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The class of self-similar solutions for laminar buoyant jets

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Abstract—Based on the analysis of a system of laminar boundary layer equations under the Boussinesq approximation, the results of a theoretical investigation of the buoyancy effects on flow and heat transfer in vertical jets and plumes are presented. Exact and approximate solutions are constructed analytically. The predictions are compared with numerical data reported by other authors.

1. INTRODUCTION

THE THEORY of viscous buoyant jets has claimed the close attention of scientists and practising engineers, as attested by a great number of original papers (see, for example, the bibliography in ref. [1] and also the review in ref. [2]) published during the past 10–15 years. This is attributed to the abundance of buoyant jet flows in nature and technology.

The most simple and traditional approach to the description of various jet flows has been based on the use of self-similar solutions. The interest in obtaining this kind of relation was primarily spurred by the possibility of overcoming mathematical difficulties posed by the non-linear character of basic equations and of gaining an insight into the nature of this phenomenon. Besides, with so many factors affecting momentum and heat transfer in viscous jets, a judicious planning of experimental investigations can hardly be possible without constructive employment of theoretical results. Therefore, the search for self-similar solutions is usually started as soon as the problem has been mathematically formulated.

For the most part, the efforts of researchers have been devoted to the development of theories of forced (momentum) jets and pure plumes, with almost no attempt at constructing analytical solutions for buoyant jets. On the whole, this is quite conceivable, at least for the reason that for a jet flow with initial non-zero momentum and buoyancy fluxes the dimensional analysis is unable to provide variables and establish the self-similarity from the problem formulation. There is even a general agreement in the literature that such problems do not have closed-form self-similar solutions.

Taking into consideration the great interest in this problem, an attempt is made in the present study to set forth the results of a theoretical analysis of laminar buoyant jets on the basis of boundary-layer equations under the Boussinesq approximation. Great emphasis is placed on the development of the method for studying these jet flows and on the possibility of its use for solving a wide range of problems.

2. EXACT SOLUTIONS

2.1. *Basic equations*

For the boundary-layer theory approximation with regard to the quadratic dependence of density on temperature, $\Delta\rho = -\rho\gamma(\Delta T)^2$, the basic equations of plane ($j = 0$) or axisymmetric ($j = 1$) vertical fluid motion in a gravitational field have the form

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y} = \frac{\nu}{y^j}\frac{\partial}{\partial y}\left(y^j\frac{\partial u}{\partial y}\right)\pm g\gamma(\Delta T)^2$$

$$\frac{\partial}{\partial x}(y^j u)+\frac{\partial}{\partial y}(y^j v) = 0$$

$$u\frac{\partial \Delta T}{\partial x}+v\frac{\partial \Delta T}{\partial y} = \frac{a}{y^j}\frac{\partial}{\partial y}\left(y^j\frac{\partial \Delta T}{\partial y}\right). \qquad (1a)$$

Note that in the coordinate system adopted, the x-axis is running vertically upward (downward). Furthermore, the plus and minus signs on the right-hand side of the first equation of system (1a) relate to the cases when the buoyancy force, and the flow induced by it, are either compounded with the initial momentum (a jet propagates vertically upward) or opposed to it (a jet propagates vertically downward).

As already remarked, in studying jet flow mathematically the flow parameters can be presented in the form of simple functional relations

$$u \sim x^{\alpha}f(y/x^{\sigma}), \quad \Delta T \sim x^{\varepsilon}h(y/x^{\sigma}). \qquad (1b)$$

However, the application of expressions (1b) in the analysis of system (1a) (with regard to corresponding boundary and integral conditions) shows that the latter admits the solution of the form of expressions (1b) only in two limiting cases: forced jets and pure plumes. It is just here that the self-similarity indices

NOMENCLATURE

- a thermal diffusivity
- C_p specific heat at constant pressure
- g gravitational acceleration
- Gr_x local Grashof number
- k thermal conductivity
- K momentum flux
- $\dot{m}$ mass flow rate
- Nu_x local Nusselt number
- Pr Prandtl number
- Q heat source strength
- q heat flux
- Re_x local Reynolds number
- T fluid temperature
- u, v vertical (axial) and radial (transverse) components of velocity vector, respectively
- x, y axial and transverse coordinates, respectively.

Greek symbols
- β, γ density parameters
- ζ parameter of mixed convection
- ν kinematic viscosity of fluid
- ρ density.

Subscripts
- m maximum
- w parameters on the wall
- 0 initial value
- ∞ parameters at infinity, in surrounding medium (specifically, in aiding flow).

α, ε and σ, which virtually prescribe the group of transformations, are defined by the problem formulation itself. In all intermediate cases the statement (1b) does not hold. This fact gave an impetus to the development of approximate analytical methods by many researchers. Some of the most frequently used of these are the integral, perturbation and several hybrid methods [3]. While these methods allow qualitative and quantitative calculations of the development of viscous jets in the field of body forces, they are far from being adequate to describe the actual flow pattern comprehensively with desired reliability, and, therefore, do not always make it possible to predict quite satisfactorily the dynamics of the process of momentum and heat transfer. The latter is a consequence of the shift in research interests toward the use of numerical methods—the most traditional approach involving numerical solution for two systems of differential equations. One of these describes the perturbation of the purely forced convection pattern (the case of a weakly buoyant plume), whereas the second describes that of a purely free convection pattern (the case of a strongly buoyant plume). The results of this kind of simulation are reflected in refs. [4, 5].

However, this approach is not devoid of its own complications. Thus, in calculations of a jet with negative buoyancy, the jet flow stagnation point is a singular point responsible for a reduced convergence or even loss of stability of a numerical solution. Under these conditions, greater importance is attached to exact analytical solutions, an acute demand for which becomes increasingly evident. They are needed not only because of the attractiveness of a search for elegant analytical expressions in non-linear problems, but also for constructing effective numerical methods from preliminary information about the behaviour of solutions of the problem investigated in different regions of its determination. In this sense, numerical and analytical methods do not exclude but complement each other.

It should be noted that all the problems in the theory of buoyant jets are classified as second-kind problems—i.e. those the formulation of which does not specify the similarity group. Just this is the major problem encountered in analysing mathematically this type of problems.

2.2. *Plane buoyant jet*

The system of equations (1a) should be integrated using the following boundary conditions

$$y = 0: \quad \frac{\partial u}{\partial y} = v = \frac{\partial \Delta T}{\partial y} = 0$$
$$y \to \infty: \quad u \to 0, \Delta T \to 0 \tag{2}$$

and integral conditions

$$Q_0 = \rho C_p \int_{-\infty}^{\infty} u \Delta T \, dy = \text{const.} \tag{3}$$

An attempt will be made to construct a direct self-similar solution of the problem. It will be sought in the class of solutions of the form

$$\psi(x, \eta) = Af(\eta)p(x), \quad \Delta T(x, \eta) = \cosh(\eta)\varepsilon(x)$$
$$\eta = Bq(x)y \tag{4}$$

where

$$A = \left(\frac{K_0 \nu}{\rho}\right)^{1/3}, \quad B = \left(\frac{K_0}{\rho \nu^2}\right)^{1/3}, \quad C = \frac{Q_0}{\rho C_p}\left(\frac{\rho}{K_0 \nu}\right)^{1/3}.$$

Substituting expressions (4) into (1a) will yield

$$f_{\eta\eta\eta}pq^3+ff_{\eta\eta}p_xpq^2-f_\eta^2pq(pq)_x\pm\omega h^2\varepsilon^2=0$$

$$\frac{1}{Pr}h_{\eta\eta}\varepsilon q^2+fh_\eta p_xq\varepsilon-f_\eta hpq\varepsilon_x=0$$

$$\omega=g\gamma\left(\frac{Q_0}{C_pK_0}\right)^2. \tag{5}$$

Assume that

$$pq^3=a(1+n\phi),\quad p_xpq^2=b(1+n\phi)$$

$$pq(pq)_x=c(1+k\phi),\qquad \phi=\pm\omega\varepsilon^2. \tag{6}$$

Then, instead of the first equation of system (5), we have

$$af_{\eta\eta\eta}+bff_{\eta\eta}-cf_\eta^2=0$$

$$anf_{\eta\eta\eta}+bnff_{\eta\eta}-ckf_\eta^2+h^2=0. \tag{7}$$

Upon compliance with boundary conditions (2) the first equation (7) is easily integrated and gives a closed-form solution

$$f=2\frac{a}{b}\alpha\tanh\alpha\eta=6\alpha\tanh\alpha\eta. \tag{8}$$

Without limiting the generality it is impossible to assume that $a/b=3$, since equation (8) involves another free constant α. Noting further that integral relation (3) yields $p\varepsilon=1$, consider the second equation of system (5) which, after elimination of εq^2, goes over into the equation for seeking $h(\eta)$

$$\frac{1}{Pr}h_{\eta\eta}+\frac{1}{3}(fh)_\eta=0. \tag{9}$$

The solution of this equation is

$$h=c_1(1-\tanh^2\alpha\eta)^{Pr}$$

$$c_1=\left[6\alpha\int_{-1}^{1}(1-t^2)^{Pr}\,dt\right]^{-1}. \tag{10}$$

Now, the form of the function $k(x)$ in equations (6) will be found. For this purpose, substitute equation (10) into equations (7) at $Pr=1$ and integrate it from $-\infty$ to ∞

$$\left(bk-bn+\frac{c_1^2}{36\alpha^4}\right)\int_{-\infty}^{\infty}f_\eta^2\,d\eta=0\rightarrow k=n-\frac{c_1^2}{36b\alpha^2}.$$

As a result, equations (6) transform into equations for finding the unknown functions p and q

$$pq(pq)_x+p_xpq^2=\pm\frac{c_1^2}{36\alpha^4}\omega\frac{1}{p^2}$$

$$q=3p_x. \tag{11}$$

It can be easily verified that the relations

$$p(x)=\frac{[(1+2\Omega x)^{3/2}-1]^{1/3}}{(3\Omega)^{1/3}},\quad \Omega=\pm\frac{c_1^2}{36\alpha^4}\omega$$

$$q(x)=\frac{(3\Omega)^{2/3}(1+2\Omega x)^{1/2}}{[(1+2\Omega x)^{3/2}-1]^{2/3}} \tag{12}$$

provide the necessary coupling in system (11). The solution of the problem can be completed if it is possible to determine the constant α in terms of the given jet flow characteristic. The latter is taken to be the momentum in the initial cross-section of the jet

$$K_0=\lim_{x\to0}\int_{-\infty}^{\infty}\rho u^2\,dy. \tag{13}$$

According to equation (13), $\alpha=(1/48)^{1/3}$. Thus, the velocity and temperature fields in a plane buoyant jet are defined by the set of equations

$$u=\left(\frac{K_0}{\rho\sqrt{v}}\right)^{2/3}\frac{6^{1/3}}{4}(1-\tanh^2\alpha\eta)\frac{(\pm3\omega)^{1/3}(1+2\omega x)^{1/2}}{[(1\pm2\omega x)^{3/2}-1]^{1/3}}$$

$$\Delta T=\frac{Q_0}{\rho C_p}\left(\frac{\rho}{K_0v}\right)^{1/3}\frac{6^{1/3}}{4}(1-\tanh^2\alpha\eta)\times\frac{(\pm3\omega)^{1/3}}{[(1\pm2\omega x)^{3/2}-1]^{1/3}}. \tag{14}$$

This set of equations shows that, the directions of free and forced convection being opposite, the jet terminates at a finite distance from the heat source ($u=0$) owing to the jet retardation in the cross-section

$$\frac{Gr_x}{Re_x^2}=0.5,\quad Gr_x=\frac{g\gamma Q_0^2x^3}{\rho^2C_p^2v^4},\quad Re_x=\frac{K_0x}{\rho v^2}$$

and its complete mixing with the surroundings. It is not difficult to observe that the jet temperature has a finite magnitude. Furthermore, the momentum flux per second through the plane perpendicular to the jet axis

$$K(x)=K_0\left(1\pm2\frac{Gr_x}{Re_x^2}\right)^{1/2}$$

commonly prescribed in the theory of jets, does not remain unchanged but increases (or decreases in the conditions of opposing thermogravitation) from section to section on moving away from the jet source. At $Gr_x/Re_x^2=0$ this relation transforms into the condition of the constancy of axial momentum [6] and relations (14) go over into the well-known solutions [7]. The case $Gr_x/Re_x^2\rightarrow\infty$ corresponds to the problem of natural convection. Relations (14) yield formulae for u and $\Delta T(Pr=1)$

$$u=\left(\frac{g\gamma Q_0^2}{\rho^2C_p^2v}\right)^{1/3}4\alpha^2(1-\tanh^2\alpha\eta)$$

$$\eta=\left(\frac{g\gamma Q_0^2}{\rho^2C_p^2v^4}\right)^{1/6}x^{-1/2}y$$

$$\Delta T=\left(\frac{Q_0^4}{\rho^4C_p^4g\gamma v^2}\right)^{1/6}\sqrt{8}\alpha^2(1-\tanh^2\alpha\eta)x^{-1/2}$$

$$\alpha = \left(\frac{3}{32\sqrt{2}}\right)^{1/3}.$$

2.3. *Axisymmetric buoyant jet*

We analyse the system of equations (1a) with boundary condition (2) and integral conditions

$$Q_0 = 2\pi\rho C_p \int_0^\infty u\Delta T y \, dy = \text{const.} \qquad (15)$$

For functions ψ and ΔT there are equations (4), where

$$A = \nu, \quad B = \frac{K_0}{\pi\rho\nu^2}, \quad C = \frac{Q_0}{2\pi\mu C_p}, \quad \eta = \frac{B}{4}y^2 q(x).$$

Then, instead of equations (1a) it is possible to obtain

$$(\eta f_{\eta\eta})pq^2 + \frac{1}{2}ff_{\eta\eta}p_x pq^2 - \frac{1}{2}f_\eta^2 pq(pq)_x \pm \omega h^2 \varepsilon^2 = 0$$

$$\frac{1}{Pr}(\eta h_\eta)_\eta q\varepsilon + \frac{1}{2}fh_\eta p_x q\varepsilon - \frac{1}{2}f_\eta hpq\varepsilon_x = 0$$

$$\omega = \frac{1}{2}g\gamma\left(\frac{Q_0}{K_0 C_p}\right)^2. \qquad (16)$$

The system of equations (16) is integrated consequently, and under boundary conditions (2) and integral conditions (15) it admits the solution of the form

$$u = \frac{K_0}{\pi\mu}\frac{3}{8}\frac{1}{(1+\alpha\eta)^2}\frac{1}{x}(1 \pm 4\omega x)^{1/2}$$

$$\Delta T = \frac{3Q_0}{8\pi\mu C_p}\frac{1}{(1+\alpha\eta)^2}\frac{1}{x}$$

$$\eta = \frac{K_0}{\pi\rho\nu^2}\frac{y^2}{4x^2}(1 \pm 4\omega x)^{1/2}$$

$$\alpha = 3/16. \qquad (17)$$

The constant α has been found on the basis of arguments that with $x \to 0$ the expression

$$\lim_{x\to 0}\int_0^\infty u^2 y \, dy = K_0/2\pi\rho \qquad (18)$$

should transform into the familiar hydrodynamic condition of axial momentum conservation [6].

Now, the changes in some integral characteristics of the axisymmetric vertical buoyant jet will be written:

mass flow rate

$$\dot{m}(x) = 8\pi\mu x;$$

axial momentum flux

$$K(x) = K_0\left(1 \pm 2\frac{Gr_x}{Re^2}\right)^{1/2}, \quad Gr_x = \frac{g\gamma Q_0^2 x}{\pi^2\rho^2 C_p^2 \nu^4},$$

$$Re = \frac{K_0}{\pi\rho\nu^2}$$

which, at $Gr_x/Re^2 = 0$, is equal to K_0 (momentum of a forced jet) and at

$$\frac{Gr_x}{Re^2} = 0.5$$

is equal to zero, i.e. the flow in the jet ceases completely. It is interesting to note that the temperature, determined by equations (17), does not depend on Gr_x. This indicates that the temperature distribution in an axisymmetric vertical buoyant jet is a weak function of Gr_x.

Furthermore, the series expansion of equations (17) in the parameter $\zeta = Gr_x/Re^2$ (the parameter of mixed convection) may give expressions of the form

$$u = \frac{K_0}{2\pi\mu}x^{-1}\left\{f_0' + \sum_{i=0}^{\infty}\frac{2}{1+i}(f_i'\eta + (1-2i)f_i')\zeta^{1+i}\right\}$$

$$\Delta T = \frac{Q_0}{2\pi\mu C_p}x^{-1}\left\{h_0 + \sum_{i=0}^{\infty}\frac{2}{1+i}(h_i'\eta - 2ih_i)\zeta^{1+i}\right\}$$

where the functions f_0' and h_0 conform to the problem of submerged axisymmetric jet development without account for buoyancy forces [12]. These formulae can be obtained (see, for example, ref. [8]) by applying a radically different approach, namely the method of small perturbations. The significance of the latter fact is twofold. First, it testifies that the perturbation method is applicable to the analysis of jet flows with allowance for buoyancy forces, i.e. those problems for which it is very difficult to obtain exact solutions. Second, the use of the method has always involved the assumption that the basic process is heat transfer by forced convection, with free convection being regarded as a perturbation. However, asymptotic expansions (provided all terms of the series are retained) also describe the behaviour of a jet flow in the region where these two types of convection are comparable in magnitude.

2.4. *Plane wall buoyant jet*

Equations (1a) can also be used for considering the problem of a viscous liquid plane jet propagating along a solid vertical impenetrable surface of temperature $T_w = T_\infty$. In this case, boundary conditions have the form

$$y = 0: \quad u = v = \Delta T = 0$$

$$y \to \infty: \quad u \to 0, \Delta T \to 0. \qquad (19)$$

The solution of this problem is sought in the form of equation (4)

$$A = (E_0\nu)^{1/4}, \quad B = \left(\frac{E_0}{\nu^3}\right)^{1/4}, \quad C = \left(\frac{Q_0^2}{E_0\nu}\right)^{1/2}$$

where the prescribed characteristic constant Q_0 is equal to

$$Q_0 = \int_0^\infty u\Delta T\left(\int_0^y u \, dy\right) dy = \text{const.} \; (Pr = 1). \qquad (20)$$

Omitting computational details, which repeat those just presented in Section 2.2, the following system of equations can be obtained:

$$f_{\eta\eta\eta}pq^3+ff_{\eta\eta}p_xpq^2-f_\eta^2pq(pq)_x\pm\omega h^2\varepsilon^2=0$$

$$\frac{1}{Pr}h_{\eta\eta}\varepsilon q^2+fh_\eta p_x\varepsilon q-f_\eta hpq\varepsilon_x=0$$

$$\omega=g\gamma\left(\frac{Q_0}{E_0}\right)^2 \tag{21}$$

the solutions of which are

$$p(x)=\frac{[(1+\frac{2}{3}\Omega x)^{3/2}-1]^{1/4}}{(\Omega)^{1/4}},\quad \Omega=\pm108\frac{c_1^2}{\alpha^4}\omega$$

$$q(x)=\frac{(\Omega)^{3/4}(1+\frac{2}{3}\Omega x)^{1/2}}{[(1+\frac{2}{3}\Omega x)^{3/2}-1]^{3/4}},\quad c_1=\frac{20}{3\alpha^2}. \tag{22}$$

It remains to determine the still unknown quantity α. For this purpose, consider the integral relation

$$E_0=\lim_{x\to 0}\int_0^\infty u^2\left(\int_0^y u\,\mathrm{d}y\right)\mathrm{d}y=\text{const.} \tag{23}$$

Substituting equations (22) into equation (23), we obtain $\alpha=(40)^{1/4}$. This results in

$$u=\left(\frac{E_0}{\nu}\right)^{1/2}\frac{\sqrt{(10)}}{3}z(1-z^3)\frac{(\pm3\omega)^{1/2}(1\pm2\omega x)^{1/2}}{[(1\pm2\omega x)^{3/2}-1]^{1/2}}$$

$$\Delta T=\left(\frac{Q_0^2}{E_0\nu}\right)^{1/2}\frac{\sqrt{(10)}}{3}z(1-z^3)\frac{(\pm3\omega)^{1/2}}{[(1\pm2\omega x)^{3/2}-1]^{1/2}}$$

$$\eta=\left(\frac{E_0}{\nu^3}\right)^{1/4}\frac{(\pm3\omega)^{3/4}(1\pm2\omega x)^{1/2}}{[(1\pm2\omega x)^{3/2}-1]^{3/4}}y$$

$$\eta=\frac{12}{\alpha}\int\frac{\mathrm{d}z}{1-z^3}$$

$$\omega x=\frac{Gr_x}{Re_x^2},\quad Gr_x=\frac{g\gamma Q_0^2x^3}{\nu^6},\quad Re_x=\frac{E_0x}{\nu^3}. \tag{24}$$

If $Gr_x/Re_x^2=0$—that is, only forced convection is considered—solutions (24) become expressions for velocity and temperature fields in a wall jet without accounting for buoyancy forces [9, 10]. The case $Gr_x/Re_x^2\to\infty$ (or $x\to\infty$) corresponds to the problem of natural convection. Relations (24) give the formulae

$$u=\left(\frac{g\gamma Q_0^2}{\nu^2}\right)^{1/4}\frac{\alpha^2}{4}z(1-z^3)x^{-1/4},\quad \eta=\frac{8}{\alpha}\int\frac{\mathrm{d}z}{1-z^3}$$

$$\Delta T=\left(\frac{Q_0^2}{g\gamma\nu^2}\right)^{1/4}\frac{\sqrt{2}}{8}\alpha^2z(1-z^3)x^{-3/4}$$

$$\eta=\left(\frac{g\gamma Q_0^2}{\nu^6}\right)^{1/8}x^{-5/8}y,\quad \alpha=\left(\frac{80\sqrt{2}}{3}\right)^{1/4}$$

which describe this process. Under the conditions of opposing thermogravitation in the section

$$\frac{Gr_x}{Re_x^2}=0.5$$

there occur the separation of flow, an infinite growth of the boundary layer thickness and a decrease of friction on the wall

$$\frac{\tau_w}{\rho V^2}=\frac{(40)^{3/4}}{72}Re_x^{-5/4}\frac{(3\zeta)^{5/4}(1+2\zeta)}{[(1+2\zeta)^{3/2}-1]^{5/4}}$$

$$\zeta=\frac{Gr_x}{Re_x^2},\quad V=\frac{E_0}{\nu^2}.$$

Moreover, when $Gr_x/Re_x^2=0.5$, $\tau_w=0$. The other flow characteristics which can be found are:

flow rate

$$\frac{\dot m}{\mu}=(40)^{1/4}Re_x^{1/4}\frac{[(1+2\zeta)^{3/2}-1]^{1/4}}{(3\zeta)^{1/4}};$$

heat transfer rate

$$Nu_x\,Re_x^{1/4}=\frac{(40)^{3/4}}{72}\frac{(3\zeta)^{5/4}(1+2\zeta)^{1/2}}{[(1+2\zeta)^{3/2}-1]^{5/4}}.$$

2.5. *Validity of the results obtained*

In order to ascertain the range of validity of the analytical solutions obtained in the present paper, a comparison with the results of the numerical solution in ref. [11] was carried out. A buoyant jet is considered which rises near a wall from a line momentum and heat source at the front edge of a vertical plate with the surface temperature of the fluid away from it, i.e. $T_w=T_\infty$. This comparison is shown in Fig. 1, where dimensionless velocities and temperatures are plotted against a dimensionless coordinate for a jet with positive buoyancy. In this case, the change in the basic parameters constitutes a smooth transition from one law $u_m\sim x^{-1/2}$, $\Delta T_m\sim x^{-1/2}$ typical of a purely forced convection flow to another power law $u_m\sim x^{-1/4}$, $\Delta T_m\sim x^{-3/4}$ corresponding to free-convective heat transfer. As is seen, the theoretical curves from the equations

$$u_1=0.863\frac{(1+2x_1)^{1/2}}{[(1+2x_1)^{3/2}-1]^{1/2}}$$

$$\Delta T_1=0.863[(1+2x_1)^{3/2}-1]^{-1/2}$$

$$u_1=u_m\left(\frac{\nu E_0}{g\gamma Q_0}\right)^{1/2},\quad \Delta T_1=\Delta T_m\left(\frac{E_0^3\nu}{g\gamma Q_0^4}\right)^{1/2}$$

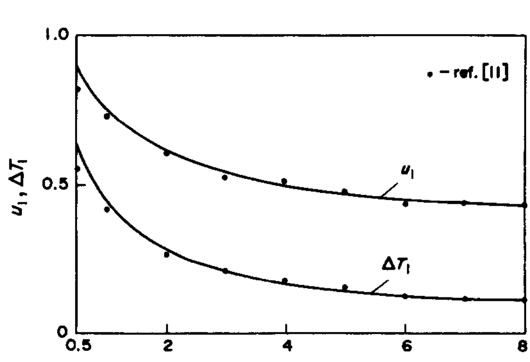

FIG. 1. Variation of maximum velocity and temperature along the near-wall buoyant jet ($Pr=1$).

$$x_1 = \frac{Gr_x}{Re_x^2}$$

agree satisfactorily with the data of the numerical simulation of ref. [11] in the entire flow region from the purely forced ($u_1 = 0.498x_1^{-1/2}$, $\Delta T_1 = 0.498x_1^{-1/2}$) to purely free convection regime ($u_1 = 0.725x_1^{-1/4}$, $\Delta T_1 = 0.513x_1^{-3/4}$) for the fluid with a Prandtl number equal to 1. As for the jet with adverse buoyancy, the numerical simulation gives a somewhat more rapid decrease in velocity temperature, etc. with approach to the stagnation point than predicted theoretically. Thus, for example, the theory predicts the flow separation to occur at $x_1 = 0.5$, whereas the numerical solution gives $x_1 = 0.45517$ (the difference is about 9%).

2.6. *Results and discussion*

Thus, a method has been suggested above on the basis of which the construction of exact self-similar solutions is possible for laminar boundary layer equations in the class of type (4) solutions, with the latter having been found only at constant Prandtl number equal to 1. The validity of the results obtained for other Prandtl numbers will now be considered. An attempt will be made to construct the solution of the problem from Section 2.3 by analysing integral expressions of the form

$$\frac{d}{dx}\int_0^\infty u^2 y\,dy = \pm g\gamma\int_0^\infty (\Delta T)^2 y\,dy$$

$$2\pi\rho C_p\int_0^\infty u\Delta T y\,dy = Q_0 \qquad (25)$$

derived conventionally from the motion and continuity equations and boundary conditions (1) and (2). In the framework of the approach suggested, it is necessary, before integrating the system of equations (25), to prescribe the form of the vertical velocity and temperature profiles. However, since in any case the shape of the profiles is not essential for the physics of the model, the discussion will be restricted only to the finding of the laws of change in the velocity, temperature and transverse dimension of a jet as a function of the longitudinal coordinate. It can easily be verified that

$$u(x,0) = \text{const.}\left(\frac{K_0}{\mu}\right)x^{-1}\left(1 \pm c_2 g\gamma\left(\frac{Q_0}{K_0 C_p}\right)^2 x\right)^{1/2}$$

$$\Delta T(x,0) = \text{const.}\left(\frac{Q_0}{\mu C_p}\right)x^{-1}$$

$$y = \text{const.}\left(\frac{\rho\nu^2}{K_0}\right)^{1/2} x\left(1 \pm c_2 g\gamma\left(\frac{Q_0}{K_0 C_p}\right)^2 x\right)^{-1/4} \qquad (26)$$

ensures the necessary relation (25). Moreover, the analysis of the system of equations (1a) indicates that the general solution should be determined in the form

$$\psi = Af(x,y)p(x)$$

$$\Delta T = \cosh(x,y)\varepsilon(x). \qquad (27)$$

Here too, however, the previously derived laws of change in the velocity and temperature along the symmetry axis, i.e. the functions $p(x)$, $q(x)$, $\varepsilon(x)$ are valid. The only difference is in the numerical coefficients (the constant c_2 is concerned) which vary with Pr. When $Pr = 1$, relations (27) and (4) yield coinciding formulae.

The analysis conducted being of a general character, it can be stated that the results obtained above represent, on the one hand, exact self-similar solutions at $Pr = 1$ and, on the other hand, the asymptotics of the behaviour of the problem solution at other Prandtl numbers.

Another important remark should be made. It was found earlier [8] (see Section 2.3) that the solution of the problem obtained as a result of a radically different approach (series expansion in small parameter powers) coincides with the expansion of an exact solution in an asymptotic series in powers of the same parameter. Consequently, this makes it quite clear that power expansions (i.e. expansions of the conventional perturbation theory) in jet problems involving buoyancy forces have the character of asymptotic series. This allows the properties of an exact solution to be found from several first terms of the expansions of sought-after functions in powers of the small parameter ζ and also from their asymptotic behaviour when $\zeta \to \infty$.

3. APPROXIMATE SOLUTIONS

3.1. *Basic equations*

The laminar boundary layer equations will now be analysed, with the linear temperature dependence of density being taken into account

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \frac{\nu}{y^j}\frac{\partial}{\partial y}\left(y^j\frac{\partial u}{\partial y}\right) \pm g\beta(\Delta T)$$

$$\frac{\partial}{\partial x}(y^j u) + \frac{\partial}{\partial y}(y^j v) = 0$$

$$u\frac{\partial \Delta T}{\partial x} + v\frac{\partial \Delta T}{\partial y} = \frac{a}{y^j}\frac{\partial}{\partial y}\left(y^j\frac{\partial \Delta T}{\partial y}\right). \qquad (28)$$

This system forms a basis for mathematical simulation. It is used (with corresponding boundary and integral conditions) to describe the development of various jet flows in the field of body forces. Moreover, the limiting flow regimes (purely forced and purely free convection) have been covered most comprehensively: investigations comprise vast theoretical, numerical and experimental data concerning the hydrodynamics and heat transfer characteristics

[3, 12]. As a rule, these relations give a satisfactorily quantitative and qualitative description of the process of jet development only at small or only at large values of the mixed convection parameter. In the region of mixed convection the data [3, 12] can hardly aid the prediction of the basic characteristics. For this reason, it is highly desirable to work out the method for constructing analytical solutions which would be valid for the entire flow region and appropriate for direct practical application.

3.2. *A plane free buoyant jet*

In order to investigate the action of body forces on forced flow in a jet, the system of basic equations (28) with boundary condition (2) and integral condition (3) is transformed into a new form by passing over from the coordinates (x, y) to the coordinates (x, η) which have the form

$$\eta = \left(\frac{K_0}{\rho v^2}\right)^{1/3} x^{-2/3} y$$

$$\zeta = \frac{g\beta Q_0 v}{C_p K_0}\left(\frac{\rho}{K_0 v}\right)^{2/3}, \quad x^{4/3} = \frac{Gr_x}{Re_x^{5/3}}. \qquad (29)$$

Moreover, the stream function and dimensionless temperature are introduced

$$\psi = \left(\frac{K_0 v}{\rho}\right)^{1/3} x^{1/3} \sum_{i=0}^{\infty} f_i(\eta)\zeta^i$$

$$\Delta T = \frac{Q_0}{\rho C_p}\left(\frac{\rho}{K_0 v}\right)^{1/3} \sum_{i=0}^{\infty} h_i(\eta)\zeta^i. \qquad (30)$$

The resulting system of transformed equations is written as

$$f_i''' + \frac{1}{3} f_0 f_i'' - \frac{1}{3}(4i-2) f_0' f_i' + \frac{1}{3}(4i+1) f_0'' f_i + h_{i-1} = A_i$$

$$A_i = \sum_{j=1}^{i-1} \left\{\frac{1}{3}[4(i-j)-1] f_j' f_{i-j}' - \frac{1}{3}(4j+1) f_j f_{i-j}\right\}$$

$$\frac{1}{Pr} h_i'' + \frac{1}{3} f_0 h_i - \frac{1}{3}(4i-1) f_0' h_i = B_i$$

$$B_i = \sum_{j=1}^{i} \left\{\frac{1}{3}[4(i-j)-1] f_j' h_{i-j} - \frac{1}{3}(4j+1) f_j h_{i-j}'\right\} \qquad (31)$$

with the boundary conditions

$$f_i(0) = 0, \quad f_i''(0) = 0, \quad h_i'(0) = 0$$

$$f_i' \to 0, \quad h_i \to 0, \quad \text{at} \quad \eta \to \infty. \qquad (32)$$

Besides, the energy conservation law (3) requires that at any ζ the following integral relations be fulfilled:

$$\int_{-\infty}^{\infty} \left\{ f_0' h_i + \sum_{j=1}^{i} f_j' h_{i-j} \right\} d\eta = 0. \qquad (33)$$

Thus, equations (31)–(33) allow the problem solution to be constructed in any approximation, i.e. make it possible to determine the functions f_i and h_i. For example, when $Pr = 2$

$$u = \left(\frac{K_0}{\rho\sqrt{v}}\right)^{2/3} x^{-1/3} \left\{ f_0' + \frac{c_1}{336\alpha^4}(5f_0''\eta + 6f_0')\zeta + \left(\frac{c_1}{336\alpha^4}\right)^2 \left(\frac{25}{2} f_0'''\eta^2 + \frac{42}{11} f_0''\eta - \frac{265}{11} f_0'\right)\zeta^2 + \dots \right\}$$

$$\Delta T = \frac{Q_0}{\rho C_p}\left(\frac{\rho}{K_0 v}\right)^{1/3} x^{-1/3} \left\{ h_0 + \frac{c_1}{336\alpha^4}(5h_0'\eta - h_0)\zeta + \left(\frac{c_1}{336\alpha^4}\right)^2 \left(\frac{25}{2} h_0''\eta^2 - \frac{343}{11} h_0'\eta + \frac{43}{11} h_0\right)\zeta^2 + \dots \right\}$$

$$f_0' = 6\alpha^2(1 - \tanh^2 \alpha\eta)$$

$$h_0 = c_1(1 - \tanh^2 \alpha\eta)^2$$

$$\alpha = (1/48)^{1/3}, \quad c_1 = \frac{5(6)^{1/3}}{16} \qquad (34)$$

where f_0' and h_0 are the familiar solutions for a forced jet. Consequently, the present approach makes use of the theory of forced jet flows as a zero approximation and this, naturally, simplifies the investigation. However, since in practical calculations the finite, though in some cases a large enough, number of series terms of equations (30) is usually known, the perturbation method is not effective in the region of strong coupling, because, apart from technical difficulties, its realization also encounters such major problems that the exactness of any solution deteriorates with an increase in the parameter ζ. Although the method is accurate enough in principle and in some cases ensures the appropriate accuracy of computation [12], the question of when and in which area the relations obtained provide reliable quantitative data has become a serious barrier to its use as a practical computational method. Naturally, this entailed a decrease of interest in this method, and even led to its underestimation.

We now turn our attention again to the analysis of equations (34). Since these expressions represent the asymptotic series expansion of an exact solution (at $Pr = 2$), it is possible to reconstruct the form of the latter

$$u = \left(\frac{K_0}{\rho\sqrt{v}}\right)^{2/3} 6\alpha^2(1 - \tanh^2 \alpha\eta) \frac{\left(1 \pm \frac{c_1}{16\alpha^4}\omega p^4\right)^{1/3}}{p}$$

$$\Delta T = \frac{Q_0}{\rho C_p}\left(\frac{\rho}{K_0 v}\right)^{1/3} c_1(1 - \tanh^2 \alpha\eta)^2 \frac{1}{p}$$

$$\eta = \left(\frac{K_0}{\rho v^2}\right)^{1/3} \frac{\left(1 \pm \frac{c_1}{16\alpha^4}\omega p^4\right)^{1/3}}{p^2} y$$

$$\omega = \frac{g\beta Q_0}{C_p K_0}\left(\frac{\rho}{K_0 v}\right)^{2/3}$$

$$x = \int \frac{3p^2\,\mathrm{d}p}{\left(1 \pm \frac{c_1}{16\alpha^4}\omega p^4\right)^{1/3}}. \tag{35}$$

These formulae will give the corresponding relations depicting the change of the maximum characteristics

$$u_1 = \frac{6^{1/3}}{4} \frac{\left(1 \pm \frac{45}{4(6)^{1/3}} t^4\right)^{1/3}}{t}$$

$$\Delta T_1 = \frac{5(6)^{1/3}}{16} \frac{1}{t}$$

$$\zeta^{3/4} = \int \frac{3t^2\,\mathrm{d}t}{\left(1 \pm \frac{45}{4(6)^{1/3}} t^4\right)^{1/3}}$$

$$u_1 = u_{\mathrm{m}}\left(\frac{\rho^2 C_p v}{g\beta Q_0 K_0}\right)^{1/4}$$

$$\Delta T_1 = \Delta T_{\mathrm{m}}\left(\frac{\rho^2 v C_p^5 K_0^3}{g\beta Q_0^5}\right)^{1/4}$$

$$\zeta = \frac{Gr_x}{Re_x^{5/3}}, \quad Gr_x = \frac{g\beta Q_0 x^3}{\rho C_p v^3}, \quad Re_x = \frac{K_0 x}{\rho v^2}. \tag{36}$$

These relations yield

when $\zeta \to 0$:

$$u_1 = 0.454\zeta^{-1/4}, \quad \Delta T_1 = 0.568\zeta^{-1/4}$$

when $\zeta \to \infty$:

$$u_1 = 0.837\zeta^{3/20}, \quad \Delta T_1 = 0.561\zeta^{-9/20}$$

in accordance with exact analytical solutions for forced and natural convective jets [7, 13]. Furthermore, compare equations (36) with the results of the numerical simulation of a plane vertical jet developing in the field of body forces. Investigations of this kind are described in ref. [5]. According to this study, in the case of opposing thermogravitation $\zeta_{\mathrm{T}} = 0.3576886$, conditions (36) give

$$\zeta_{\mathrm{T}} = \frac{1}{5}\left(\frac{5}{6}\right)^{1/4} \frac{\Gamma\left(\frac{3}{4}\right)\Gamma\left(\frac{2}{3}\right)}{\Gamma\left(\frac{17}{12}\right)} = 0.357685$$

the quantity $\Delta T_1(\zeta_{\mathrm{T}})$ being equal to 0.89573 and

$$\Delta T_1(\zeta_{\mathrm{T}}) = \frac{15}{16}\left(\frac{5}{6}\right)^{1/4} = 0.895727$$

respectively.

3.3. *Axisymmetric buoyant jet*

The present problem is still the only one in the class of equations (28) for which an exact solution can be constructed at $Pr = 2$ [15]

$$u = \frac{K_0}{\pi\mu}\frac{3}{8}\frac{1}{(1+\alpha\eta)^2}\frac{1}{x}\sqrt{\left(1 \pm \frac{40}{9}\omega x^2\right)}, \quad \alpha = \frac{3}{16}$$

$$\Delta T = \frac{Q_0}{\pi\mu C_p}\frac{5}{8}\frac{1}{(1+\alpha\eta)^4}\frac{1}{x}$$

$$\eta = \frac{K_0}{\pi\rho^2}\frac{y^2}{4x^2}\bigg/\sqrt{\left(1 \pm \frac{40}{9}\omega x^2\right)}, \quad \omega = \frac{g\beta Q_0 \pi\mu}{C_p K_0^2}$$

$$\omega x^2 = \frac{Gr_x}{Re^2}, \quad Gr_x = \frac{g\beta Q_0 x^2}{\pi C_p \rho v^3}, \quad Re = \frac{K_0}{\pi\rho v^2}. \tag{37}$$

It would be highly important to find a closed solution at arbitrary Prandtl numbers, but this involves great mathematical difficulties posed by the non-linear character of the equations, with the main difficulty presented by the finding of the function $h(x, y)$ (equations (27)). Naturally, a rigorous calculation of this temperature field characteristic is fraught with the same difficulties as the solution of the initial problem. However, for the latter an approximate solution can be constructed. In fact, taking as a basis the results of the investigation of equations (28), (2) and (15) by the method of small perturbations at different Prandtl numbers ($Pr = 2$ [15], 6.7, 10.0, 20.0 [16]), it is possible to obtain

$$u_{\mathrm{m}} = \frac{K_0}{\pi\mu}\frac{3}{8}x^{-1}(1+c_2\omega x^2)^{1/2}$$

$$\Delta T_{\mathrm{m}} = \frac{Q_0}{2\pi\mu C_p}\frac{2Pr+1}{4}\frac{1+\frac{3Pr+4}{2Pr+1}\omega x^2}{1+\frac{11Pr+10}{8Pr}\omega x^2}x^{-1}. \tag{38}$$

The first noteworthy fact here is that equations (38) describe a continuous change of the jet maximum parameters over the entire flow range from the regime of purely forced convection to free-convective heat transfer. When $x \to \infty$, equations (38) simplify and take the form

$$u_{\mathrm{m}} = \left(\frac{g\beta Q_0}{\pi\mu C_p}\right)^{1/2} f_0'(0), \quad \Delta T_{\mathrm{m}} = \frac{Q_0}{2\pi\mu C_p} h_0(0) x^{-1}$$

$$f_0'(0) = \frac{3}{8}c_2^{1/2}, \quad c_2 = \frac{160Pr(13Pr+28)}{9(20Pr^2+167Pr+18)}$$

$$h_0(0) = \frac{2Pr(3Pr+4)}{11Pr+10}. \tag{39}$$

Since equations (39) comply with the solution of the problem of free convective flow above a point heat source, the range of possible Prandtl numbers can be determined. For example, when $Pr = 0.01$ and 10.0

$$h_0(0) = 0.00797, \quad h_0(0) = 5.667$$

whereas exact numerical calculations [13] yield

$$h_0(0) = 0.00795, \quad h_0(0) = 5.61.$$

Asymptotic evaluation of equations (39) for $Pr \to \infty$ gives $h_0(0)/Pr = 0.545$ and the familiar numerical calculation [17] in this case gives $h_0(0)/Pr = 0.522$.

Finally, at $Pr = 1$ and 2, equations (39) transform into the analytical correlations [18] obtained previously.

3.4. *Axisymmetric buoyant plume*

In Section 3 the problem of the development of a vertical submerged jet was considered and solved. Since the method used to investigate the system of equations (28) with the boundary conditions

$$y = 0: \quad v = \frac{\partial u}{\partial y} = \frac{\partial \Delta T}{\partial y} = 0$$

$$y \to \infty: \quad u \to u_\infty, \quad \Delta T \to 0 \tag{40}$$

and integral conditions

$$Q_0 = 2\pi\rho C_p \int_0^\infty u\Delta T y \, dy = \text{const.} \tag{41}$$

is close in structure to the corresponding method described above, the final formulae will be given directly as

$$\Delta T_m = \frac{Q_0}{2\pi\mu C_p} \frac{Pr}{2} \frac{1 \pm c_3\omega}{1 \pm c_4\omega} x^{-1}$$

$$u_m = u_\infty (1 \pm c_2\omega)^{1/2}$$

$$\omega = \frac{g\beta Q_0}{2\pi\mu C_p u_\infty^2} = \frac{Gr_x}{2Re_x^2} \tag{42}$$

or, in dimensionless form

$$u_1 = (1 + c_2\zeta)^{1/2}, \quad \Delta T_1 = \frac{Pr}{2} \frac{1 + c_3\zeta}{1 + c_4\zeta}$$

$$u_1 = \frac{u_m}{u_\infty}, \quad \Delta T_1 = \frac{2\pi\mu C_p x \Delta T_m}{Q_0}, \quad \zeta = \frac{Gr_x}{2Re_x^2} \tag{43}$$

where the values of the constants c_2, c_3 and c_4 at different Prandtl numbers are given in Table 1.

Table 1

Pr	c_2	c_3	c_4
0.7	0.868	0.274	0.200
7.0	1.817	0.665	0.585
50.0	2.847	1.825	1.705
100.0	3.266	1.985	1.865

Figures 2–4 display the results of the comparison of the analytical and numerical solutions to the problem of propagation of a laminar jet originating from a point heat source moving vertically (falling) in a viscous fluid. It is seen that the data from numerical simulation [19, 20] and from theoretical curves (equations (43)) agree satisfactorily in the entire flow region in the regime of assisting and opposing thermal gravitation.

3.5. *Plane wall buoyant jet*

In conclusion, a summary of the main results will be given for the near-wall jet propagating upward from a line heat source and for momentum on the frontal edge of a vertical impenetrable surface under different boundary conditions.

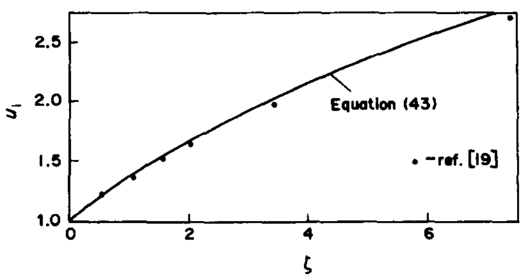

FIG. 2. Dependence of u_1 on ζ. Comparison between the predicted and numerical data [19] for the case of assisting thermal gravitation ($Pr = 0.72$).

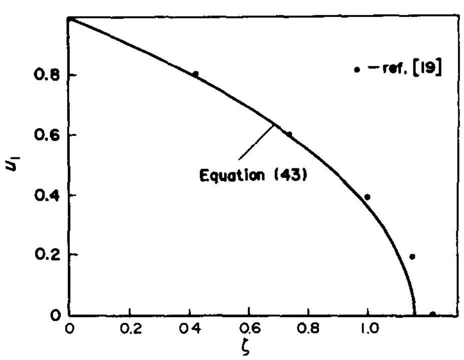

FIG. 3. Dependence of u_1 and ΔT_1 on ζ. Comparison between the predicted and numerical data [19] for the case of opposing thermal gravitation ($Pr = 0.72$).

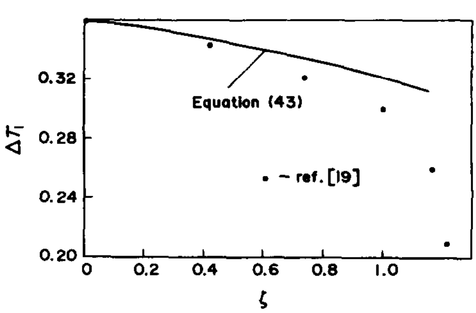

FIG. 4. Dependence of ΔT_1 on ζ. Comparison between the predicted and numerical data [20] for the case of assisting thermal gravitation ($Pr = 7.0$).

3.5.1. *Isothermal surface*. The final formulae are

$$u_m = \left(\frac{E_0}{\nu}\right)^{1/2} \frac{(10)^{1/2}}{4^{4/3}} x \frac{(\pm 3c_2\omega)^{3/4}(1 \pm c_2\omega x^2)^2}{[(1 \pm c_2\omega x^2)^3 - 1]^{3/4}}$$

$$\omega = \frac{g\beta\Delta T_w \nu}{E_0}, \quad \omega x^2 = \frac{aGr_x}{Re_x}, \quad Gr_x = \frac{g\beta\Delta T_w x_3}{\nu^2}$$

$$Re_x = \frac{E_0 x}{\nu^3}. \tag{44}$$

In this problem, of paramount importance, together with the velocity and temperature distribution, is the local Nusselt number, which can be written as

$$\frac{Nu_x}{Re_x^{1/4}} = [-h_0'(0)](3c_2)^{7/8} \frac{\zeta^{7/8}(1+c_2\zeta)^2}{[(1+c_2\zeta)^3 - 1]^{7/8}}$$

$$[-h_0'(0)] = \frac{(40)^{1/4}}{12}\left[\int_0^1 (1-z^3)^{Pr-1}\,dz\right]^{-1}. \tag{45}$$

Assuming in equations (44) and (45) that $\zeta = Gr_x/Re_x = 0$ and ∞, we obtain the data for the purely forced and purely free convection regimes, which are two limiting cases of the mixed convection regime.

3.5.2. *Adiabatic surface*. In the case of a laminar buoyant jet developing along an adiabatic vertical surface, the analytical dependence of u_m, ΔT_w on the free convection parameter $\zeta = Gr_x/Re_x^{5/4} = \omega x^{7/4}$ can be represented by an approximate formula

$$u_m = \left(\frac{E_0}{\nu}\right)^{1/2} \frac{(10)^{1/2}}{4^{4/3}} x^{3/4} \times \frac{\left(\pm \frac{12}{5}c_2\omega\right)^{5/7}(1 \pm c_2\omega x^{7/4})^{7/5}}{[(1 \pm c_2\omega x^{7/4})^{12/5} - 1]^{5/7}}$$

$$\Delta T_w = \left(\frac{Q_0^4}{\rho^4 C_p^4 E_0 \nu}\right)^{1/4} h_0(0) \frac{\left(\pm \frac{12}{5}c_2\omega\right)^{1/7}}{[(1 \pm c_2\omega x^{7/4})^{12/5} - 1]^{1/7}}$$

$$\omega = \frac{g\beta Q_0 \nu^2}{\rho C_p (E_0\nu)^{5/4}}$$

$$h_0(0) = \frac{1}{(40)^{1/4}}\left[\int_0^1 (1-t^{3/2})^{Pr}\,dt\right]^{-1}. \tag{46}$$

3.5.3. A constant heat flux is set on the wall for the boundary conditions of the form

$$y = 0: \quad u = v = 0, \quad q_w = -k\frac{\partial \Delta T}{\partial y}$$

$$y \to \infty: \quad u \to 0, \quad \Delta T \to 0. \tag{47}$$

Analysis of the system of equations (28) yields

$$u_m = \left(\frac{E_0}{\nu}\right)^{1/2} \frac{(10)^{1/2}}{4^{4/3}} x^{7/4} \times \frac{\left(\pm \frac{16}{5}c_2\omega\right)^{9/11}(1 \pm c_2\omega x^{11/4})^{11/5}}{[(1 \pm c_2\omega x^{11/4})^{16/5} - 1]^{9/11}}$$

$$\Delta T_w = \left(\frac{q_w^4 \nu^3}{k^4 E_0}\right)^{1/4} h_0(0) x^{-7/4} \times \frac{[(1 \pm c_2\omega x^{11/4})^{16/5} - 1]^{10/11}}{\left(\pm \frac{16}{5}c_2\omega\right)^{10/11}(1 \pm c_2\omega x^{11/4})^{11/5}}$$

$$h_0(0) = \frac{4}{(40)^{1/4}} \frac{\Gamma(Pr+a)\Gamma(Pr+b)\Gamma(1/3)}{\Gamma(2/3)\Gamma(1-a)\Gamma(1-b)}$$

$$\omega x^{11/4} = \frac{Gr_x^*}{Re_x^{5/4}} = \zeta \tag{48}$$

where it is assumed that

$$\omega = \frac{g\beta\nu^3 q_w}{k(E_0\nu)^{5/4}}, \quad a+b = \frac{2}{3} - Pr, \quad ab = 2Pr$$

$$Gr_x^* = \frac{g\beta q_w x^4}{k\nu^2}.$$

On the basis of the results of equations (48) and conversion to the local Nusselt number Nu_x it is possible to obtain

$$\frac{Nu_x}{Re_x^{1/4}} = [h_0(0)]^{-1} \frac{\left(\frac{16}{5}c_2\zeta\right)^{10/11}(1+c_2\zeta)^{11/5}}{[(1+c_2\zeta)^{16/5} - 1]^{10/11}}. \tag{49}$$

It should be noted that the solution of the problem of plane buoyant jet propagation along a vertical surface, i.e. equations (44)–(49), determines the self-similar asymptotics only accurate to the constant c_2, which, generally speaking, is a function of Pr ($c_2 = f(Pr)$). For this quantity to be actually found, it is necessary to solve the problem by the method of small perturbations within a wide range of Prandtl numbers.

4. CONCLUSIONS

Based on laminar boundary layer equations under the Boussinesq approximation, theoretical results are presented for the effect of buoyancy forces on hydrodynamics and heat transfer in vertical buoyant jets and plumes. The method for analysing the interrelated system of equations is developed and used as a basis to show the possibility of the existence of exact self-similar solutions for some jet problems. The solutions obtained allowed insight to be gained into the problems of calculation for viscous buoyant jets, and, moreover, turned out to be a link between the data obtained previously. It appeared that the power expansions (the ordinary expansions of the perturbation theory) have the nature of asymptotic series. The latter enables the reconstruction of the main properties of exact solutions on the basis of several first terms of the sought-after functions expanded in small parameter powers.

This procedure corresponds to the summation of a certain infinite succession entering into the main series. The integration of the chain of equations obtained is not the result, but rather the starting point of solution. The emphasis is shifted to the second stage, i.e. to the study of the expansions constructed.

The analysis of various jet problems from the above viewpoint has made it possible to obtain a number of new results and also to extend substantially the class of problems susceptible to analytical examination. The authors hope that the suggested theory of the construction of exact and approximate self-similar solutions is also of practical significance.

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LA CLASSE DES SOLUTIONS AFFINES POUR LES JETS LAMINAIRES FLOTTANTS

Résumé—On présente les résultats d'une étude théorique, basée sur l'analyse d'un système d'équations de couche limite laminaire dans l'approximation de Boussinesq, pour les effets de flottement sur l'écoulement et le transfert de chaleur dans des jets et des panaches verticaux. On construit analytiquement des solutions exactes et approchées. Les calculs sont comparés avec des données numériques fournies par d'autres auteurs.

EINE KLASSE VON LÖSUNGEN FÜR LAMINARE AUFTRIEBSSTRÖMUNGEN

Zusammenfassung—Das System der laminaren Grenzschichtgleichungen wird nach Boussinesq vereinfacht und analytisch gelöst. Auf dieser Grundlage wird der Einfluß des Auftriebs auf Strömung und Wärmeübergang in senkrechten Strahl- und Konvektionsströmungen untersucht und dargestellt. Exakte und Näherungslösungen werden analytisch bestimmt. Die Ergebnisse werden mit numerisch berechneten Daten anderer Autoren verglichen.

КЛАСС АВТОМОДЕЛЬНЫХ РЕШЕНИЙ ДЛЯ ЛАМИНАРНЫХ ПЛАВУЧИХ СТРУЙ

Аннотация—Представлены результаты теоретического исследования влияния эффектов плавучести на течение и теплообмен в вертикальных струях и следах на основе анализа системы уравнений ламинарного пограничного слоя в приближении Буссинеска. Построены точные и приближенные аналитические решения. Результаты расчетов сопоставлены с численными данными других авторов.