\delta^2)^{5/2}}{1+(\delta^2/2)}\left[1 - \frac{6\pi^2(1-\delta^2)\delta^4}{10[1+(\delta^2/2)]}\left(\frac{d_m}{\lambda}\right)^2\right]. \quad (20)$$

Comparison of (20) and (12) shows that the second term in brackets in (20) is the correction due to nonzero values of σ/λ. At first sight it might appear that, due to the $1/\lambda^2$ dependence, the correction term could easily be very large if λ were small enough. However, the amplitude of the roughness usually drops off rapidly with increasing spatial frequency. For example, (18) implies that as λ decreases, $\delta = \text{constant} \times \lambda^s$, where s lies between 0.5 and 1.5. However, as indicated above, realistic values of s lie between 1 and 1.5 (corresponding to surface fractal dimensions between 2 and 2.5). Hence we see that if λ is small, δ will necessarily be small also and the correction term will remain bounded. For example, consider the "worst case", $s = 1$, for which $\delta = C\lambda$. The correction term in (20) then scales as $\lambda^4/\lambda^2 = \lambda^2$, and will therefore be very *small* for small spatial wavelengths. This is analogous to the fact, well-known to fluid-flow engineers, that for *laminar* flow in a pipe, small-scale roughness has no effect on the hydraulic resistance. This insensitivity of laminar pipe flow to small-scale roughness is illustrated in the "Moody–Nikuradse" charts [4, p. 580], in which the "friction factor" is plotted against the Reynolds number. The curves corresponding to different values of the "relative roughness" do not diverge until the turbulent (high Reynolds number) regime.

The most stringent condition that we can derive by requiring the correction term to be small is actually found by considering the *largest* wavelength roughness. We first use (16) to replace d_m with $\sqrt{2}\sigma/\delta$, and note that, over the range of definition of δ, which is $0 < \delta < 1$, the maximum value of the term that multiplies $(\sigma/\lambda)^2$ is 2.39. Therefore, if we want to restrict the relative magnitude of the correction term to 10% of the value predicted by lubrication theory, we must have $2.39\,(\sigma/\lambda)^2 < 0.10$, which implies $\lambda > 5\sigma$. This condition is much less restrictive than the condition that was postulated by Brown [2], which was $\lambda > 50\sigma$. If we agree that a sinusoidal surface can be considered "smooth" over lengths not greater than about $\lambda/10$, say, this new criterion can (very roughly) be viewed as requiring the surfaces to be smooth over lengths on the order of σ. Further evidence supporting this conclusion can be found in the work of Pozrikidis [21], who used a boundary-integral method to study the same problem that Hasegawa and Izuchi [11] analyzed by perturbation methods. Although Pozrikidis did not solve for the flow rate, he did show that as long as $\lambda > 5\sigma$, the streamlines will adjust to the curvature of the wall, and no eddies will be generated within the sinusoidal bumps. Since the existence of such eddies (see Fig. 11 of [22]) would cause a breakdown of the quasi-1-D lubrication assumption, the results of Pozrikidis corroborate our conclusion that the lubrication assumption will not become invalid as long as $\lambda > 5\sigma$.

SUMMARY AND CONCLUSIONS

The lubrication approximation has been used to study the permeability of rough-walled rock fractures. A major purpose of this study was to develop an understanding of how the hydraulic aperture depends on the statistics of the aperture distribution. When the various aperture elements are in series, the effective hydraulic aperture is given by $d_h = \langle d^{-3} \rangle^{-1/3} < d_{mean}$, while if they are arranged in parallel, $d_h = \langle d^3 \rangle^{1/3} > d_{mean}$. Since the assumption of series (or parallel) resistances vastly underestimates (or overestimates) the actual effective conductance [14], a further averaging of these two values is needed. This can be achieved using either the geometric mean of the series and parallel conductances, or the geometric mean of the (narrower) Hashin–Shtrikman bounds; both methods provide similar estimates of the effective hydraulic aperture.

For fractures whose apertures vary sinusoidally (or in a sawtooth manner), analytical expressions were found for $\langle d^3 \rangle$ and $\langle d^{-3} \rangle$, in terms of $d_{mean} = \langle d \rangle$ and the standard deviation of the aperture σ which is equal to $\sqrt{\langle d^2 \rangle - \langle d \rangle^2}$. The motivation for analyzing these models was that they permit analytical treatment, while still capturing some of the qualities of "roughness". For these models, the predicted hydraulic aperture is always less than the mean aperture, by an amount that increases with increasing roughness. In fact, the resulting relations between d_h/d_m and σ/d_m are very similar to that found numerically by Brown [8] and Patir and Cheng [3], who studied fractures with highly irregular roughness profiles. Our analytical expression for d_h for the sinusoidal model fell within the spread of values found by Brown in his various stochastic realizations.

An attempt was also made to estimate the conditions under which the lubrication approximation would be valid in treating flow through fractures. Although this approximation has been frequently used, quantitative estimates of its accuracy have not been available. By examining the higher-order solutions to the Navier–Stokes equations for flow through a sinusoidally-varying channel, developed by Hasegawa and Izuchi [11], we have been able to make some comments regarding this question. Deviations from the permeability predicted under the lubrication approximation seem to become appreciable only when the spatial wavelength of the dominant roughness component becomes on the order of (or smaller than) the amplitude of that roughness. This implies that the condition hypothesized by Brown [2], which was that the fracture should be smooth over distances of at least 50σ, was in fact much too conservative. If our conclusion is correct, then the use of the lubrication approximation would be justified for many real fractures, and the more difficult Navier–Stokes analysis could be avoided.

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