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Atmospheric refraction effects in Earth remote sensing

Peter D. Noerdlinger *

SM & A, Code 902, Building 32, Goddard Space Flight Center, Greenbelt, MD 20771, USA

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Abstract

Just as refraction moves the apparent positions of stars from their true ones, it slightly distorts the view of Earth from space, as well as affecting the angle at which sunlight or moonlight illuminates its surface. The astronomer's problem is to correct the apparent position of a star for refraction, relating the position as observed from Earth to the true position. By contrast, the space observer's problem is to obtain the true (refracted) surface zenith angle z' of illumination or of viewing, when the zenith angle z_0 of the ray in space is known, and to correct for the apparent horizontal displacement of the surface point being viewed. This paper solves the problem of the refraction angle for a spherical atmosphere by a simple, analytic solution, depending *only* on the surface index of refraction μ_0 namely: $\sin(z_0) = \mu_0 \sin(z')$. The problem of the apparent horizontal displacement of the point viewed is also solved analytically, but approximately, because the result depends weakly on an assumed vertical structure of the atmosphere. The results are useful primarily in cases where observation must be done at large zenith angle, or low Sun angle.

Keywords: spaceborne imagery; stmospheric refraction; BRDF; satellite observations; Earth limb observations; terminator observations

1. Introduction

1.1. The problem

In space sensing of terrestrial data in optical wavelengths, atmospheric refraction affects the angle of solar (or lunar) illumination at the earth, the viewing angle, and the apparent position of the lookpoint, or terrestrial point being viewed. Rays of light striking the atmosphere are refracted toward the zenith as they descend, and outgoing rays away from the zenith as they ascend. In both cases, the ray is closer to the zenith lower in the atmosphere. Thus, refraction introduces small corrections to both the angles used in defining the bi-directional reflectance distribution function, or BRDF (Burgess and Pairman, 1997).

The present work develops an analytic method that determines the angle of atmospheric refraction and the apparent displacement of the point of Earth intersection due to refraction, assuming a spherically symmetric atmosphere. The equation for the angle is rigorous, and the approximation for the displacement is accurate within a tolerance comparable to the variations to be expected from the weather.

The method requires only four equations: one rigorous conservation law for the product of the geocentric distance of a point on the ray, the index of refraction at that point, and the sine of the zenith angle there; one geometric equation usually used, in

* Fax: +1-301-614-5268; E-mail: pnoerdli@gsfcsrvr4.gsfcmo.ecs.nasa.gov

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astronomy, to correct eclipse and occultation predictions for the heights of observatories, and two empirical formulas for refraction, spliced together at 83.9° zenith angle. The empirical formulas are needed only for determining the displacement of the ray along the earth's surface, not for the refraction angle. A simple atmospheric model is provided that simulates typical conditions at sea level and allows extension of the algorithm to other altitudes.

1.2. Previous analyses

Existing studies and algorithms for refraction (Chauvenet, 1885; Garfinkel, 1944; Hohenkerk et al., 1992; Stone, 1996) were all derived from ground based measurements of the heavens and are expressed in terms of the *refracted* (Earth surface) zenith angle. Such algorithms are unsuited for space observation, where the *unrefracted* zenith angle is known. For ground-based observations, the lateral displacement of the refracted ray from the prolonged unrefracted ray *along the vertical* was calculated in order to correct eclipse predictions for the height of the observatory. In the case of space observation, the *vertical* displacement is of less interest, but the observer needs to know how far the true point of Earth contact of the ray is moved *along the horizontal* by refraction. The two problems are related (Section 2.2.3). The analysis presented here resulted from re-analysing the classical refraction theories, designed for terrestrial observation, in order to derive corrections to geolocation algorithms (Noerdlinger, 1994) in the Mission to Planet Earth EOSDIS (NASA, 1998) Science Data Processing Toolkit (NASA, 1997), an extensive set of software used in processing remote sensing.

At first glance, it might appear difficult to replace the various ad hoc recipes for refraction with one that depends only on the angle of incidence. It is not possible to assume plane geometry — a spherical atmosphere must be used. Indeed, the Moderate Resolution Imaging Spectrometer (MODIS) experiment (Barnes and Salomonson, 1993; Hoyt and Storey, 1994; MODIS, 1998) of Mission to Planet Earth relies on a FORTRAN finite difference program, originally written by Dr. Douglas Hoyt of Research Data (RDC), and modified by Mr. James Storey of STX, The authors kindly provided their program, which has been used as a check (Section 4) on the present analytic work. A listing of the program is available from the author on request. It has been long recognised (Chauvenet, 1885) that there are certain conservation laws at work, even for a spherical atmosphere. It has been possible, by adding a little further analysis to Chauvenet's, to devise an equation that yields the zenith angle at the surface analytically, as a function of the unrefracted angle of incidence and the surface index of refraction only. The displacement of the terrestrial point from the unrefracted one is then found using empirical approximations. The extremely simple atmosphere model used here serves only to obtain the surface index of refraction for any specified elevation off the geoid; it is not used to integrate differential equations for the ray.

1.3. Limitations of scope and assumptions

1.3.1. Geometrical restrictions

In the astronomer's problem, differential refraction distorts a highly refracted image and spreads each point into a small, vertical spectrum of the source; similarly, for the space observer, the bending and displacement of near grazing rays will be strongly wavelength and angle dependent. The results of this paper can be adapted to such problems, but the present study is confined to a single ray at a single wavelength. Limb sounders employ rays that refract around the earth without striking the surface, a case not treated here. This study does not directly treat airborne observation, but it is adaptable, for our key result is a simple conservation law, Eq. (9), which holds in any case. All geometrical calculations are based on a spherical Earth and a spherically symmetric atmosphere. The oblateness of the earth should be taken into account in finding the earth point of interest from the original space data but once that is done, the loss of accuracy by reverting to a spherical atmosphere herein is negligible. Thus, the latitude dependence that is introduced later idealises the problem, by assuming that within the region traversed by the ray, a spherically symmetric approximation suffices. This is justified by noting that a ray which traverses a substantial range in latitude must be a grazing ray, and so must be subject to large fluctuations from weather that would dominate the error in using a spherical model. Fluctuations due to weather cannot be treated by the methods presented here. See also Section 1.2.

1.3.2. Use of mean wavelength and atmospheric conditions

All the algorithms derived here will be for monochromatic light, ignoring scattering and absorption. The approximate equations adapted from various sources for the classical total refraction as a function of the refracted angle, are for ''white light'', which we can assume is centred in the approximate range 460 to 530 nm. In Section 2.2.3.2 selected references are given to the recent literature on refraction, which could be used to modify our equations for different wavelengths, temperatures, and different water vapour contents.

1.3.3. Terrain effects are represented only as altitude differences

Although our model allows for the variation of index of refraction due to changes in altitude, it does not, by itself, account for the effect of terrain geometry on the intersection of the ray with the earth's surface. The geometric displacement from the intersection of the ray with an ideal ellipsoid is often a much larger effect than refraction! In imaging a plateau at 2000 m altitude, at zenith angle of, say, 45° the lookpoint would be displaced 2000 m due to geometry, and only about 4 m due to refraction. There are simple ways to handle the geometrical correction for a rather flat plateau. For example, assuming that one knows the approximate location, one can find the intersection of the line of sight with an earth ellipsoid inflated to the required altitude. If the terrain is jagged, however, iterative methods appear to be needed (Nishihama et al., 1997). In principle, for this case, the refractive vertical correction to the ray path could be significant, because it could cause a ray to hit or to miss a mountain peak, for example. It is believed that Eq. (2) or Eq. (10), below, could be adapted to dealing with the vertical displacement of the ray, because Eq. (2) is used to correct eclipse observations in a like manner, and Eq. (10) is its equivalent for space sensing. These equations, however, have not been tested in the space sensing context. Either one of them requires an empirical equation for refraction, such as is developed in Section 2.2.3, Eqs. (17) and (18).

2. Analysis

2.1. Problem definition and notation

We shall refer to the zenith angle [1] of the ray in space, prolonged, where it meets the earth, as the ''unrefracted zenith angle'', and that at the surface as the ''refracted (or true) zenith angle''. Table 1 gives the notation for the derivations, including the atmospheric model in Appendix A. The symbol '' $\triangle$ '' indicates a triangle and not an increment. Angles are in rad unless otherwise specified. The lapse rate is defined unsigned but acquires a negative sign in use. The asterisk '' $*$ '' is occasionally used for multiplication, when the juxtaposition of symbols could be confusing. As in common usage, due to the near axial symmetry of the earth, the terms ''spheroid'' and ''ellipsoid'' are used interchangeably for figures used to represent the earth.

In Fig. 1, a distant observer D views point P′, which is displaced by distance d from the intersection of the unrefracted ray $\overline{\mathrm{DP}}$ with the earth. Point O is Earth centre. Note that there are three different angles that we must relate: z_0 is the zenith angle at the intersection of the idealised unrefracted ray with Earth, z the zenith angle of the same (unrefracted) ray at the vertical of the true, or refracted, Earth intersection, and z' is the zenith angle of the refracted ray at the earth's surface. In this discussion, ''Earth's surface'' means the end of the ray, be it on the ellipsoid, on some terrain, or on a low cloud layer. Note that the difference between z and z_0 is due to the sphericity of the earth, and that, as it descends, the ray also travels laterally. Of course, the known quantity is z_0, and the required quantity is z'. The angle z is of interest for two reasons only: (1) the difference between z and z_0 is needed to determine the displacement d, and (2) the original data on refraction were tabulated in terms of z' and fitted to a function that gave $(z - z')$ from z', without regard to z_0. Indeed, the conventional angle of refraction is z-z', but *from the standpoint of space observation,*

[1] The units for the zenith angles and angle of refraction are left open when they appear within a trigonometric function, or within a linear, homogeneous equation. When the units are important, as when numerical values are given, they are specified in each case.

Table 1
Notation for refraction and the atmosphere models

Symbol	Definitions of symbols and values of constants
A (m)	Earth radius (spherical model)
a (m)	Altitude off the geoid
a_{tropop} (m)	Altitude of tropopause
d (m)	Displacement of the point viewed, along Earth's surface
d_{Ang} (rad)	Angular displacement of the point viewed about Earth centre
densFac	Ratio of density to that at sea level ρ/ρ_0
east	Eastward unit vector on the Earth's surface
f (Pa)	Partial pressure of water vapour (1 Pa = 1 N/m^2 = 0.01 mb)
g_0 (m/s^2)	Mean sea level acceleration of gravity = 9.805
H (deg)	Refracted ray's elevation off horizon at Earth's surface
h (m)	Altitude from observation point to the unrefracted ray
Molec_{mean}	Mean tropospheric molecular weight ~ 28.825 (see Appendix A)
N	Unit vector normal to the spheroid
north	Northward unit vector on the Earth's surface
P (Pa)	Pressure
P_{mb} (mb)	Pressure
q (m)	Geocentric distance of any point along the ray
$r_{L,trop}$ (K/m)	Tropospheric temperature rate with altitude (lapse rate)
Refr (rad)	Conventional refraction angle $= z - z'$
Refr(z')	Refr as a function of z' (rad)
Refr_{deg}	Conventional refraction angle in ° $= 180\,(z - z')/\pi$
Refr_{arcsec}	Conventional refraction angle in s of arc
Refr_0 (rad)	Refraction angle $= z_0 - z'$
R_{gas} (J (k mol)$^{-1}$ K^{-1})	Ideal gas constant[a] = 8314.3
T (K)	Temperature
T_C (°C)	Temperature
$T_{sealevel}$ (K)	Mean sea level temperature
T_{tropop} (K)	Temperature at the tropopause
tempFac	Ratio of temperature to that at sea level $T/T_{sealevel}$
$\mathrm{tempFac}_{tropop}$	Ratio of temperature at tropopause to that at sea level $T_{tropop}/T_{sealevel}$
$\hat{\mathbf{u}}$	Unit vector along the outward ray
$\hat{\mathbf{u}}_h$	Projection of the $\hat{\mathbf{u}}$ on the horizontal plane
v (m^3/kg)	Specific volume of air; reciprocal of the density
W (m)	Density scale height
z	Zenith angle in space, relative to **N** at true Earth intersection
$z(q)$	Running zenith angle of refracted ray relative to **N**
z_{0r}	Running zenith angle of unrefracted ray relative to **N**
z_0	Zenith angle in space, relative to normal at base of unrefracted ray
z'	Zenith angle of refracted ray at the Earth's surface, relative to **N**
Γ	Polytropic index such that $\rho \propto T^{-\Gamma}$ in the troposphere
λ	Latitude
Λ	Longitude
μ	Refractive index of air (a function of a)
μ_0	Refractive index of air at the point of observation
ρ (kg/m^3)	Density of the atmosphere
ρ_0 (kg/m^3)	Density of the atmosphere at zero altitude
$\langle \rho_0 \rangle$	Global average, over latitude, of ρ_0
ψ	Azimuth of the outward ray projected on the Earth

[a]We use the k-mole (kilo mole) value here, which is not an SI value. The SI system adopts a value effectively based on the gram mole (kilogram mole divided by 1000), tantamount to an exception, for compatibility with older chemical data.

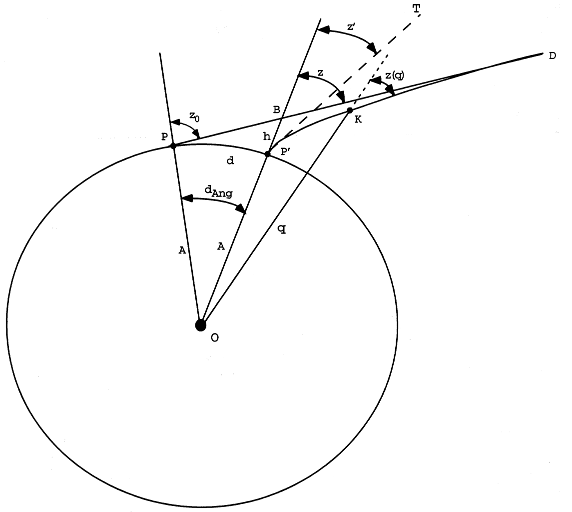

Fig. 1. Geometry of terrestrial refraction.

the change in the zenith angle due to refraction is $z_0 - z'$.

Again referring to Fig. 1, we see that $z_0 > z > z'$. The difference between z_0 and z is, however, quite small. Note that for very distant astronomical objects, there is a ray parallel to line $\overline{DP}$, which if unrefracted, would meet the earth with zenith angle z (not z_0) at P′. Because this ray could have been used in place of $\overline{DP}$, it is understandable that no distinction was made, traditionally, between z_0 and z, except when discussing eclipses. In Fig. 44 of Chauvenet (1885), point D is the same as ours, but we omit the line $\overline{OD}$, which he draws, and we include the line $\overline{OP}$, which he omits. Line $\overline{P'T}$ is tangent to the ray at P′. Our usage of h, q, and μ is identical with Chauvenet's. In common with him, q is the geocentric distance of any point K on the refracted ray, but we denote the value of the zenith angle at K as $z(q)$, at variance with Chauvenet's notation i for that angle.

The displacement of the point where the ray meets the earth from its unrefracted intersection can be described by either of two quantities d or d_{Ang}; the former is a linear measure on the surface and the latter is an angular measure about earth centre. We need to derive one of these, say along with z', from z_0, eliminating z. Three equations in the four variables d_{Ang}, z', z, and z_0 are needed, which will leave one degree of freedom, allowing for the independent variation of z_0. We shall get the relation of z' and z from a modification of a standard refraction algorithm (Section 2.2.3.1), and the two equations for d_{Ang} and $z'(z_0)$ from the theory of refraction in a spherical atmosphere. It will turn out that a very simple relation between z' and z_0 exists, independent of the details of atmospheric structure, while to determine the displacement d_{Ang} requires knowledge of the angle z, whose value does depend on the structure and must be found from z' using empirical or semi-empirical formulas for $z(z')$.

Two well-known analytic approximations for $z(z')$ will be used for different ranges of the angles. The motivation for switching between these approximations will be discussed in Section 2.2.3.1 after the underlying theory. For the present, we refer to such algorithms generically as offering a function $\mathrm{Refr}(z') = z - z'$ as a function of z'.

2.2. *Derivation of the refraction algorithm*

2.2.1. *Geometrical description and previously known results*

To review the geometry, consider Fig. 1. The unrefracted ray $\overline{DP}$ would have struck the earth at P, making angle z_0 with the vertical there in the absence of an atmosphere. It is refracted so as to strike the earth at P′, at an angle z' from the vertical there. The unrefracted ray meets the local vertical from P′ at the angle z, which is usually denoted the unrefracted angle, because conventionally the rays from a distant celestial object are parallel. A line through P′, and parallel to $\overline{DP}$, to illustrate the point, has been omitted as inappropriate to the case of Earth remote sensing.

Again, the (horizontal) displacement of the ray is in a vertical plane containing the ray and is in the sense that the actual (refracted) ray will meet the earth at a displacement of $d = d_{\mathrm{Ang}} * A$ m from the geometrical (unrefracted) position, on the side towards the nearest horizon. The angle d_{Ang} is the angle in radians that the displacement in meters, d subtends at Earth centre.

In Eq. (149) and Eq. (564) of Chauvenet (1885), one finds two remarkable equations, rigorous for a spherical atmosphere. The first,

$$q \mu \sin[z(q)] = \mathrm{const.} \tag{1}$$

follows directly from Snell's law, and the second is

$$1 + h/A = \mu_0 \sin(z')/\sin(z). \tag{2}$$

In these equations, μ is the index of refraction at any altitude, and μ_0 that at P′. The variable q is the geocentric distance, and will be used as a *running* variable along the ray, so that Chauvenet's relation $q = A + h$ will not, in general, hold except at point B. Eq. (1) will be shown to lead directly to an analytic relationship between z_0 and z'. Eq. (2) was originally intended to correct eclipse and occultation calculations for the height of the observatory, but it will be used to solve the problem of determining d_{Ang}, and hence d. The problem addressed here, to determine the effect of refraction on the observation of Earth from space, is complicated by the fact that neither z nor z′ is the known angle z_0. If we can determine the difference z_0-z, we can solve $\triangle$POB for angle $d_{\mathrm{Ang}} \equiv \angle\mathrm{POB} = \angle\mathrm{POP}'$, which is the rad equivalent of d. Summing angles around $\triangle$POB and using the fact that z is the same as angle $\angle$PBO, we see that, with z in rad,

$$\mathrm{POP}' + (\pi - z_0) + z = \pi \tag{3}$$

or

$$d_{\mathrm{Ang}} = d/A = z_0 - z \ (\mathrm{rad}) \tag{4}$$

2.2.2. Solution for the angle of refraction

Eq. (2) looks temptingly as if we could use it to get z' from z, or vice versa, but it does not contain enough information; we shall need an empirical equation for $z(z')$. Eq. (1) comes directly from Snell's law, but it also expresses a conservation law that we shall exploit. Remember that it deals with the angle between the ray and the *local* vertical — a vertical that swings round the earth from $\overline{\mathrm{OD}}$ to $\overline{\mathrm{OP}}'$ as the ray descends. Following up on this idea, we shall see that although it cannot give the refraction $z - z'$, Eq. (1) relates z_0 and z'. Note that Eq. (1) becomes that of a straight line in polar coordinates as $q \to \infty$, $\mu \to 1$. Thus, outside the atmosphere, the Eq. (1) is rigorous but seemingly of little interest. Yet, it is most useful as a reference equation for *comparing* the relationship of zenith angle to geocentric distance along the unrefracted line $\overline{\mathrm{DP}}$ and the refracted ray $\overline{\mathrm{DP}}'$. Along the straight line $\overline{\mathrm{DP}}$, if z_{0r} is the running zenith angle (angle of the ray to the radius), then

$$q \sin(z_{0r}) = b \tag{5}$$

where b is a constant, equal to the distance closest approach of the line $\overline{\mathrm{DP}}$ to O. But from $\triangle$POB, applying the previous equation at P,

$$b = A \sin(z_0) \tag{6}$$

We have used the fact that $\angle$OPB has the same sine as its supplement, z_0. Furthermore, by Eq. (1), with $z(q)$ the running value of the zenith angle along $\overline{\mathrm{DP}}'$, and μ the running index of refraction, the actual ray has the equation

$$q \mu \sin[z(q)] = b_1 \tag{7}$$

where b_1 is another constant. But as $q \to \infty$, $\mu \to 1$, so the values of $z(q)$ become the same on the refracted and unrefracted rays there. Therefore, the two constants b_1 and b are *equal*,

$$b_1 = b \tag{8}$$

When the comparison is at the same geocentric radius, the factor q cancels when we use Eq. (8) to equate the left hand sides of Eqs. (5) and (7). Thus, $\sin(z_{0r}) = \mu \ \sin[z(q)]$. Applying this equation at $q = A$, we see that

$$\sin(z_0) = \mu_0 \sin(z') \tag{9}$$

where μ_0 is the surface index of refraction. Eq. (9) closes the system of equations. The remarkable simplicity of this equation makes one wonder why there are many complicated analyses (e.g., Garfinkel, 1944, 1967), depending on atmospheric models and many fits depending on refraction data. The reason is that for the classical case of terrestrial observation, z_0 is not known, and precisely when the refraction becomes large, at large zenith angles, then the difference between z_0 and z becomes significant. Even when point D is very far away, so that a ray parallel to $\overline{\mathrm{DP}}$ will meet P′ at angle z, it is only z that is then known from tabulated data on the star or planet; to know z_0 one must also know d or d_{Ang}. The fact that Eq. (9) seems previously unknown, although both Chauvenet (1885) and Garfinkel knew of Eq. (7), may be largely due to their not having plotted point P and defined angle z_0 — an angle of interest only to space observers. The work of Stone (1996) comes quite close to deriving Eq. (9), but is again designed for terrestrial use and omits use of point P and angle z_0. Stone optimises his approximations for highly accurate results at zenith angles of 75° or less, because outside that range, few useful astronomical observations can be made, and the reliability of any algorithm is also less. This is *complementary* to what is needed for space observation, for low Sun angles, or to observe near Earth limb, should that become necessary.

2.2.3. Horizontal displacement

Chauvenet's second equation, our Eq. (2), relates z and z_0. This can be verified by using the law of sines on $\triangle$POB, yielding

$$(A+h)/A = \sin(z_0)/\sin(z) \tag{10}$$

which agrees with Eq. (2) because $\sin(z') = \mu_0 \sin(z_0)$. We use Eq. (4) rather than Eq. (2) or Eq. (10), which are equivalent to it, being based on the same triangle, because we do not need h in the present discussion. Because all available data and approximations relate z and z', without regard to z_0, we first obtain z' from Eq. (9) as

$$z' = \arcsin\left[\sin(z_0)/\mu_0\right]. \tag{11}$$

This gives the surface zenith angle, which defines the angle of refraction as

$$\mathrm{Refr}_0 = z_0 - z' \tag{12}$$

To obtain d, using Eq. (4), we need z, which must be obtained by applying some empirical equation. There is a wide choice (Chauvenet, 1885; Garfinkel, 1967; Allen, 1973; Hohenkerk et al., 1992).

2.2.3.1. Spliced empirical approximations. The problem of refraction received attention from such well-known astronomers as Bradley, Argelander, and Bessel. We shall select a relatively simple equation, in view of the inevitable variations due to the weather. Our choice, equation [2] (H3.283-1) from Hohenkerk et al. (1992), while of good reputation near the horizon, is not usable near the zenith, as it gives a nonzero refraction value there. Therefore, we shall first develop analytical results for small z, where the refraction tends to zero. Obtaining the correct analytic behaviour $\mathrm{Refr} \equiv z - z' \to 0$ smoothly as $z \to 0$ is important so that the displacement d will behave reasonably near the zenith.

Clearly, as $z_0 \to 0$, the difference in z and z_0 must tend to 0. For small zenith angles, the earth curvature is also unimportant, so we can estimate d using a plane parallel atmosphere approximation. Assuming a single homogeneous layer, whose thickness is one density scale height W, Fig. 2 shows that $d \sim (z - z')\, W$ (rad). Even though we get the linear dimension d from a plane parallel model, for such small angles we can still interpret it in the spherical Earth context, using Eq. (4), as $d = (z_0 - z)A$ (rad). Thus,

$$\text{For } z \to 0,\ z_0 - z \sim (W/A)(z - z') \tag{13}$$

Although this equation is approximate, it is good enough for small zenith angles and it gives a good picture of why the difference between z_0 and z, although small, is so significant. Also, it enables us to calculate analytically an approximate formula for $z - z'$. To do this we apply Eq. (9) in the limit of small z and z_0, to yield $z_0 \sim \mu_0\, z'$. Combining with Eq. (11), we see that

$$\text{For } z \to 0, \mathrm{Refr} \to \left\{(\mu_0 \cdot 1)/\left[1 + (W/A)\right]\right\} z' \tag{14}$$

It is now possible to derive analytically the leading constant in the approximation of Allen (1973)

$$\mathrm{Refr}_{\mathrm{arcsec}} = 58.3 \tan(z') - 0.067 \tan^3(z') \tag{15}$$

Using a mean sea level index of refraction $\mu_0 = 1.0002904$, (from Hoyt's model in the MODIS software; this seems slightly large, as explained in Section 2.2.3.2), $A = 6371000$ m (the mean Earth radius), and $W = 8591.7$ m, based on the pressure scale height of 8434.5 m of Allen (1973), and lapse rate of 0.0065 K/m we find from Eq. (14):

$$\text{For } z \to 0, \mathrm{Refr}_{\mathrm{arcsec}} \quad 0.0002900\, z' \text{ rad}$$
$$= 59.82\, z' \text{ arc sec} \tag{16}$$

with z' in rad. Using $\tan(z') \sim z'$ (rad) and ignoring the cubed term, we see that Allen's 58.3 arc s is explained as the value of $(\mu_0 - 1)/[1 + (W/A)]$ for a slightly smaller index of refraction than ours. We thus choose Allen's approximation for small zenith angles, changing the value 58.3 to 59.82. For larger zenith angles, we use a minor adaptation of the

[2] We denote equation number # from Hohenkerk et al. (1992) as H#.

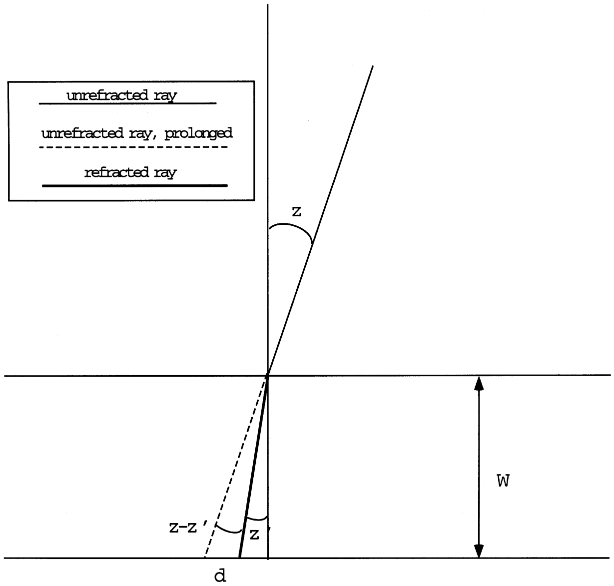

Fig. 2. Simplified model for estimating refraction near the zenith.

standard algorithm to relate z' and z from Hohenkerk et al. (1992), Eq. H3.283-1 (Eq. H3.283.2 was compared and found to offer no advantage). Finally, to make a smooth transition at $z = 1.465$ rad, it was necessary to change the coefficient of $\tan^3(z')$ of Allen (1973) very slightly. The final equation used for zenith angles less than 1.465 rad (83.94°) is

$$\mathrm{Refr} = \{(\mu - 1.0)/[1 + (W/A)]\} \times [\tan(z') - 0.00117 \tan^3(z')] \quad (\text{for } H > 6.06°) \tag{17}$$

where H is the elevation of the ray at surface level, $H = (180/\pi)((\pi/2) - z')°$. Note that the derivation in Eqs. (13)–(16) is only to validate Allen's constant and to assure continuity with Eq. H3.283-1; therefore it is assumed that z and z' are small. The overall effect is that Allen's numerical constant can be replaced consistently by the factor $(\mu - 1.0)/[1 + (W/A)]$, so that the dependence on altitude or on the assumed baseline index of refraction, μ, is consistent, but the peculiar dependence on the tangent and its cube cannot be derived this way. The equation actually used is Eq. (17), in which $(\mu - 1.0) = (\rho/\rho_0)(\mu_0 - 1.0)$. In order to validate against Hoyt, we use $\mu_0 = 1.0002904$, and this was actually used in the EOSDIS software, but see Section 2.2.3.2.

For larger zenith angles, we use an equation adapted from Eq. H3.283-1, which reads:

$$\mathrm{Refr} = (0.0167°)[(0.28 * P_{\mathrm{mb}})/T] / \{\tan[(H + 7.31/(H + 4.4)]\} \quad (\text{for } H < 6.06°) \tag{18}$$

Let us eliminate both H and the factor $(0.28 * P_{\mathrm{mb}}/T)$, which is really a complicated representation of the air density, a variable probably used because pressure and temperature are easily measured at surface stations. To an excellent approximation, $0.28 * P_{\mathrm{mb}}/T$ is unity at sea level for $T = T_{\mathrm{sealevel}}$, so it can be replaced with

$$0.28\, P_{\mathrm{mb}}/T \to \rho/\rho_0 \tag{19}$$

In the EOSDIS software (NASA, 1997), the value of ρ/ρ_0 is taken from an atmospheric model described in Appendix A. Finally, when the zenith angle z is obtained from one of the two empirical equations, then d_{Ang} is found from Eq. (4), which completes the solution, since $d = A * d_{\mathrm{Ang}}$.

2.2.3.2. Corrections to the index of refraction. The subject of index of refraction in air is important as a correction to many high precision laboratory measurements, and is relevant to terrestrial geodesy as well; see, e.g., Remmer (1994). Those who hope to improve on the equations just presented, undaunted by the effects of the weather, may wish to consult Birch and Downs (1994). Values from that paper, for dry air at 15°C, are about 1.000277, a bit less than Hoyt's 1.000290, and the corrections for moist air further *reduce* the values slightly. Birch and Downs' Eq. (2) for dry air, at pressure 101 325 Pa, can be put in the form

$$\mu_0 - 1 = \frac{1}{1 + 0.003661 T_C} \times \Big\{ 1.05442 \times 10^{-8} \times \left[1 + \frac{4053(0.601 - 0.00972 T_C)}{4 \times 10^6} \right] \times \left[8342.54 + \frac{15\,998}{38.9 - \sigma^2} + \frac{2\,406\,147}{130 - \sigma^2} \right] \Big\} \quad (20)$$

where T_C is the Celsius temperature and σ is the vacuum wavenumber in μm^{-1}, from wavelength 350 nm to 650 nm. Extending beyond the Birch and Downs upper wavelength limit to some of MODIS's infrared bands would reduce the index of refraction slightly further. Birch and Downs's Eq. (3) gives for the change in μ_0 when the partial pressure of water vapour is f Pa,

$$\delta\mu = -f(3.7345 - 0.0401) * 10^{-10} \quad (21)$$

If refined values such as the above are used for μ, the constants used in Eqs. (16)–(19) would, in principle, have to be adjusted. Inasmuch as these equations only affect the horizontal displacement, such extra work would be of questionable value.

2.3. Increments in latitude and longitude

Finally, the equations are given for transforming d_{Ang} to changes in latitude λ and longitude Λ. The trace of the ray path $\overline{DP}$ on the earth is a vector lying ψ rad East of North, where ψ is the azimuth of the ray. The azimuth ψ can be found as follows. Let $\hat{\mathbf{u}}$ be the unit normal along $\overline{PD}$ (the line $\overline{DP}$ reversed), and $\hat{\mathbf{u}}_h$ its projection on the horizontal plane at P. N will stand for the normal to the ellipsoid at P,

$$\mathbf{N} = [\cos(\lambda)\cos(\Lambda), \cos(\lambda)\sin(\Lambda), \sin(\lambda)]. \quad (22)$$

All this analysis is correct for an ellipsoidal rather than a spherical earth if λ is taken to stand for the geodetic latitude. Clearly,

$$\hat{\mathbf{u}}_h = \hat{\mathbf{u}} - (\hat{\mathbf{u}} \cdot \mathbf{N})\,\mathbf{N} \quad (23)$$

It is not necessary to normalise the vector $\hat{\mathbf{u}}_h$ for the remaining operations; however, if the ray is at zenith, $\hat{\mathbf{u}}_h$ will be zero and the azimuth is indeterminate. In this case, of course, the displacement is zero, so the calculation is of no interest. Nevertheless, one must be careful in software to avoid division by meaninglessly small quantities when near this condition. Next, define North and East unit vectors on the ellipsoid by

$$\mathbf{north}_x = -\sin(\lambda)\cos(\Lambda) \quad (24a)$$

$$\mathbf{north}_y = -\sin(\lambda)\sin(\Lambda) \quad (24b)$$

$$\mathbf{north}_z = \cos(\lambda) \quad (24c)$$

$$\mathbf{east}_x = -\sin(\Lambda) \quad (24d)$$

$$\mathbf{east}_y = \cos(\Lambda) \quad (24e)$$

$$\mathbf{east}_z = 0.0 \quad (24f)$$

The azimuth of $\hat{\mathbf{u}}_h$, and hence of the ray is then:

$$\psi = \mathrm{atan}\left(\mathbf{east} \cdot \hat{\mathbf{u}}_h / \mathbf{north} \cdot \hat{\mathbf{u}}_h\right) \quad (25)$$

With ψ known, the displacement of the earth intersection of the ray, P′, is as given in Table 2, where $d = A d_{Ang}$. Thus, the increments in latitude and longitude are as shown in Table 3. The last expression in Table 3 is singular at the North and South poles and should not be used there. The displacement of the ray can be assumed to be South at the North pole

Table 2
Vector displacement on surface

Direction	Value
North	$d\cos(\psi)$
East	$d\sin(\psi)$

Table 3
Displacement in latitude and longitude

Direction	Value
Latitude (λ)	$d_{\mathrm{Ang}} \cos(\psi)$
Longitude (Λ)	$d_{\mathrm{Ang}} \sin(\psi)/\cos(\lambda)$

and North at the South pole but when starting at either pole, the longitude (not its increment) must be found from atan$(y_{\mathrm{ray}}/x_{\mathrm{ray}})$ where $(x_{\mathrm{ray}}, y_{\mathrm{ray}}, z_{\mathrm{ray}})$ are the components of the vector $\overline{\mathrm{PD}}$ in Earth centred rotating (ECR) coordinates, i.e., right handed rectangular coordinates with the z axis North and the x axis lying in the semi-plane of North and the Greenwich meridian.

3. Sample results and discussion

Table 4 exemplifies results at sea level, using a conversion of 6 371 000 m per rad for the displacement, a global mean surface temperature of 288.115 K, and a global mean tropospheric height of 10 500 m. The local Earth curvature radius could be used in place of 6 371 000 m if the difference were considered significant.

Note that the linear displacement at 88° zenith angle is about 17.5 km — very substantial. Because of the very approximate atmosphere model, this number could vary by perhaps 25% depending on weather in temperate and tropical regions; in the Arctic it would be considerably smaller. Comfort-

Table 4
Refraction results at sea level

Zenith angle in space z_0 (°)	Zenith angle at surface z' (°)	Refraction $(z_0 - z')$ (°)	Linear displacement d (m)
10	9.9971	0.0029	0.55
20	19.9939	0.0061	1.22
30	29.9904	0.0096	2.22
40	39.9860	0.0140	3.98
45	44.9834	0.0166	5.46
50	49.9802	0.0198	7.73
55	54.9762	0.0238	11.40
60	59.9712	0.0288	17.85
61	60.9700	0.0300	19.70
62	61.9687	0.0313	21.82
63	62.9674	0.0326	24.26
64	63.9659	0.0341	27.08
65	64.9643	0.0357	30.37
70	69.9543	0.0457	58.38
75	74.9380	0.0620	136.07
76	75.9334	0.0666	166.73
77	76.9281	0.0719	207.38
78	77.9220	0.0780	262.47
79	78.9147	0.0853	338.98
80	79.9061	0.0939	448.38
81	80.8955	0.1045	610.33
82	81.8825	0.1175	860.32
83	82.8658	0.1342	1266.53
84	83.8437	0.1563	1970.64
85	84.8133	0.1867	2974.07
86	85.7687	0.2313	4858.39
87	86.6977	0.3023	8677.42
88	87.5698	0.4302	17 538.46
89	88.2951	0.7049	41 818.33
90	88.6191	1.3809	113 429.26

ingly, however, at an incident zenith angle as large as 85.25° (85.05° refracted), our result for the displacement, 3.33 km, agrees with the result from Hoyt's finite difference program (2.91 km) within 15%. The displacement at 90° incidence, over 113 km, is only suggestive and could easily vary by 50%.

4. Validation

The refraction angle and displacement were compared with Hoyt's finite difference program at sea level, 10 km, and 20 km altitude. This program divides the atmosphere into seven homogeneous spherical layers. At each layer boundary, Snell's law is used to obtain the refractive change in zenith angle, while within a layer the angle is changed by a trigonometric algorithm to account for the curvature of the layer. The results at sea level are shown in Figs. 3 and 4, which speak for themselves. The results at altitude are nearly as close in relative error, and better in absolute error, so they are not presented. It is clear that the theory given here is adequate for refraction corrections in most remote sensing applications. Exceptions were noted in Section 1.2. The biggest problem remaining is that terrain corrections exceed the refraction corrections in most cases, and the interaction of the two has not been studied. Problems with limb sounders also invite generalisation of our results. This might be possible by mating two cases where $z = 90°$, $z_0 \approx$ 88.6°, but the two would meet at some nonzero altitude, for which one would have to solve by iteration. Ray tracing programs along the lines of Hoyt's would probably be considered preferable for such work, but the analytic method given herein is simpler and faster for space observation of the earth's surface at zenith angles of a few degrees to about 85°.

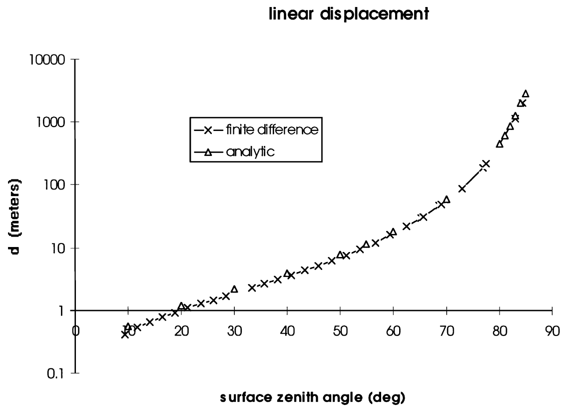

Fig. 4. Comparison of surface displacement with the MODIS finite difference program.

5. Conclusions

It is possible to correct the viewing angle and illumination angle for remote space sensing of sur-

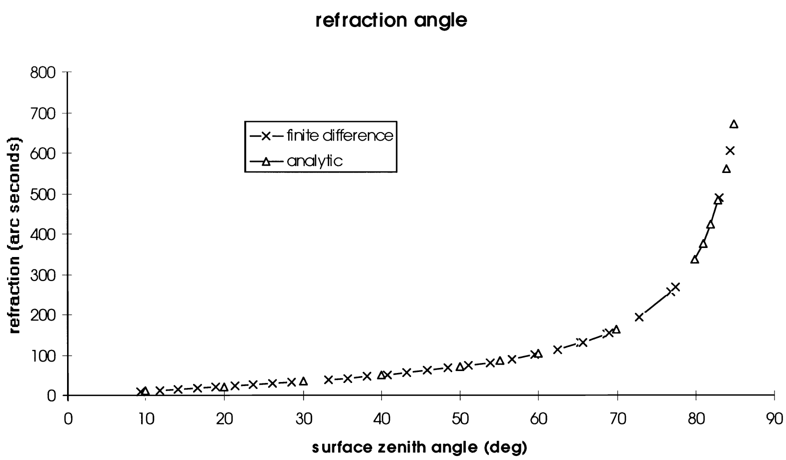

Fig. 3. Comparison of refraction angle with the MODIS finite difference program.

face phenomena accurately with Eq. (9). Using this result, it is possible to correct for the apparent displacement of the point being viewed with Eqs. (4), (11) and (12), and either Eq. (17) or Eqs. (18) and (19). To obtain numerical results at any altitude and latitude, it is necessary to use either local data or an atmospheric model, for example the one described in Appendix A, to obtain the index of refraction at the actual surface being viewed. For smooth terrain at any altitude, these methods suffice, but for work in rugged terrain, it would be necessary first to correct the surface location with an elevation model (Nishihama et al., 1997). In very jagged areas, and at low viewing angles, the correction for elevation of the line of sight intersection with the surface could interact with refraction in such a way as to require use of Eq. (2) or Eq. (10) as well. The further adaptation of these methods to limb sounding and aerial imagery seem worth exploring.

Acknowledgements

The author is indebted to Dr. Douglas Hoyt and Mr. James Storey for their finite difference refraction program.

Appendix A. Atmospheric model

The atmospheric model used for refraction calculations is based on the following idealised equations (see Table 1 for definitions). Again, the only purpose of this model is to yield the index of refraction at any altitude, so that Eq. (9) can be applied at the actual Earth intersection of the unrefracted ray. If available, local data could always be used instead of this model, to advantage.

A.1. Barometric law

Ignoring the inverse square law variation of the acceleration of gravity with altitude, as well as Earth oblateness and centrifugal force, the barometric law reads

$$\mathrm{d}P/\mathrm{d}a = -\rho g_0 \tag{A1}$$

where d is the differential operator and P the pressure.

A.2. Ideal gas law

For the ideal gas law

$$Pv = R_{\mathrm{gas}} T/\mathrm{Molec}_{\mathrm{mean}} \tag{A2}$$

the value of $\mathrm{Molec}_{\mathrm{mean}}$ was taken as the molecular weight of dry tropospheric air, based on 1 part water to 80 dry air by number (Allen, 1973).

A.3. Mean lapse rate and density variation

The lapse rate is the rate of change of temperature with altitude. Assuming an empirical global mean adiabatic lapse rate of 0.0065 K/m (Allen, 1973; Gill, 1982) in the troposphere, we get at altitude [3] a

$$T(a) = T_{\mathrm{sealevel}} - a * r_{\mathrm{L,trop}}, \quad a < a_{\mathrm{tropop}} \tag{A3}$$

Letting

$$\mathrm{tempFac} = 1.0 - r_{\mathrm{L,trop}} * a/T_{\mathrm{sealevel}} \tag{A4}$$

we integrate Eqs. (A1) and (A2), using Eq. (A3) to obtain, in the troposphere,

$$\mathrm{densFac} = \mathrm{tempFac}^{\Gamma} \tag{A5}$$

where

$$\Gamma = (\mathrm{Molec}_{\mathrm{mean}} * g_0)/(R_{\mathrm{gas}} * r_{\mathrm{L,trop}}) - 1 = 4.123 \tag{A6}$$

Above the tropopause, an isothermal atmosphere was assumed, matched at the tropopause. In the stratosphere, we assume a constant temperature and scale height and so have an exponential atmosphere with the same scale height as at the tropopause,

$$\mathrm{densFac} = (\mathrm{tempFac}_{\mathrm{tropop}})^{\Gamma} \exp[(a - a_{\mathrm{tropop}}) \times (\mathrm{Molec}_{\mathrm{mean}} * g_0)/(R_{\mathrm{gas}} * T_{\mathrm{tropop}})]. \tag{A7}$$

The first factor sets the correct scale to match at the tropopause.

[3] Because the altitude is used *only* to obtain the air pressure, which is then used to obtain the surface index of refraction, altitude should be referred to geoid.

Table 5
Latitude dependence on the tropopause altitude and the sea level temperature

Latitude (°)	a_{tropop} (Allen) (m)	a_{tropop} (fitted) (m)	$T_{sealevel}$ (Allen) (K)	$T_{sealevel}$ (Fitted) (K)
0	17 000	17 786.1	300	299.35
10	16 600	16 194.8	299	298.53
20	15 500	14 681.1	297	296.12
30	13 700	13 244.9	293	292.18
40	11 800	11 886.1	286	286.83
50	9800	10 604.9	279	280.24
60	9000	9401.13	271	272.60
70	8100	8274.85	263	264.15
80	7800	7226.06	255	255.15
90	n/a	6254.76	248	245.86

A.4. Refractive index variation with altitude

For comparison with Hoyt and Storey (1994) the index of refraction is assumed to be given by

$$\mu = 1.0 + 0.0002905(\rho/\rho_0)$$
$$= 1.0 + 0.0002905 * \text{densFac} \quad \text{(A8)}$$

This is not quite consistent with the Clausius–Mossotti relation $(\mu^2-1)/(\mu^2+2) = \text{const} \times \rho$, but agrees within 3.5 parts per billion between -1000 and $+25\,000$ m altitude — well above any known terrain. See also Section 2.2.3.2. This concludes the altitude dependence of the atmospheric model. We next present a simple latitude dependent models of the quantities a_{tropop}, ρ_0, and $T_{sealevel}$. The composition of the atmosphere was obtained from Allen (1973), p. 119. The atmospheric model is used only to get the index of refraction at sea level. Latitude dependence is based on the following fits to the sea level temperature and mean scale height as functions of latitude, from the table of Allen (1973) on p. 121. The US Standard Atmosphere (NOAA, 1976) is dry, which seemed less realistic. The effect of humidity on the molecular weight is small, but effect of water vapour on the lapse rate is considerable. The latitude dependence of the lapse rate is apparently minor (Gill, 1982). Fits to the tropopause height vs. latitude and surface temperature are given in Table 5. These were used in late releases of the Science Data Processing Toolkit (NASA, 1997). Alternate columns give values from Allen (1973) and the fitted Eqs. (A9) and (A10). The Mathematica program (Wolfram, 1996), was used to perform least squares fits to the data of Allen (1973), yielding

$$a_{tropop} = 17\,786.1 - 9338.96|\lambda| + 1271.91\lambda^2 (\text{m}) \quad \text{(A9)}$$

$$T_{sealevel} = 245.856 + 53.4894\cos(\lambda)\ (\text{K}) \quad \text{(A10)}$$

where λ is the latitude. Using mean atmospheric pressure as a constant, and the ideal gas law, Eq. (A2), one can derive from Eq. (A10) the dependence of density on latitude. Forcing the mean over latitude (weighted by $\cos(\lambda)$ for solid angle) of $\rho_0/\langle\rho_0\rangle$ to be unity, where $\langle\rho_0\rangle$ is the global average, a least squares fit yields

$$\rho_0/\langle\rho_0\rangle = 1.14412 - 0.185488\cos(\lambda) \quad \text{(A11)}$$

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