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# Stability analysis of control system having PD type of fuzzy controller

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**Abstract**

Stability of a control system having PD type of fuzzy controller (a two-input and single-output controller) is analyzed in this paper using Yakubovich's method. In this analysis, the controller is treated as a two-input and two-output controller, and the nonlinearity of the controller is characterized as the sector condition. Absolute stability conditions are derived for the control system as three conditions; two are conditions for either one of control modes, P or D, is independently operating; the third is the condition for two modes are simultaneously active. The former two conditions are nothing but Popov's conditions for two SISO systems. A new graphical method is proposed to solve the latter condition. This method is explained illustratively using an example, and the results obtained are compared numerically with those by other graphical methods as well as simulation results.

*Keywords:* Control theory; Fuzzy control system; Absolute stability; Yakubovich's criterion; Popov's trajectory

## 1. Introduction

Applications of fuzzy logic controllers have been gaining significant practical usages, since Mamdani [7] utilized this type of controller for the control of a steam engine. However, the systematic design methodology of fuzzy controllers has not been formulated yet; some efforts to establish the methodology have been started [2, 4, 9, 12]. If a systematic design algorithm is formulated, practical applications of fuzzy control system can be greatly widened.

One important prerequisite for the fuzzy control to widen its application areas is to devise the method for the stability analysis of the control system. Generally, the fuzzy controller can be regarded as a kind of nonlinear controller as some researchers have pointed out [4, 5]. If a controlled plant is linear, some of the traditional stability analysis methods for the nonlinear control systems [1, 10, 13, 16] could be applicable to fuzzy control systems. Several approaches in this line have already been started [3–6, 11, 14]; for example,

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Kirckert and Mamdani utilized the describing technique [4], Hojo et al. [3] and Maeda et al. [6] used the phase plane method, Tanaka and Sugeno [11] made use of Lyapunov's direct method, and Kitamura proposed the extended circle criterion for this analysis [5]. The results obtained by these respective methods, however, have not been compared to each other yet.

In this paper, Yakubovich's method [13] is applied for the analysis of a control system having a PD type of fuzzy controller, which has two inputs and a single output, by transforming the system into a two-input and two-output fuzzy control system having a linear plant; a Lure's control system. A fuzzy controller derived by an indirect reasoning [6, 8] is used here, and the nonlinearity of the controller can be characterized by the sector condition. Yakubovich's method is universally applicable to the absolute stability analysis of Lure's system which consists of two parts; an asymptotically stable linear part; static nonlinear part, input–output relations of which are characterized by sector conditions.

Absolute stability conditions derived for the control system are represented by three conditions; two of them are those when either of two control modes, P or D mode, is independently operating to the system, and the last one is that when two modes are active simultaneously. It will be made clear that the former conditions are nothing but Popov's conditions for two SISO (single-input and single-output) systems, and a new graphical algorithm for solving the latter condition will be proposed. Analysis is carried out by solving an example; a second order oscillatory plant with a dead time is assumed; the results obtained are compared numerically with those by other graphical methods as well as simulation results.

In this paper, boldfaced italic capital and small letters denote matrices and vectors, respectively, and nonboldfaced italic small letters denote scalars, as the notations.

## 2. Yakubovich's stability criterion

In this section, the absolute stability analysis method of the nonlinear control system proposed by Yakubovich [13] is explained briefly to be easily applicable to analyze the stability of the fuzzy control systems.

In the method, the following system is considered.

$$\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{b}\boldsymbol{u}, \quad \boldsymbol{\sigma} = \boldsymbol{c}\boldsymbol{x} \quad \text{(linear dynamic plant)}, \tag{1a}$$

$$\boldsymbol{u} = -\boldsymbol{\phi}(\boldsymbol{\sigma}) \quad \text{(nonlinear static controller)}, \tag{1b}$$

where $\boldsymbol{x} = [x_j] \in R^{v \times 1}$ is a state variable vector of the system, $\boldsymbol{\sigma} = [\sigma_j] \in R^{m \times 1}$ is an output vector of the plant, $\boldsymbol{\phi} = [\phi_j] \in R^{n \times 1}$ is an output vector of the controller, $\boldsymbol{A} \in R^{v \times v}$, $\boldsymbol{b} \in R^{v \times n}$, and $\boldsymbol{c} \in R^{m \times v}$. It is assumed that the linear plant is controllable, observable, and stable, that is, $\boldsymbol{A}$ is a stable (Hurwitz) matrix.

The system defined in Eq. (1) can be written as

$$\boldsymbol{\sigma} = -\boldsymbol{G}(s)\boldsymbol{\phi}(\boldsymbol{\sigma}), \tag{2}$$

where $\boldsymbol{G}(s)$ is the transfer function of the linear part of the system, and it is defined as

$$\boldsymbol{G}(s) \equiv \boldsymbol{c}(s\boldsymbol{I} - \boldsymbol{A})^{-1}\boldsymbol{b}. \tag{3}$$

The stability analysis procedure of this system can be formulated as follows:

(1) It is assumed that it would be possible to find homogeneous quadratic forms $W_j$ in the input $\sigma_j$ $(j = 1, \dots, m)$ and the output $\phi_j$ $(j = 1, \dots, n)$ of nonlinear elements, and also that during the operation of the system the $\sigma_j$ and the $\phi_j$ satisfy the relations

$$W_j = 0 \quad (j = 1, \dots, k_1), \tag{4a}$$

$$W_j \geqslant 0 \quad (j = k_1 + 1, \dots, k_1 + k_2 = k). \tag{4b}$$

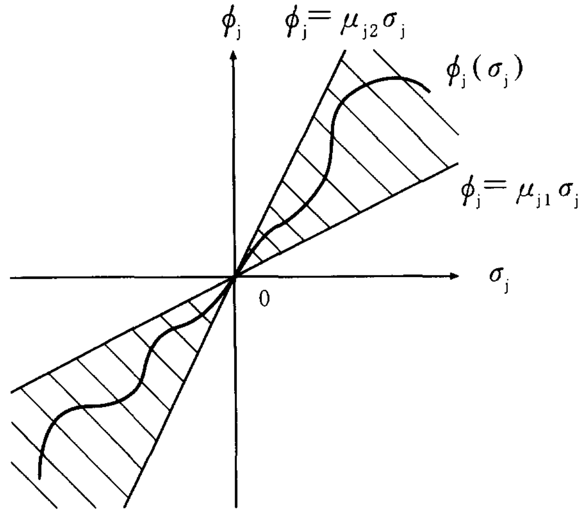


Fig. 1. Sectors enclosing nonlinear function $\phi(\sigma)$.

For example, let the output $\phi_j$ and the input $\sigma_j$ always be situated on the $\sigma_j$–$\phi_j$ plane between the rays $\phi_j = \mu_{j1}\sigma_j$ and $\phi_j = \mu_{j2}\sigma_j$, as shown in Fig. 1, where $\mu_{j1}$ and $\mu_{j2}$ ($\mu_{j1} < \mu_{j2}$) are constant positive numbers. Then, the relation $\mu_{j1} \leqslant \phi_j/\sigma_j \leqslant \mu_{j2}$ is satisfied, i.e., the following quadratic form $W_j$ can be found:

$$W_j \equiv (\phi_j - \mu_{j1}\sigma_j)(\mu_{j2}\sigma_j - \phi_j) \geqslant 0. \tag{5}$$

The nonlinear element $j$ can be characterized by a sector $[\mu_{j1}, \mu_{j2}]$, where parameters $\mu_{j1}$ and $\mu_{j2}$ are usually called as sector parameters.

(2) By introducing parameters $\tau_j$ ($j = 1, \dots, k$) a new quadratic form $W$ is defined by

$$W(\boldsymbol{\phi}, \boldsymbol{\sigma}) \equiv \sum_{j=1}^{k} \tau_j W_j. \tag{6}$$

(3) The following quadratic form in the complex variables $\boldsymbol{\phi}$ with coefficients depending a complex parameter $s$ is setup.

$$F(s, \boldsymbol{\phi}) = \mathrm{Re}\left[ s \sum_{h=1}^{n} \theta_h \phi_h^* \sigma_{qh} \right] + W(\boldsymbol{\phi}, \boldsymbol{\sigma}), \tag{7}$$

where $\boldsymbol{\sigma} = -\boldsymbol{G}(s)\boldsymbol{\phi}(\boldsymbol{\sigma})$, $\sigma_{qh} = -G_{qh,h}(s)\phi_h$, $G_{qh,h}(s)$ means a $qh$ column $h$ row element of $\boldsymbol{G}(s)$, and $\phi_h^*$ is a complex conjugate of $\phi_h$.

System (1) is absolute stable if the quadratic form $F$ is negative definite for admissible parameters $\tau_j$, $\theta_j$ and for all $\omega$ ($-\infty \leqslant \omega \leqslant \infty$). Appendix shows how parameters $\tau_j$ and $\theta_j$ must be selected.

## 3. Control system having a PD type fuzzy controller

Many types of fuzzy controllers have been already proposed. They can be classified according to the reasoning method and the defuzzification method. In this paper, a fuzzy controller proposed by Murakami and Maeda [8] is studied because its input–output relation can be expressed in an analytical form, which is derived by the indirect reasoning and the defuzzification by the maximum grade.

Fuzzy control rules for the position algorithm of this controller having P and D modes are given as a pair as follows:

$$R_{P_i}: \text{ IF } e_i \text{ is } P_i \text{ THEN } u \text{ is } P_{ui} \tag{8a}$$

$$R_{N_i}: \text{ IF } e_i \text{ is } N_i \text{ THEN } u \text{ is } N_{ui} \quad \text{for } i = 1,2. \tag{8b}$$

In the rules, $e_i$ means $e$ or $de/dt$, where $e$ is a control error and $u$ is a controller output. Characters $P$ and $N$ are the positive and the negative fuzzy sets, respectively. Membership functions for $P_i$, $P_{ui}$ and $N_i$, $N_{ui}$ are defined as follows:

$$\lambda_{P_i} = (1/\pi)\tan^{-1}(c_i e_i) + 0.5, \tag{9a}$$

$$\lambda_{N_i} = -(1/\pi)\tan^{-1}(c_i e_i) + 0.5, \tag{9b}$$

$$\lambda_{P_{ui}} = (1/2b_i)u + 0.5, \tag{9c}$$

$$\lambda_{N_{ui}} = -(1/2b_i)u + 0.5, \tag{9d}$$

where $c_i = \tan(0.45\pi)/a_i$. Parameters $a_i$ and $b_i$ are the values of inputs and outputs in the universe of discourse for which value of membership function is 0.95 and 1.0, respectively.

Eqs. (8) and (9) show that the controller is characterized by a smaller number of the membership functions than the direct reasoning method: two pairs of symmetrical function for one control input. Output of the controller can be obtained by a geometrical method [14] as well as the method described in [8] as follows:

$$u = K_c\{\tan^{-1}(c_1 e_1) + \tan^{-1}(c_2 e_2)\}, \tag{10}$$

where $K_c = 2b_1 b_2/\pi(b_1 + b_2)$, $e_1 = e$, and $e_2 = de/dt$. Eq. (10) shows this controller is characterized as a nonlinear PD type.

Let us consider the sector condition of the controller. Output $u$ of the fuzzy controller given by Eq. (10) cannot be written in the form of $u = \phi(e)$, because the controller has two inputs $e_1$ and $e_2$. Then, the controller is here regarded as having two inputs and two outputs:

$$u = \phi = K_c\left\{\sum_{i=1}^{2} \tan^{-1}(c_i e_i)\right\}, \tag{11}$$

which can be split as $\phi_1$ and $\phi_2$ as follows:

$$\phi_i = K_c\{\tan^{-1}(c_i e_i)\}, \quad i = 1,2. \tag{12}$$

Each $\phi_i$ is enclosed between a sector $[0, \mu_i]$, where $\mu_i$ is the slope of the straight line touching the curve $\phi_i$ at the origin of the $e_i$–$\phi_i$ plane:

$$\mu_i = K_c c_i. \tag{13}$$

Here, the fuzzy controller represented by Eq. (10) is used for the discussion of the stability of the fuzzy control system, which can be illustrated using a block diagram, as Fig. 2. However, if the output of the controller can be written as the sum of the outputs of P mode and D mode just as Eq. (10), the bellow analysis is proper independently of the type of fuzzy controller.

The control system having this controller can be translated to a Lure's system shown in Fig. 3, assuming the asymptotic stability for the linear part, where $e_i = -\sigma_i$. Let $G_i$ ($i = 1,2$) be the transfer function from the

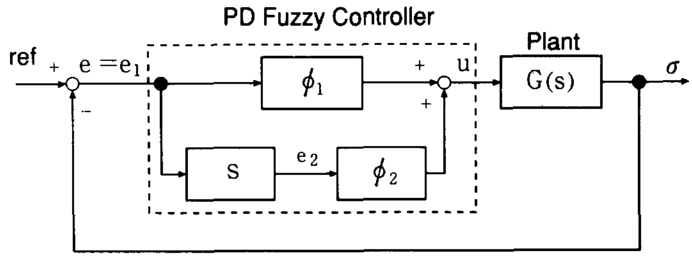


Fig. 2. PD type fuzzy control system.

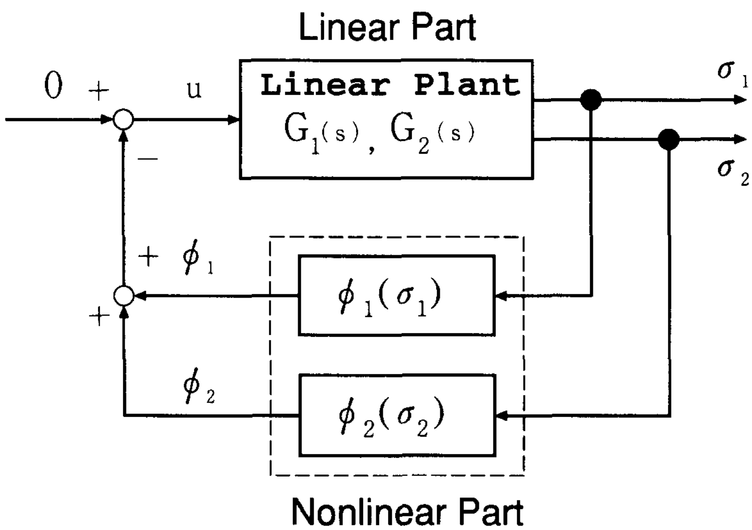


Fig. 3. Fuzzy control system transformed to a Lure's system.

input of the plant to its output $\sigma_i$, we have

$$\sigma_1 = -G_1(s)\{\phi_1(\sigma_1) + \phi_2(\sigma_2)\}, \tag{14a}$$

$$\sigma_2 = -G_2(s)\{\phi_1(\sigma_1) + \phi_2(\sigma_2)\}, \tag{14b}$$

where $G_1(s) = G(s)$ and $G_2(s) = sG(s)$. Therefore, the output of the control system can be written as

$$\boldsymbol{\sigma} = -\begin{pmatrix} G_1 & G_1 \\ G_2 & G_2 \end{pmatrix}\boldsymbol{\phi}, \tag{15}$$

where $\boldsymbol{\sigma} = (\sigma_1, \sigma_2)^{\mathrm{T}}$ and $\boldsymbol{\phi} = (\phi_1, \phi_2)^{\mathrm{T}}$.

## 4. Stability condition

In this section, stability condition of the fuzzy control system shown in Fig. 3 is derived using Yakubovich's method explained in Section 2. As the nonlinearity of the controller can be represented by

$0 \leqslant \phi_j/\sigma_j \leqslant \mu_j$ due to Eq. (12) and the homogeneous quadratic forms $W_i$ ($i = 1, 2$) can be described by the form of Eq. (5), parameters $\tau_i$ are nonnegative (see Eq. (A.1b) in the Appendix). Moreover, parameters $\theta_i$ are also nonnegative, because the controller satisfies the condition of Eq. (A.2a), as shown in the Appendix.

Application of Eq. (7) for the analysis of this system gives the following equation:

$$F = -\operatorname{Re}[\boldsymbol{\phi}^* \boldsymbol{\pi}(j\omega)\boldsymbol{\phi}], \tag{16}$$

where $\boldsymbol{\pi}(j\omega) = \boldsymbol{\tau}_{\rm d}\boldsymbol{\mu}_{\rm d}^{-1} + \operatorname{Re}[(\boldsymbol{\tau}_{\rm d} + j\omega\boldsymbol{\theta}_{\rm d})\boldsymbol{G}(j\omega)]$. Parameters $\boldsymbol{\tau}_{\rm d}$, $\boldsymbol{\mu}_{\rm d}^{-1}$ and $\boldsymbol{\theta}_{\rm d}$ are diagonal matrices having $[\tau_1, \tau_2]$, $[\mu_1^{-1}, \mu_2^{-1}]$ and $[\theta_1, \theta_2]$ as diagonal elements, respectively. And, this equation can be rewritten as

$$F = -\boldsymbol{\phi}^*\{\boldsymbol{\pi}(j\omega) + \boldsymbol{\pi}^*(j\omega)\}\boldsymbol{\phi}/2. \tag{17}$$

For the absolute stability of the control system, $F$ in Eq. (17) is required to be negative definite. Since $\boldsymbol{\pi}(j\omega) + \boldsymbol{\pi}^*(j\omega)$ is a Hermite matrix, inequalities (18) and (19) are obtained by applying Sylvester's criterion for the positive definiteness of the matrix for $\tau_1 \geqslant 0$, $\tau_2 \geqslant 0$, $\theta_1 \geqslant 0$, $\theta_2 \geqslant 0$.

$$\tau_1\mu_1^{-1} + \operatorname{Re}[(\tau_1 + j\omega\theta_1)G_1(j\omega)] > 0 \tag{18}$$

$$\begin{aligned} &4\{\tau_1\mu_1^{-1} + \operatorname{Re}[(\tau_1 + j\omega\theta_1)G_1(j\omega)]\}\{\tau_2\mu_2^{-1} + \operatorname{Re}[(\tau_2 + j\omega\theta_2)G_2(j\omega)]\} \\ &\quad - \{\operatorname{Re}[(\tau_1 + j\omega\theta_1)G_1(j\omega)] + \operatorname{Re}[(\tau_2 + j\omega\theta_2)G_2(j\omega)]\}^2 \\ &\quad - \{\operatorname{Im}[(\tau_1 + j\omega\theta_1)G_1(j\omega)] - \operatorname{Im}[(\tau_2 + j\omega\theta_2)G_2(j\omega)]\}^2 > 0 \quad (-\infty \leqslant \omega \leqslant \infty). \end{aligned} \tag{19}$$

Stability analysis is divided into three steps to bring out clearly the algorithm involved in the analysis.

### 4.1. Step A

The first condition, Eq. (18), is nothing but Popov's stability condition for a single-input and single-output system, in our case the system consists of $\phi_1$ and $G_1$. As another necessary requirement in order to satisfy Eq. (19), the following condition is obtained:

$$\tau_2\mu_2^{-1} + \operatorname{Re}[(\tau_2 + j\omega\theta_2)G_2(j\omega)] > 0. \tag{20}$$

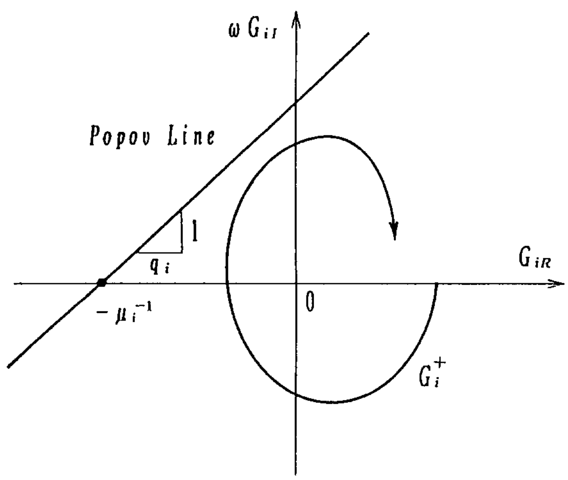


Fig. 4. Modified Nyquist trajectory $G_i^+$ and Popov straight line.

This is derived from the following two facts: Eq. (18): the fact that last two terms of inequality (19) are always negative. By dividing two conditions (18) and (20) by $\tau_1$ and $\tau_2$, respectively, the following conditions can be derived:

$$\mu_i^{-1} + \mathrm{Re}[(1 + j\omega q_i)G_i(j\omega)] > 0, \quad i = 1,2, \tag{21}$$

where $q_i = \theta_i/\tau_i$.

Parameter values $\mu_i^{-1}$ and $q_i$ that are obtained by drawing the modified Nyquist trajectories and the Popov straight lines for condition (21) as the critically stable values are defined as $\tilde{\mu}_i^{-1}$ and $\tilde{q}_i$, respectively. The modified Nyquist trajectory is called the Popov trajectory. Fig. 4 shows a typical example of the modified Nyquist trajectory $G_i^+$ and a Popov straight line. Characters $G_{i\mathrm{R}}$ and $G_{i\mathrm{I}}$ means a real part and an imaginary part of the $G_i$, respectively.

### 4.2. Step B

By dividing condition (19) by $\tau_2^2$, and defining $\tau_1/\tau_2 = \tau$, the following inequality can be obtained.

$$f(\tau) = A(\omega)\tau^2 + B(\omega)\tau + C(\omega) < 0, \tag{22}$$

where

$$A(\omega) = \mathrm{Re}^2[\Gamma_1(j\omega)] + \mathrm{Im}^2[\Gamma_1(j\omega)],$$

$$B(\omega) = 2\{\mathrm{Re}[\Gamma_1(j\omega)]\mathrm{Re}[\Gamma_2(j\omega)] - \mathrm{Im}[\Gamma_1(j\omega)]\mathrm{Im}[\Gamma_2(j\omega)]\}$$
$$- 4\{\mu_1^{-1} + \mathrm{Re}[\Gamma_1(j\omega)]\}\{\mu_2^{-1} + \mathrm{Re}[\Gamma_2(j\omega)]\},$$

$$C(\omega) = \mathrm{Re}^2[\Gamma_2(j\omega)] + \mathrm{Im}^2[\Gamma_2(j\omega)], \qquad \Gamma_1(j\omega) = (1 + j\omega q_1)G_1(j\omega), \qquad \Gamma_2(j\omega) = (1 + j\omega q_2)G_2(j\omega).$$

Let $F(\tau)$ be defined by putting $\mu_i^{-1} = \tilde{\mu}_i^{-1}$ and $q_i = \tilde{q}_i$ (for $i = 1,2$) in $f(\tau)$, where parameters $\tilde{\mu}_i^{-1}$ and $\tilde{q}_i$ are determined in Step A. Sufficient condition for satisfying condition (22) is that there must exist at least one $\tau$ such that $F(\tau)$ is negative for all positive $\omega$. If we assume all coefficients $A(\omega)$, $B(\omega)$, and $C(\omega)$ of $F(\tau)$ are bounded for $0 \leqslant \omega \leqslant \infty$, we can expect that the upper value of $F(\tau)$ is bounded for a finite value of $\tau$ so that we can find a curve enveloping all $F(\tau)$ curves in the $\tau$–$F(\tau)$ plane from the upside for $0 \leqslant \omega \leqslant \infty$. The envelope curve can be used to check whether or not Eq. (22) is satisfied: if there exists a negative region in the envelope, we guarantee Eq. (22) is satisfied; otherwise, Step C is applied.

### 4.3. Step C

Let $F_1(\tau)$ express $f(\tau)$ on selecting

$$\mu_i^{-1} = \tilde{\mu}_i^{-1} + k_i \quad (k_i > 0), \qquad q_i = \tilde{q}_i \quad (i = 1,2). \tag{23}$$

Then the following relation can be obtained.

$$F_1(\tau) = F(\tau) - 4\{k_2(\mu_1^{-1} + \mathrm{Re}[\Gamma_1]) + k_1(\mu_2^{-1} + \mathrm{Re}[\Gamma_2])\}\tau - 4k_1k_2\tau. \tag{24}$$

Let a value of the envelope curve obtained in Step B for an arbitrary $\tau = \hat{\tau}$ be $\varepsilon$ ( $> 0$). If a value of $k_1k_2$ is determined as shown in condition (25), then a parameter $\tau$ satisfying condition (22) exists without any failure, because the second term on the right-hand side of Eq. (24) is not negative due to conditions (18) and (20).

$$k_1k_2 > \varepsilon/4\hat{\tau}. \tag{25}$$

It is clear from condition (25) that a value of $\varepsilon/\hat{\tau}$ should be minimum for guaranteeing a wider stable region of the control system. The minimum value can be chosen if a straight line passing through the origin of this graph and contacting with the envelope curve is selected, a slope of which helps estimate the minimum value of $k_1 k_2$.

## 5. Illustrative analysis

Here stability analysis is concretely done using a simple example. A second order plant with delay is taken, transfer function of which is given by

$$G(s) = K\omega_n^2 e^{-Ls}/(s^2 + 2\xi\omega_n s + \omega_n^2), \tag{26}$$

where $K$ is a plant gain, $\omega_n$ is a natural frequency, $\xi$ is a damping factor, and $L$ is a dead time, respectively. In the present analysis, the values of these parameters were selected as $K = 2.5$, $\omega_n = 1.0$ rad/s, $\xi = 0.4$, and $L = 0.5$ s.

Analysis will be followed as stated in Section 4, Steps A–C.

### 5.1. Step A

By drawing the two modified Nyquist trajectories and Popov's lines, Figs. 5 and 6, the values of parameters $\tilde{\mu}_i^{-1}$ and $\tilde{q}_i$ for $i = 1, 2$ are obtained as follows:

$$\tilde{\mu}_1^{-1} = 1.420, \quad \tilde{q}_1 = 0.809, \quad \tilde{\mu}_2^{-1} = 0.728, \quad \tilde{q}_2 = 0.150. \tag{27}$$

### 5.2. Step B

Using the values of $\tilde{\mu}_i^{-1}$ and $\tilde{q}_i$ for $i = 1, 2$, an envelope curve is drawn as shown in Fig. 7.

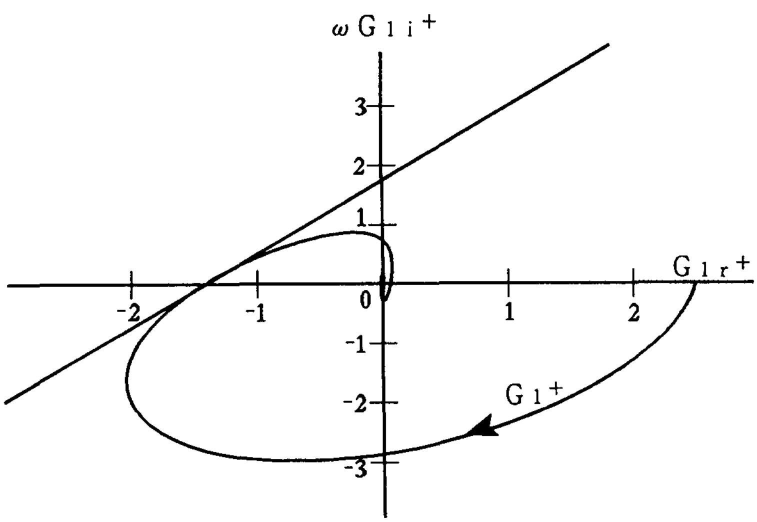


Fig. 5. Modified Nyquist trajectory and Popov straight line for $G_1(s) = G(s)$.

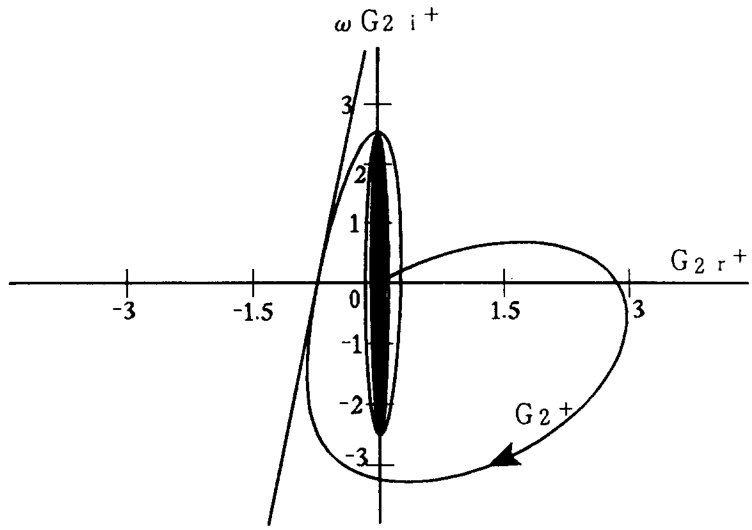


Fig. 6. Modified Nyquist trajectory and Popov straight line for $G_2(s) = sG(s)$.

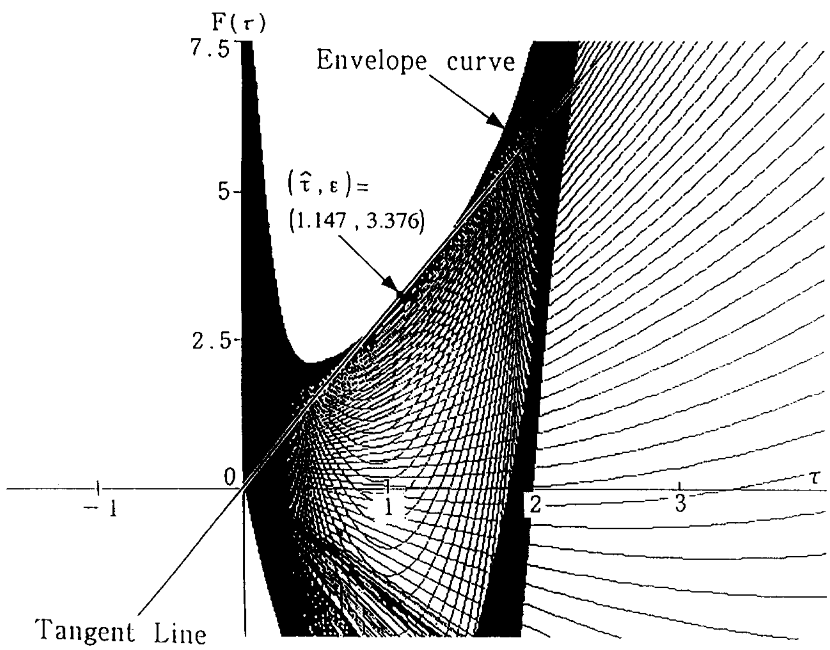


Fig. 7. Envelope curve.

### 5.3. Step C

Values of $\varepsilon$ and $\hat{\tau}$ are determined as 3.376 and 1.147, respectively, by drawing a line passing through the origin of the graph and contacting with the envelope curve, and the following condition is

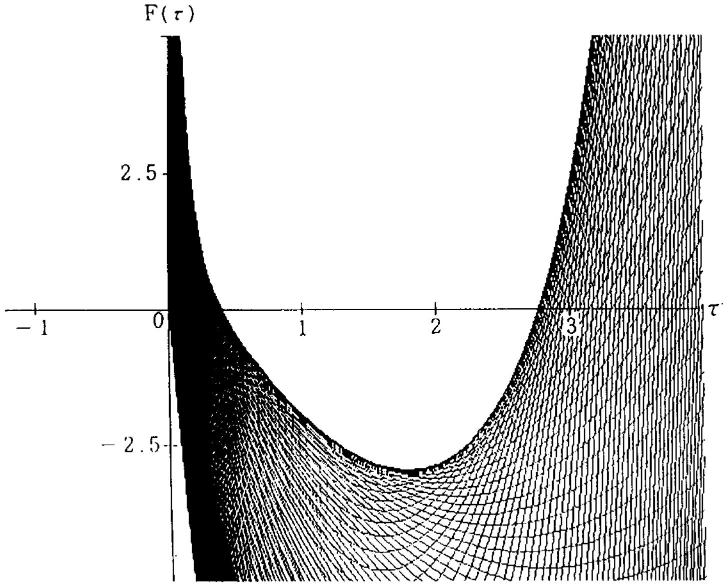


Fig. 8. Envelope curve obtained for $k_1 = k_2 = 0.857$.

obtained:

$$k_1 k_2 > 0.735. \tag{28}$$

Various sets of $\mu_1$ and $\mu_2$ guaranteeing the stability of the system are determined by using Eqs. (23), (27), and condition (28). For example, if the value of $k_1$ and $k_2$ is selected as $k_1 = k_2 = 0.8574$, the following conditions as for the values of parameter $\mu_1$ and $\mu_2$ can be derived:

$$\mu_1 < 0.439, \qquad \mu_2 < 0.630. \tag{29}$$

Critical values obtained in these conditions (29) are substituted into Eq. (22) to draw a new envelope curve as shown in Fig. 8. This figure shows that there exists $\tau$'s ($0.42 < \tau < 2.76$) such that condition (22) is satisfied.

However, a minimum value of the envelope curve in Fig. 8 is much less than 0, this fact implies a possibility of a better solution. A better solution can be obtained by reducing the values of $k_1$ and $k_2$ and repeatedly drawing envelope curves by trial and error until condition (22) is marginally satisfied. For example, if $k_1 = k_2$ is assumed, these values are obtained after several iteration as follows:

$$k_1 = k_2 > 0.630. \tag{30}$$

Finally, the following condition was obtained:

$$\mu_1 < 0.487, \qquad \mu_2 < 0.736. \tag{31}$$

It is clear that this condition is better than the condition shown in (29). The envelope curve drawn in the condition $k_1 = k_2 = 0.630$ is shown in Fig. 9. In this procedure, the values of parameters $\tilde{q}_1$ and $\tilde{q}_2$, which are obtained by Eq. (27) in Step A, are fixed.

It should be noted that the present method described in Section 4 can give the relation between the parameters $\mu_1$ and $\mu_2$, which correspond to the proportional gain and the differential time of the controller

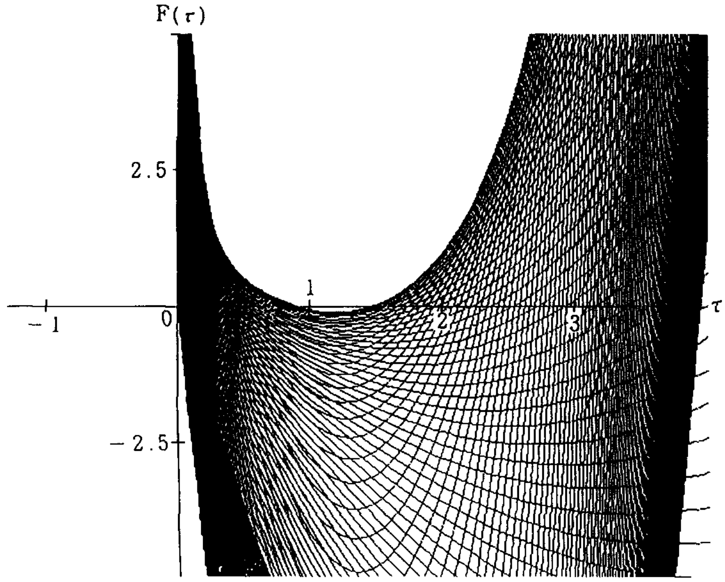


Fig. 9. Envelope curve obtained for $k_1 = k_2 = 0.630$.

respectively as shown in Eqs. (23) and (25), although the result obtained by this method is more conservative than that by a cut-and-try method.

## 6. Comparison of the results

Results obtained by the present method are compared with the results by other graphical methods, namely Shankar's method [10] and Kitamura's method [5], as well as simulation results. These other methods and a simulation method will be briefly explained here.

### *6.1. Shankar's method*

This method can analyze the stability of nonlinear MIMO (multiple–input and multiple–output) control system. The method can be applied for a PD type of fuzzy control system, because this system can be regarded as a two-inputs and two-outputs system. Stability conditions can be obtained as follows

$$\mu_1^{-1} + \mathrm{Re}[\Gamma_1(j\omega)] > \gamma/2, \tag{32}$$

$$\mu_2^{-1} + \mathrm{Re}[\Gamma_2(j\omega)] > \gamma/2, \tag{33}$$

where

$$\gamma^2 = \{\mathrm{Re}[(1 + j\omega\theta_1)G_1(j\omega)] + \mathrm{Re}[(1 + j\omega\theta_2)G_2(j\omega)]\}^2 + \{\mathrm{Im}[(1 + j\omega\theta_1)G_1(j\omega)] - \mathrm{Im}[(1 + j\omega\theta_2)G_2(j\omega)]\}^2.$$

Either of these conditions can be solved by drawing a modified Nyquist trajectory shifted by $\gamma/2$ and Popov's line. This method has two disadvantages: one of them is that it is difficult to obtain a solution

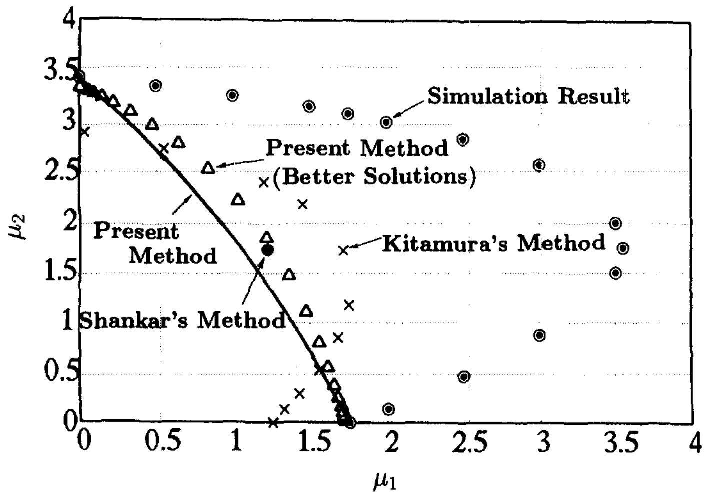


Fig. 10. Comparisons of the analysis results.

satisfying the above inequalities, because the value of $\gamma$ depends upon all the $\theta_i$ ($i = 1, 2$), where $\theta_i^{-1}$ means the slope of Popov's line; another is that the method gives only one set of $\mu_1$ and $\mu_2$.

### 6.2. Kitamura's method

This method is explained in connection with the present example. In his analysis, utilizing the relation

$$u = \phi(e_1, e_2) = K_c\{\tan^{-1}(c_1e_1) + \tan^{-1}(c_2e_2)\} \leqslant K_c(c_1e_1 + c_2e_2) \tag{34}$$

and introducing a new variable $\sigma = (c_1e_1 + c_2e_2)/(c_1^2 + c_2^2)^{1/2}$, an extended sector condition is obtained as follows:

$$0 \leqslant \phi(e_1, e_2)/\sigma \leqslant k, \tag{35}$$

where $k = K_c(c_1^2 + c_2^2)^{1/2}$. Then, the controller nonlinearity can be characterized by a sector $[0, k]$.

The stability condition, called extended circle criterion, is given by

$$k_2^{-1} + \mathrm{Re}[(c_1 + j\omega c_2)G_1(j\omega)] > 0. \tag{36}$$

If the values $c_1$ and $c_2$ corresponding to the input membership functions, as explained in Eq. (9), are given, the value of $k$ can be estimated by drawing the Nyquist trajectory of $(c_1 + j\omega c_2)G_1(j\omega)$ and Popov's line having an infinite slope. Parameters $\mu_1$ and $\mu_2$ can be related to parameters $K_c$ and $c_i$ ($i = 1, 2$) as follows:

$$\mu_1 = K_c c_1 = kc_1(c_1^2 + c_2^2)^{-1/2}, \tag{37}$$

$$\mu_2 = K_c c_2 = kc_2(c_1^2 + c_2^2)^{-1/2}. \tag{38}$$

### 6.3. Simulation method

Actual values of $\mu_1$ and $\mu_2$ for the stability of fuzzy control system are estimated through computer simulations. For the given values of $c_1$ and $c_2$, controller gain $K_c$ is established by carrying out numerical simulations applying the Runge–Kutta method. Values of $\mu_1$ and $\mu_2$ are estimated according to Eqs. (37) and (38).

In Fig. 10 the boundaries of the stability regions of the system guaranteed by five methods are shown in the $\mu_1$–$\mu_2$ plane. A solid line means the boundary by the present method. Symbols △, ●, × and ◉ mean the results by the present better solution, Shankar's method, Kitamura's method and computer simulations, respectively. The following is suggested:

(1) All methods except the present method can give only one set of $\mu_1$ and $\mu_2$, namely one point in the $\mu_1$–$\mu_2$ plane, in one trial of each analysis. Especially, Shankar's method can give only one solution as described above.

(2) All results are conservative compared with the result by computer simulations.

(3) The present method and Kitamura's method are complementary to each other. The former gives better results than the latter when either of two control modes, P and D, is more active than another, and the situation is reverse when both control modes are active. This fact was verified in other examples which were examined.

## 7. Conclusion

Absolute stability conditions were derived for the control system having a PD type of fuzzy controller using Yakubovich's method. The conditions were represented by three conditions; two are conditions for either one of two control modes, P or D, is independently operating; the third is the condition for two modes are simultaneously active. The former two conditions are nothing but Popov's conditions for two SISO systems. A new graphical method was proposed to solve the latter condition. Present analysis method was applied to the second order oscillatory plant with dead time to estimate the upper bounds of the sector parameters, and the better solutions were also searched through trial and error in order to bring out steps involved in the stability analysis of the two-input and two-output control system. Comparisons of the results were also carried out with other graphical methods as well as computer simulation. The present method has the advantage in the meaning that the result obtained by this method can give the stability region in a controller's gain parameter plane (the $\mu_1$–$\mu_2$ plane), while other methods can give the only one point in the plane by one trial of each analysis.

In this paper, the assumption that all coefficients $A(\omega)$, $B(\omega)$, and $C(\omega)$ of $F(\tau)$ are bounded for $0 \leqslant \omega \leqslant \infty$ was used, as shown in Section 4. The condition that this assumption can be held must be mathematically examined in more detail to extend the applicable field of this stability analysis method, for example, a control system having a PID type of fuzzy controller.

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## Appendix

Values of parameters $\tau_j$ and $\theta_h$ must be determined by the following rules according to Yakubovich's method.

*Selection of values of parameters $\tau_j$*

Admissible values of parameters $\tau_j$ are determined as follows:

$$\tau_j\text{: arbitrary} \quad \text{for } j = 1, \ldots, k_1, \tag{A.1a}$$

$$\tau_j \geqslant 0 \quad \text{for } j = k_1 + 1, \ldots, k_1 + k_2 = k. \tag{A.1b}$$

*Selection of values of parameters $\theta_h$*

Parameter $\theta_h$ is selected as follows:

$$\theta_h \geqslant 0 \quad \text{if } \phi_h(\sigma_{qh}) \text{ satisfies the positivity condition,} \tag{A.2a}$$

$$\theta_h \leqslant 0 \quad \text{if } -\phi_h(\sigma_{qh}) \text{ satisfies the positivity condition,} \tag{A.2b}$$

$$\theta_h = 0 \quad \text{if neither } \phi_h \text{ nor } -\phi_h \text{ satisfies the positivity condition,} \tag{A.2c}$$

where it is said that a nonlinearity $\phi_j = \phi_j(\sigma_h)_t$ satisfies a positivity condition with respect to the argument $\sigma_h$, if for any initial value $\phi_j|_{t=0}$ and for other values of parameters or functions on which $\phi_j$ possibly depends, we can find a constant $\delta > 0$ such that

$$\int_0^t \phi_j \,\mathrm{d}\sigma_h \geqslant -\delta, \quad \text{when } t \geqslant 0. \tag{A.3}$$

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