Analysis of Hydraulic Conductivity Averaging Schemes for One-Dimensional, Steady-State Unsaturated Flow

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Abstract

The evaluation of average properties over a block or element is required for most numerical methods used for the solution of unsaturated flow equations. In this study, it is shown that the accuracy of a particular averaging scheme depends on the type of model used for unsaturated material properties. A systematic and thorough error analysis of the averaging techniques for some commonly used soil characteristic curves is performed. It is shown that the two-point Gaussian quadrature is consistently more accurate than the arithmetic, geometric, and harmonic means for all types of constitutive models.

Introduction

Numerical methods are widely used for modeling flow through unsaturated porous media. Difficulty in finding analytical solutions due to the nonlinearity of the governing equations, and availability of high speed computers have both contributed heavily towards the popularity of numerical methods. These methods, however, have their own drawbacks, such as lack of convergence of the iterative solutions, difficulty in selection of averaging method for properties over discrete blocks or elements and choice of appropriate space and time discretization.

Various weighting schemes to evaluate the "average" conductivity over a finite-size block or element have been suggested in the past. Schnabel and Richie (1984) demonstrate that the pressure and moisture profiles generated by using various weighting schemes to determine the average parameters, do not intersect one another. Therefore, we assume that the error observed in the results obtained from any weighting scheme using just one element would represent the overall error for that scheme. The relative ordering of errors resulting from various averaging schemes are obtained by comparing their results for a single element. This assumption enables us to simulate a large number of problems and perform a detailed error analysis. A comparison of numerical errors obtained using a single element and that from a large number of elements showed that this assumption is true in all the cases.

Background

For transient unsaturated flow problems, various schemes for estimating interblock hydraulic conductivity in the finite-difference method were compared by Haverkamp and Vauclin (1979). These schemes included the arithmetic, geometric, and harmonic means of the nodal conductivities; the conductivity corresponding to the arithmetic, geometric, and harmonic means of nodal heads; upstream weighting, series expansion, and linear extrapolation. The weighting scheme for the moisture capacity term was not discussed. They used a constitutive model utilized by Haverkamp et al. (1977) to represent the soil characteristic curve (Gardner, 1958), compared their results with a quasi-analytical solution and concluded that the geometric mean of conductivity values is the best weighting scheme. They also mentioned that the space increment has negligible effect on the accuracy when using the geometric mean. The relative order of errors from different schemes was also found to be insensitive to the element size.

For horizontal and vertical flow through porous media, various averaging schemes were compared by Schnabel and Richie (1984) for the case of fixed pressure head at both the top and the bottom boundaries. These schemes included the arithmetic, geometric, and harmonic means of conductivity values; upstream weighting and an integrated conductivity. Constitutive equations similar to the Brooks and Corey (1966) equation were used, and an analytical solution was utilized to compare the errors of different weighting schemes. They concluded that the choice of weighting scheme is most critical where large pressure gradients occur. The integrated mean, when possible, was considered the best choice and the geometric mean was the next best. The harmonic mean resulted in maximum errors and its use was not recommended.

Warrick (1991) examined alternative strategies for the determination of the interblock conductivity to arrive at the correct Darcian velocity for the case of specified head at both boundaries. He found that the calculated velocities can, in some case, be as high as 130 times the actual velocity if the arithmetic mean of conductivities is used. For extremely large gradients, the geometric mean resulted in a flow much smaller than the actual flow. Moreover, upstream

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weighting was found to be no better. An alternative weighting scheme using variable weights was found to work well. However, the weights are dependent upon the pressure and the gradient, and have to be computed during the simulation resulting in about 50% increase in computational time.

Li (1993) used Simpson's one-third rule to obtain the average conductivity over an element for the finite-element method and found it to be better than the arithmetic and geometric mean schemes. It, however, requires the evaluation of the conductivity value at an additional point in the middle of the element and will marginally increase the computational time. He applied his simplified Newton iteration scheme to both specified-head and specified-flux conditions at the top, and compared the numerical results with "dense grid" solutions. The method was found to converge faster than the Picard iteration scheme, while the computational effort per iteration was nearly the same.

In this study, we compare the steady-state results obtained from various weighting schemes to determine the best scheme for averaging of block conductivities. We assume, as discussed earlier, that the error for one element would be representative of the overall error. The exponential form of the constitutive relations (Gardner, 1958) is analyzed first, since an analytical solution for the steady-state profile is available (e.g., Yeh, 1989). Both fixed-head-type and flux-type boundary conditions are examined. Comparison of the performance of these schemes for some other relations (Brooks and Corey, 1966; Gardner, 1958; Haverkamp et al., 1977; van Genuchten, 1980) is then undertaken. Only infiltration problems are analyzed in this study, and upward flow is not considered. Analysis of errors for various types of constitutive relations and inclusion of two-point Gaussian integration scheme for the averaging of conductivity are believed to be significant additions to the existing literature on the subject.

Governing Equation

The governing equation for steady-state one-dimensional infiltration through a porous medium can be written as

$$\frac{\partial q_*}{\partial z_*} = \frac{\partial}{\partial z_*}\left[K_*(\psi_*)\left(\frac{\partial \psi_*}{\partial z_*} + 1\right)\right] = 0 \qquad (1)$$

where z_* is the vertical coordinate (L), positive upward; q_* represents the Darcy velocity (L/T), positive downward; ψ is the pressure head (L); and K is the hydraulic conductivity (L/T) which is a function of the pressure head under unsaturated conditions ($\psi < 0$). The boundary conditions are

$$\left.\begin{aligned} &\text{at } z_* = 0: && \psi_* = \psi_{*0} && (2a)\\ &\text{at } z_* = L_*: && \psi_* = \psi_{*1} \\ &\text{or} && K_*(\psi)\left(\frac{\partial \psi_*}{\partial z_*} + 1\right) = q_{*1} && (2b)\end{aligned}\right\}$$

where ψ_{*0} is the constant pressure head (L) at the bottom, ψ_{*1} is the specified pressure head (L) at the top for the fixed-head boundary condition, q_{*1} is the constant rate of infiltration (L/T) for the flux-type boundary condition, and L_* is the length of modeled domain (L). For convenience, equation (1) and boundary conditions (2) are written in dimensionless form as

$$\frac{\partial}{\partial z}\left[K(\psi)\left(\frac{\partial \psi}{\partial z} + 1\right)\right] = 0 \qquad (3)$$

and

$$\left.\begin{aligned} &\text{at } z = 0: && \psi = \psi_0 && (4a)\\ &\text{at } z = L: && \psi = \psi_1 \\ &\text{or} && K(\psi)\left(\frac{\partial \psi}{\partial z} + 1\right) = q && (4b)\end{aligned}\right\}$$

in which the pressure heads and the lengths are made nondimensional by multiplying with a parameter α having dimensions of 1/L (this parameter depends upon the type of constitutive relation used) and the flux and conductivity are made dimensionless through a division by the saturated hydraulic conductivity K_s (L/T). Thus

$$\psi = \alpha\,\psi_* \quad z = \alpha\, z_* \quad K = \frac{K_*}{K_s} \quad \text{and} \quad q = \frac{q_{*1}}{K_s} \qquad (5)$$

We now apply the Galerkin technique to equation (3) to obtain the discretized equations. Details of the procedure are not given here as it can be found in a number of textbooks (e.g., Istok, 1989) and technical papers (e.g., Yeh et al., 1993). Application of the technique to equation (3) results in the following matrix equation

$$[A]\{\psi\} + \{G\} - \{Q\} = 0 \qquad (6)$$

where $\{\psi\}$ is the nodal pressure vector, A is the conductivity matrix, G is the gravity vector, and Q is the flux vector. The matrix and the vectors are given by

$$\begin{aligned} a_{ij} &= \int_0^L K \frac{\partial N_i}{\partial z}\frac{N_j}{\partial z}\,dz \\ g_i &= \int_0^L K \frac{\partial N_i}{\partial z}\,dz \\ q_i &= [N_i\, q]_0^L \end{aligned} \qquad (7)$$

in which N_i and N_j are shape functions. Note that the vector Q, in the absence of any source or sink, will have nonzero entries only in its first and the last rows, i.e., corresponding to the bottom and the top boundaries. Assuming a linear variation of pressure within an element, the derivatives of the shape functions would be constant over the element and the element matrix [A] and vector {G} can be written as

$$\begin{aligned} [a] &= \frac{\bar{K}}{\Delta z}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \\ \{g\} &= \bar{K}\begin{Bmatrix} -1 \\ 1 \end{Bmatrix} \end{aligned} \qquad (8)$$

with

$$\bar{K} = \frac{1}{\Delta z}\int_0^{\Delta z} K\,dz$$

in which Δz is the element length.

Averaging Schemes

A variety of schemes can be used to average the conductivity values over an element in equation (8) on the basis of the pressure heads at the two nodes. The schemes analyzed in this study are:

1. Arithmetic mean of conductivities (AC)

$$\bar{K} = 0.5(K_i + K_{i+1}) \tag{9}$$

where K_i and K_{i+1} are the nodal conductivities.

2. Geometric mean of conductivities (GC)

$$\bar{K} = (K_i\, K_{i+1})^{0.5} \tag{10}$$

3. Harmonic mean of conductivities (HC)

$$\bar{K} = \frac{1}{0.5\left(\frac{1}{K_i} + \frac{1}{K_{i+1}}\right)} \tag{11}$$

4. Conductivity at the arithmetic mean of pressure head (AP)

$$\bar{K} = K[0.5(\psi_i + \psi_{i+1})] \tag{12}$$

where ψ_i and ψ_{i+1} are the nodal pressure values.

5. Conductivity at the geometric mean of pressure head (GP)

$$\bar{K} = K[(\psi_i\, \psi_{i+1})^{0.5}] \tag{13}$$

6. Conductivity at the harmonic mean of pressure head (HP)

$$\bar{K} = K\left(\frac{1}{0.5\left(\frac{1}{\psi_i} + \frac{1}{\psi_{i+1}}\right)}\right) \tag{14}$$

7. Conductivity at the upstream node (UC)

$$\bar{K} = K[\mathrm{Max}(\psi_i, \psi_{i+1})] \tag{15}$$

8. Numerical integration of conductivity (NC)

$$\bar{K} = \sum_g W_g\, K(\psi_g) \tag{16}$$

where g denotes the Gaussian quadrature points, and W is the weight. Only two-point quadrature is utilized in this study as the computational effort is comparable to the other schemes. Simpson's three-point rule as used by Li (1993) is not considered for the numerical integration as it requires more computational time and is expected to have the same order of accuracy as the two-point Gauss quadrature. The weights are given by $W_1 = W_2 = 0.5$ and the quadrature points are located at $0.21\,\Delta z$ and $0.79\,\Delta z$.

9. Integrated conductivity (IC)

$$\bar{K} = \frac{1}{\psi_{i+1} - \psi_i} \int_{\psi_i}^{\psi_{i+1}} K(\psi)\, d\psi \tag{17}$$

These schemes are referred by their two-letter symbols from here on. Scheme IC requires the functional relationship between the conductivity and pressure to be analytically integrable which may not be possible for most constitutive relationships.

Methodology

Following Warrick and Yeh (1990), we can examine the case of constant flux by starting from the bottom node, where pressure is known, and obtain the pressure at the next node, ψ_a, from

$$\frac{\bar{K}}{\Delta z}(\psi_a - \psi_0) + \bar{K} - q = 0 \tag{18}$$

Equation (18) is simply the finite-difference form of Darcy's law assuming that the pressure ψ varies linearly with z. Since K depends both on ψ_0 and ψ_a, an iterative scheme has to be used to arrive at the correct value of ψ_a. Here, Newton iterations are applied to equation (18) to obtain

$$f(\psi_a) = \frac{\psi_a - \psi_0}{\Delta z} + 1 - \frac{q}{\bar{K}} = 0 \tag{19}$$

and its derivative

$$f'(\psi_a) = \frac{q}{\bar{K}^2}\frac{\partial \bar{K}}{\partial \psi_a} + \frac{1}{\Delta z} \tag{20}$$

The iterative procedure itself can then be described by

$$\psi_a^{new} = \psi_a^{old} - \frac{f(\psi_a^{old})}{f'(\psi_a^{old})} \tag{21}$$

with the starting value of ψ_a taken equal to ψ_0. Since we consider only infiltration problems and not cases with upward flux, ψ_a is constrained to be between $\psi_0 - \Delta z$ and 0.

Once the pressure at this node, ψ_a, is obtained by using equation (19), the pressure at the next node, ψ_b, is found by repeating the procedure with ψ_a replacing ψ_0. Thus at each iteration, there is only one unknown, thereby avoiding the formulation of a derivative matrix as is done in traditional methods. Also, instead of solving a system of equations, one equation at a time is solved thus reducing the computational time and storage requirements. A comparison of the conventional finite-element method and the proposed node-by-node scheme is performed in the next section. It should be mentioned that a numerical solution of desired accuracy for the specified flux-type boundary condition can be obtained by writing equation (3) as

$$z = \int_{\psi_0}^{\psi(z)} \frac{1}{\frac{q}{K} - 1}\, d\psi \tag{22}$$

and then using standard numerical integration programs to find the z corresponding to any ψ. It will, however, require more computation time as typically 20 or more quadrature points are used for an accurate solution (Warrick, 1991). Also, the results are not obtained at uniform intervals of depth.

For the specified head boundary condition, the numerical value of "average" conductivity is computed from the applicable scheme [equations (9)-(17)] and the analytical value is obtained by dividing the value of q, obtained from the appropriate equation in the Appendix, by the head gradient. Two different values of ψ_0, corresponding to $K_0 =$ 0.1 and 10^{-5}, respectively, are used and the specified head at

the top is varied from $\psi_0 - \Delta z$ to saturation. Two values of Δz, 0.1 and 0.01, are used for the numerical simulations. The constitutive relations used to describe the variation of the hydraulic conductivity with the pressure head are listed in the Appendix along with the analytical solutions for some particular cases. Where possible, the errors in the numerical scheme are obtained by comparison with the analytical solution. The error is quantified by the difference in the order of magnitude of the numerical and analytic solutions:

$$\text{error} = \log \frac{\text{Numerical value}}{\text{Analytical value}} \tag{23}$$

Thus, an error value of +1 indicates that the numerical value is one order of magnitude larger than the analytic value and a value of −1 represents a numerical value an order of magnitude smaller than the analytic value. This definition of error is used because it treats the positive and negative errors in a symmetric manner. It should be mentioned that an accurate numerical integration could have been used to obtain the "exact" results, but to apply scheme IC, analytically integrable parameter values are chosen.

For steady-state flow, the flux type and the specified head boundary condition are numerically equivalent. To solve the specified head case, we can solve a sequence of specified flux problems and match the obtained head at the top with the given value. Thus, it is expected that the ordering of the errors from different averaging schemes would be similar irrespective of the boundary condition. A numerical example confirming this is described in the next section.

Results and Discussion

The errors for a single element are analyzed first for all the constitutive relations and then a multielement problem is solved to demonstrate the efficiency of the node-by-node solution method over the conventional finite-element method.

Exponential Relation

The exponential constitutive relation [equation (A1)] is used first, and the average conductivity obtained from the numerical solution is compared with the analytical solution using (A3) and the head gradient. The case where both the boundaries are fixed head boundaries is analyzed first. An excellent analysis of such problems was performed by Warrick (1991) for the exponential and van Genuchten type constitutive equations. The range of parameter values was, however, not very extensive and not all types of averaging schemes were analyzed. The logarithmic errors for various schemes are shown in Figure 1 for various values of the specified head at the top, ψ_1, and with $\Delta z = 0.1$. Due to the exponential form of the conductivity-pressure relations, schemes GC and AP are identical for completely unsaturated conditions. As expected, the integrated scheme (IC) gives the best results. The results from scheme NC are much better than all other schemes. Schemes AC, GC, and GP are also seen to be sufficiently accurate, except near saturation where GP has large errors. Also worth noting is the fact that for infiltration in a "wetter" soil, i.e., when ψ_1 is less than ψ_0, all schemes give almost exact results. This is due to the fact that the bounds on ψ_1 under such cases are ψ_0 and $\psi_0 - \Delta z$ and for small values of Δz, all the averaging methods would be sufficiently accurate. It should be mentioned that the error in the numerical solution is not solely due to the averaging scheme. From equation (18), the numerical solution for q is a multiplication of the "average" conductivity with the head gradient. This gradient is obtained by assuming a linear variation of head with distance. For infiltration into very dry soils, the gradient would be very large and, more importantly, the pressure profile would be very nonlinear. Using a gradient based on the linear variation of pressure thus introduces an error into the numerical solution. In this study, we do not attempt to separate the two effects, the averaging of conductivity and the nonlinearity of pressure profile; we lump them together to obtain the numerical error.

Similar results are obtained for $\Delta z = 0.01$ except that the schemes NC and IC are almost exact. Comparison of results for $\Delta z = 0.1$ and $\Delta z = 0.01$ (not shown) demonstrates that the relative ordering of the accuracies of various schemes is independent of the grid size. The results for ψ_0 corresponding to $K_0 = 10^{-5}$ further confirm this behavior. The ordering, however, is seen to be dependent on the value of ψ_0 from Figures 1 and 2, particularly for schemes GP and HP. Also, Figure 2 shows that for the exponential relation, the average conductivity obtained from schemes GP and HP may be larger or smaller than the exact value. Therefore, a statement like "the use of geometric mean of pressure values will result in the underprediction of the flux" cannot be made.

Figure 3 shows the variation of error for the specified flux condition at the top. The bottom boundary is kept at $K_0 = 0.1$ and the flux at the top is varied from 0 to a maximum value which will result in saturation at the top. Comparison of Figures 1 and 3 shows that the type of boundary condition specified at the top, whether constant head or constant flux, does not affect the relative order of accuracy of various averaging schemes. Both the specified head and the specified flux-type boundary conditions are analyzed for the other constitutive relations also, but the results for the flux-type

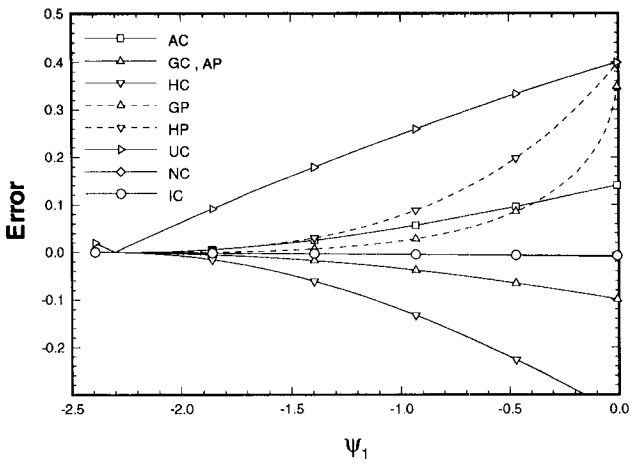

Fig. 1. Error for the exponential model for $K_0 = 0.1$ and $\Delta z = 0.1$.

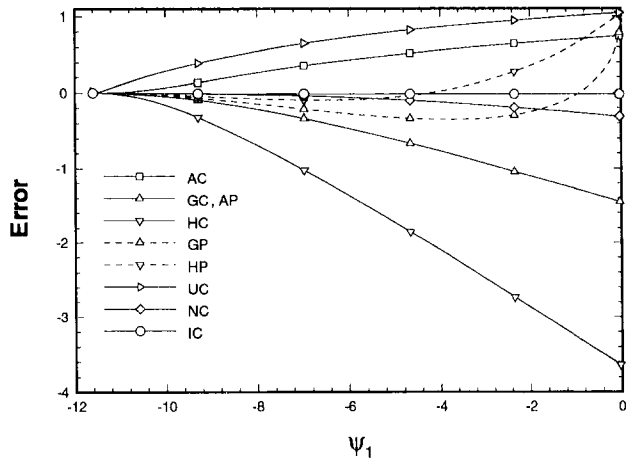

Fig. 2. Error for the exponential model for $K_0 = 10^{-5}$ and $\Delta z = 0.1$.

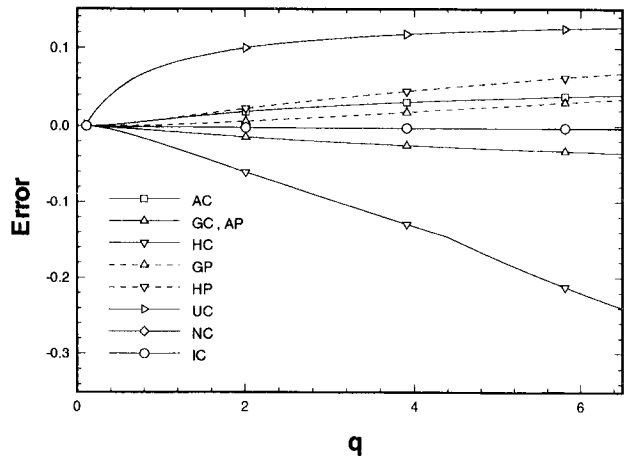

Fig. 3. Error for the exponential model for $K_0 = 0.1$ and $\Delta z = 0.1$ for the specified flux boundary condition at the top.

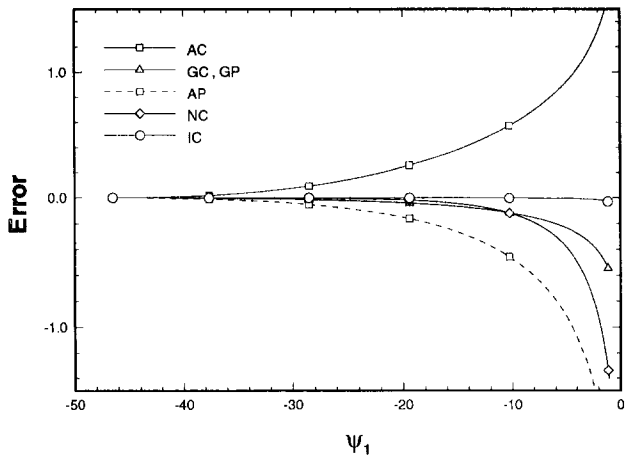

Fig. 4. Error for the power law for $K_0 = 10^{-5}$ and $\Delta z = 0.1$.

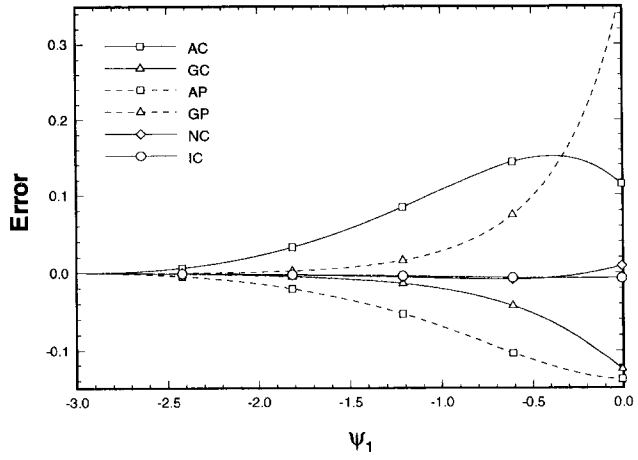

Fig. 5. Error for the Gardner model for $K_0 = 0.1$ and $\Delta z = 0.1$.

boundary conditions are not presented for the sake of brevity. Also, the results for $\Delta z = 0.01$ are not shown because the behavior of the error is similar to the case where $\Delta z = 0.1$.

Power Law

To ascertain the effect of the constitutive relations, we next use the power law form (A4) for the conductivity-pressure relation and repeat our simulations. Since a general analytical solution is not available for this case, the particular case of $n = 3$ is considered for the analysis. This enables us to use scheme IC and the exact solution is then given by (A6). Due to the form of equations, schemes GC and GP are identical in this case for unsaturated conditions. Also, equation (A6) has to be solved iteratively for q. For this purpose, the starting value of q is taken as that given by equation (18) with the average conductivity equal to the integrated conductivity.

From Figure 4 it is seen that scheme IC gives almost exact results. Numerical integration (NC) and geometric means (GC and GP) are better than the rest of the schemes. Note that near saturation the geometric means (GC and GP) perform better than numerical integration (NC). Similar results are obtained for $K_0 = 0.1$ except that NC is consistently better than all other schemes, even near saturation.

Gardner Model

A modified form of the power law, which was found to fit the available data better, was suggested by Gardner (1958) and can be written as equation (A7). Analytical solutions for various values of the exponent were obtained in that paper and one of them is listed in the appendix (A9). For the comparison of the various averaging schemes, we use a value of 2 for the exponent and use equation (A9) as the exact solution to get the error in the numerical solution. Figure 5 shows the logarithmic errors for different averaging schemes for $K_0 = 0.1$. Schemes NC and IC are again seen to be the best. The geometric mean (GC and GP) appears to be the best averaging method out of the remaining schemes in this case. Similar trends are observed for $K_0 = 10^{-5}$, except that the geometric mean outperforms the Gaussian integration (NC).

van Genuchten Model

The model proposed by van Genuchten (1980) [equation (A10)] is probably the most widely used constitutive

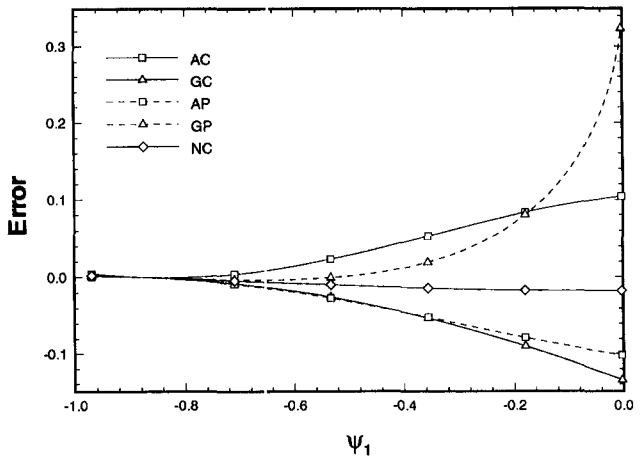

Fig. 6. Error for the van Genuchten model for $K_0 = 0.1$ and $\Delta z = 0.1$.

model at present. Therefore, even though an analytical solution is not available, we apply the various averaging schemes to this model and compare the resulting numerical errors. The "exact" solution is obtained by numerical integration of the governing equation written in the form of Darcy's law. A value of 2 for the exponent is used and the numerical errors are shown in Figure 6 for $K_0 = 0.1$. Again, the scheme NC seems to provide the best results in this case and the geometric mean (GC and GP) appears to be better than the rest. Similar results are obtained for $K_0 = 10^{-5}$.

Multielement Problem

A number of problems using different constitutive relationships and a wide range of values for the specified head at the bottom boundary and specified flux at the top boundary are solved using both the conventional finite-element method and the node-by-node method with various element sizes. In addition to requiring considerably less computer storage, the node-by-node method is found to be anywhere from 8 to 14 times more efficient in terms of the CPU time. Figure 7 shows the pressure profiles obtained for a 10-element case

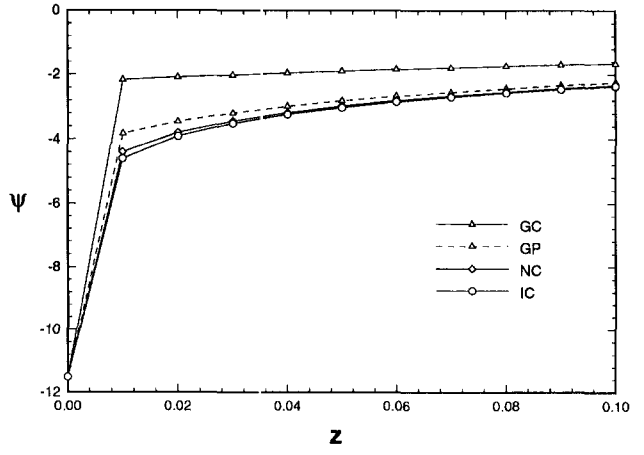

Fig. 7. Pressure profile for the multielement problem using exponential model with $K_0 = 10^{-5}$ and $q = 1$.

for infiltration in a column of dimensionless length equal to 0.1. Exponential constitutive relationship is used for the conductivity-pressure curve and the bottom and top boundary conditions are $K_0 = 10^{-5}$ and $q = 1$, respectively. The results from both the conventional finite-element method and the node-by-node method are identical, and the results using the integrated conductivity (IC) are virtually indistinguishable from the analytical solution. Only the four schemes with smallest errors are shown in Figure 7, which clearly shows the validity of assuming the single-element error to be representative of the overall error of a scheme.

Summary and Recommendations

Various averaging schemes to determine the block conductivity values for finite-size elements have been examined. The error obtained from the numerical model is found to depend upon not only the averaging scheme but also the type of constitutive model used to describe the dependence of soil hydraulic conductivity on suction. The general observation for all constitutive models is that the upstream conductivity scheme provides an upper bound, as is obvious from its definition, and the harmonic conductivity scheme provides a lower bound for the exact value. The integrated conductivity, as expected, is closest to the analytical solution and should be used whenever possible (cf., Schnabel and Richie, 1984), e.g., for the exponential model. However, for most practical cases it will not be possible to obtain the analytical expression for the integrated conductivity. Then, the Gaussian integration using two integration points is consistently the best scheme under most flow situations and should be preferred over all other schemes as the computation time is of the same order. For some situations (e.g., Figure 4), however, it may underpredict the flux by an order of magnitude. In these conditions, use of more integration points is suggested, though it will increase the computational burden. The geometric mean is found to be more accurate than the arithmetic or harmonic means for most of the flow situations considered in this study and in some cases outperforms the Gaussian integration scheme (see Figure 4).

We have not attempted to quantify the effect of the parameter n in the constitutive conductivity relations and have considered a single value of n for this study. Further work is needed to address this issue. Also, only steady-state infiltration problems are analyzed here. For transient conditions, an additional term involving the averaging of the moisture capacity or the moisture content (depending on whether it is a pressure-formulation or a mixed form of the governing equation) will have to be considered. Moreover, analytical solutions for transient flow situations over a finite domain are not widely available. For the exponential model, analytical solutions for homogeneous and two-layer soil (Srivastava and Yeh, 1991) can be used to extend the present work.

Acknowledgment

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Appendix

Various soil-characteristic equations used in this study, and the analytical solutions used for comparing the numerical accuracy are listed below:

(a) Exponential model:

$$K = e^{\psi} \quad \text{for} \quad \psi \leq 0 \quad \text{and} \quad K = 1 \quad \text{for} \quad \psi \geq 0 \tag{A1}$$

$$K_{IC} = \frac{e^{\psi_1} - e^{\psi_0}}{\psi_1 - \psi_0} \tag{A2}$$

$$q = \frac{e^{\psi_1} - e^{\psi_0 - \Delta z}}{1 - e^{-\Delta z}} \tag{A3}$$

in which K_{IC} is the integrated conductivity, and ψ_0 and ψ_1 are the pressure values at the bottom and top, respectively, of the soil column of length Δz.

(b) Power law:

$$K = (-\psi)^{-n} \text{ for } \psi \leq -1 \text{ and } K = 1 \text{ for } \psi \geq -1 \tag{A4}$$

in which n is a fitting parameter.

For n = 3,

$$K_{IC} = \frac{\psi_1 + \psi_0}{2\psi_1^2\psi_0^2} \tag{A5}$$

$$\left[\frac{1}{3} \ln|-q^{1/3}\psi - 1| - \frac{1}{6} \ln(1 - q^{1/3}\psi + q^{2/3}\psi^2) - \frac{1}{(3)^{1/2}} \arctan\left(\frac{1 - 2q^{1/3}\psi}{(3)^{1/2}} \right) \right]_{\psi_0}^{\psi_1} = -\Delta z\, q^{1/3} \tag{A6}$$

(c) Gardner model:

$$K = \frac{1}{1 + (-\psi)^n} \quad \text{for} \quad \psi \leq 0$$

$$\text{and} \quad K = 1 \quad \text{for} \quad \psi \geq 0 \tag{A7}$$

For n = 2,

$$K_{IC} = \frac{\arctan(-\psi_0) - \arctan(-\psi_1)}{\psi_1 - \psi_0} \tag{A8}$$

for $\psi_1 < \dfrac{\psi_0}{1 - \psi_0 \Delta z}$ i.e., q < 1:

$$\left| \frac{\psi_0 - A_1}{\psi_0 + A_1} \frac{\psi_1 + A_1}{\psi_1 - A_1} \right| = e^{-2qA_1\Delta z} \tag{A9a}$$

for $\psi_1 > \dfrac{\psi_0}{1 - \psi_0 \Delta z}$ i.e., q > 1:

$$\arctan(A_2\psi_1) - \arctan(A_2\psi_0) = A_2(q - 1)\Delta z \tag{A9b}$$

with $A_1 = [(1 - q)/q]^{1/2}$ and $A_2 = [q/(q - 1)]^{1/2}$.

(d) van Genuchten model:

$$K = \frac{\{1 - (-\psi)^{n-1}[1 + (-\psi)^n]^{-m}\}^2}{[1 + (-\psi)^n]^{m/2}} \tag{A10}$$

where m = 1 − 1/n.

No analytical solutions could be obtained. Some approximate analytical solutions are available in the literature (Zimmerman and Bodvarsson, 1989) but an iterative solution using numerical integration is preferred.

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