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A NOTE ON THE DETERMINATION OF THE THERMAL PROPERTIES OF A MATERIAL IN A TRANSIENT NONLINEAR HEAT CONDUCTION PROBLEM

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ABSTRACT

The determination of the temperature dependent thermal properties and the temperature distribution inside a heat conducting material when heat flux boundary conditions are prescribed is investigated. Assuming that the material has a known constant thermal diffusivity then the heat conduction problem is linearised by employing the Kirchhoff transformation and additional measurements of the temperature at an arbitrary space location are imposed in order to render a unique solution. The dependence of the thermal conductivity with the temperature is obtained as the sum of an infinite series, whilst the temperature solution is obtained implicitly and is then calculated numerically. The characteristics of the solutions with respect to the spatial position where the sensor is located is also discussed.

Introduction

Frequently in practice we are required to solve the problem in which a finite slab of material of thickness L is initially at a constant temperature T_0 and for time $t > 0$, the boundary $x = 0$ is kept insulated, whilst the boundary $x = L$ is subject to a prescribed heat flux $q(t)$, see Özişik [1]. Many materials have the property that the thermal conductivity, $k(T)$, and the heat capacity, $C(T)$, are dependent on the temperature, T, but they are directly proportional, i.e. the thermal diffusivity, $a = k(T)/C(T)$, is a known positive constant, see for example Hills and Hensel [2] and Wrobel and Brebbia [3]. The mathematical formulation of this one-dimensional transient nonlinear heat conduction problem may be written in the form

$$k(T)\,\frac{\partial T}{\partial t} = a\,\frac{\partial}{\partial x}\left(k(T)\,\frac{\partial T}{\partial x} \right) \quad \text{in} \quad (0,L)\times(0,\infty) \tag{1a}$$

$$\frac{\partial T}{\partial n} = 0 \quad \text{on} \quad x = 0\ , \quad t \in (0,\infty) \tag{1b}$$

$$k(T)\,\frac{\partial T}{\partial n} = q \quad \text{on} \quad x = L\ , \quad t \in (0,\infty) \tag{1c}$$

$$T = T_0 \quad \text{at} \quad t = 0\ , \quad x \in [0,L] \tag{1d}$$

where n is the outward normal on the boundaries x = 0 and x = L and, for simplicity, it was assumed that there is no heat generation inside the domain. Without any reduction in the generality of the problem we can assume that $T_0 = 0$, since otherwise, for any constant T_0, we can always treat an analogous problem in terms of the shifted temperature $(T-T_0)$. In addition, physical constraints require that the unknown thermal conductivity, k(T), and the temperature, T, be positive. Clearly, in problem (1) there is insufficient information to be able to determine uniquely the unknown continuous thermal conductivity function, k, which is a positive function of a single variable, and the unknown temperature function, T, which is a positive function of the space and time variables and has continuous derivatives. The mathematical problem of determining k(T) and T as formulated in problem (1) is inverse and ill-posed since it will not, in general, have a unique solution. Consequently, additional constraints on the temperature are required. In this note we assume that the temperature is measured at an arbitrary location $x = x_0 \in [0,L]$, namely

$$T(x,t) = f(t) \quad \text{at} \quad x = x_0\ , \quad t \in (0,\infty) \tag{2}$$

Under further assumptions on the functions q(t) and f(t) the mathematical problem formulated in the set of equations (1) and (2) becomes well-posed, see Cannon and Duchateau [4], i.e. the existence, the uniqueness and the continuous dependence upon the data are satisfied.

Mathematical Analysis

Prior to this investigation Cannon and Duchateau [4], Cannon [5,6] obtained formal solutions for the transient nonlinear and linear and steady nonlinear heat conduction problems, respectively. This note applies to some of these analyses and to the particular case of practical interest expressed by equations (1) and (2).

In order to solve equations (1) and (2) we linearise the nonlinear partial differential heat conduction equation (1a) using the Kirchhoff transformation, namely

$$\Psi = \int_0^T k(\xi)\, d\xi \qquad (3)$$

Under the transformation (3), equations (1) and (2) reformulate as

$$\frac{\partial \psi}{\partial t} = a\,\frac{\partial^2 \psi}{\partial x^2} \quad \text{in} \quad (0,L)\times(0,\infty) \qquad (4a)$$

$$\frac{\partial \psi}{\partial n} = 0 \quad \text{on} \quad x = 0\ , \quad t \in (0,\infty) \qquad (4b)$$

$$\frac{\partial \psi}{\partial n} = q \quad \text{on} \quad x = L\ , \quad t \in (0,\infty) \qquad (4c)$$

$$\psi = 0 \quad \text{at} \quad t = 0\ , \quad x \in [0,L] \qquad (4d)$$

$$\psi(x,t) = \Psi(T(x,t)) \quad , \quad (x,t) \in [0,L]\times[0,\infty) \qquad (5)$$

Now the problem formulated in equations (4) is well-posed and involves only the unknown transformed temperature ψ, given by expression (5). In addition, the governing partial differential heat conduction equation (4a) is linear and when solved subject to the boundary and initial conditions (4b), (4c) and (4d), posseses the analytical solution, see Hills and Hensel [2],

$$\psi(x,t) = \frac{(3x^2-L^2)}{6L}\, q(t) - q(t)\,\frac{2}{L}\sum_{n=1}^{\infty} \frac{(-1)^n}{\lambda_n^2}\cos(\lambda_n x) +$$

$$\frac{1}{L}\int_0^t q(\tau)\, d\tau + \frac{2}{L}\sum_{n=1}^{\infty} (-1)^n \cos(\lambda_n x) \left(\int_0^t q(\tau)\, \exp(-\lambda_n^2(t-\tau))\, d\tau \right) \qquad (6)$$

where $(x,t) \in [0,L]\times[0,\infty)$ and

$$\lambda_n = n\pi/L \quad , \quad n = 1,2,3,\ldots \qquad (7)$$

and, for simplicity, the constant thermal diffusivity a has been taken to be unity. Once the solution ψ, as given by expression (6), is obtained the use of equations (2), (3) and (5) results in

$$\psi(x_0,t) = \int_0^{f(t)} k(\xi)\, d\xi \quad , \quad t \in (0,\infty) \qquad (8)$$

By differentiating expression (8) with respect to t, it follows that

$$k(f(t)) = \frac{\partial\psi}{\partial t}(x_0,t) \;/\; f'(t) \quad , \quad t \in (0,\infty) \tag{9}$$

whenever $f'(t) \neq 0$. Further, if the function $f(t)$ is invertible then, from expression (9), it follows that

$$k(t) = \frac{\partial\psi}{\partial t}(x_0,f^{-1}(t))\;(f^{-1})'(t) \quad , \quad t \in (0,\infty) \tag{10}$$

Once the thermal conductivity function k, as given by expression (10) is obtained then the Kirchhoff transformed function Ψ is determined by integrating equation (3), namely

$$\Psi(T) = \int_0^T \frac{\partial\psi}{\partial t}(x_0,f^{-1}(\xi))\;(f^{-1})'(\xi)\;d\xi \tag{11}$$

Further, if the function $\Psi(T)$ is invertible then, from expression (5), it follows that

$$T(x,t) = \Psi^{-1}(\psi(x,t)) \quad , \quad (x,t)\;[0,L]\times[0,\infty) \tag{12}$$

where the function Ψ is given by expression (11).

Results

In [4], the equations corresponding to expression (10) represented the conclusion of the above method without any application to a specific situation. To relate and understand how the technique can be applied in a particular problem the following case has been investigated, noting that the final inversion must be undertaken numerically. Consider the theoretical problem of determining the thermal conductivity $k(T)$ and the temperature T from the data:

$$k(T)\,\frac{\partial T}{\partial n} = q(t) = 1 \quad \text{on } x = L \;, \quad t \in (0,\infty) \tag{13}$$

$$T(x,t) = f(t) = e^t - 1 \quad , \quad \text{at } x = x_0 \;, \quad t \in (0,\infty) \tag{14}$$

Based on expression (13) the analytical solution given by expression (6) for the transformed temperature ψ results, after some algebra, in

$$\psi(x,t) = \frac{3x^2-L^2}{6L} + \frac{t}{L} - \frac{2}{L}\sum_{n=1}^{\infty} \frac{(-1)^n}{\lambda_n^2}\cos(\lambda_n x)\exp(-\lambda_n^2 t) \;, \quad (x,t)\in[0,L]\times[0,\infty) \tag{15}$$

Introducing expressions (14) and (15) in expression (10) produces, after some calculations, the analytical solution for the thermal conductivity in an explicit form, namely

$$k(t) = \frac{1}{L(1+t)} + \frac{2}{L}\sum_{n=1}^{\infty} (-1)^n \cos(\lambda_n x_0) \left(1 + t \right)^{-1-\lambda_n^2} , \quad t \in (0,\infty) \qquad (16)$$

Figure 1 shows the thermal conductivity k(T) as a function of temperature T for various space locations $x_0 \in \{0, 0.5, 1\}$ where, for simplicity, the thickness of the slab, L, has been taken to be unity. The number of terms in the series expansion (16) was taken to be 10 which was found to be sufficiently large such that any further increase in the number of terms would not affect the accuracy of any of the results presented in this note. From this figure it can be seen that the thermal conductivity k(T) is strongly dependent on x_0 in the region $0 < T < 0.5$, whilst in the region $T \geq 0.5$ the function k(T) has the same behaviour for any values of x_0.

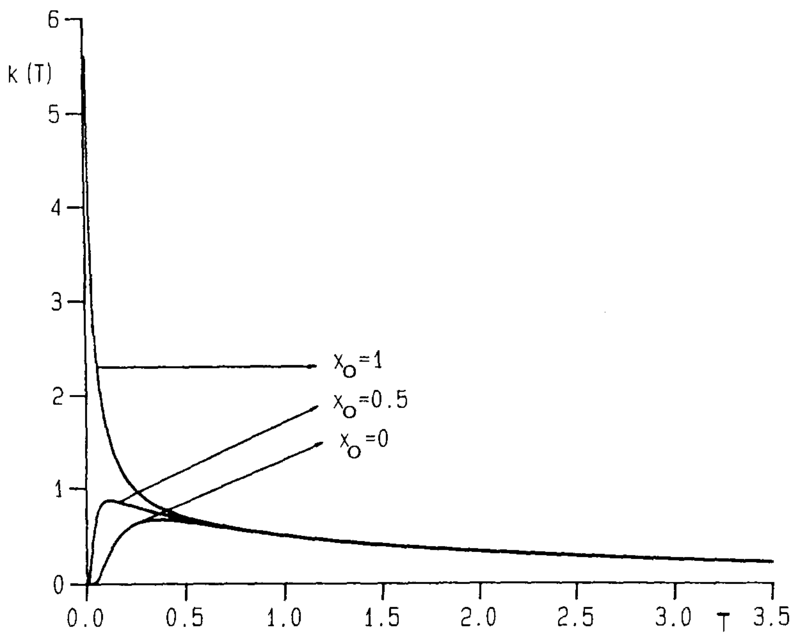

FIG. 1

The thermal conductivity k(T) as a function of the temperature T for various space locations $x_0 \in \{0, 0.5, 1\}$.

Also, it can be observed that, in general, the thermal conductivity has a temperature point of maximum, T_{max}, which decreases as x_0 increases, corresponding to a maximum value $k_{max} = k(T_{max})$ which increases as x_0 increases. Finally, as the point x_0 approaches the boundary on which the heat flux q(t) is applied the function k(T) becomes unbounded near the points of zero temperature.

Introducing expressions (14) and (16) in expression (11) results, after some algebra, in

$$\Psi(T) = \frac{1}{L}\ln(1+T) + \frac{3x_0^2 - L^2}{6L} - \frac{2}{L}\sum_{n=1}^{\infty}(-1)^n\cos(\lambda_n x_0)\,\lambda_n^{-2}\left(1+T\right)^{-\lambda_n^2} \qquad (17)$$

Based on expression (12), accompanied by the inversion of equation (17), the temperature T can be determined. However, the inversion of equation (17) seems analytically impossible and therefore a numerical scheme implemented in the NAG routine C05NCF, which numerically solves a system of nonlinear

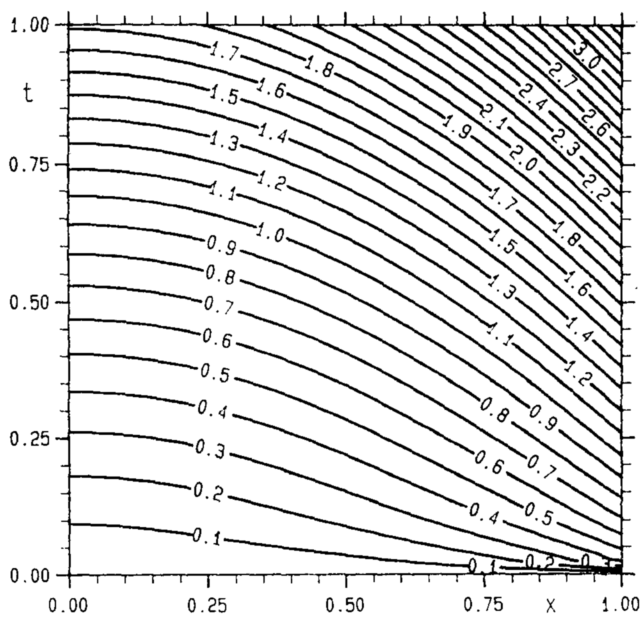

FIG. 2a

Lines of constant temperature for $x_0 = 0$.

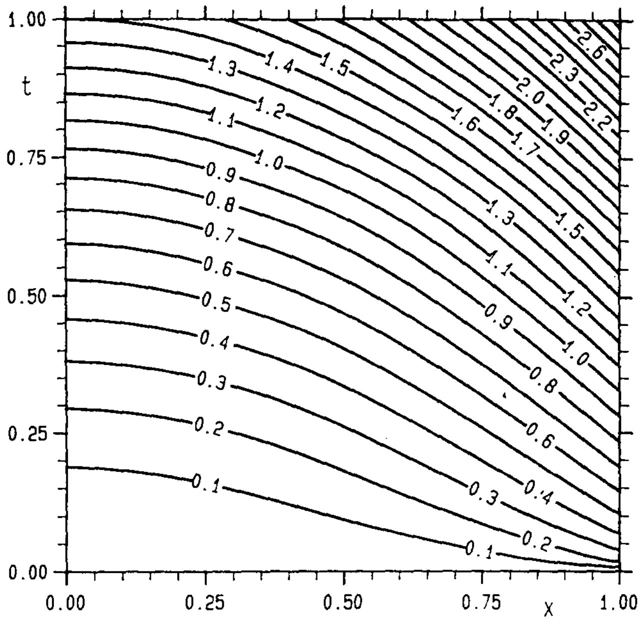

FIG. 2b

Lines of constant temperature for $x_o = 0.5$.

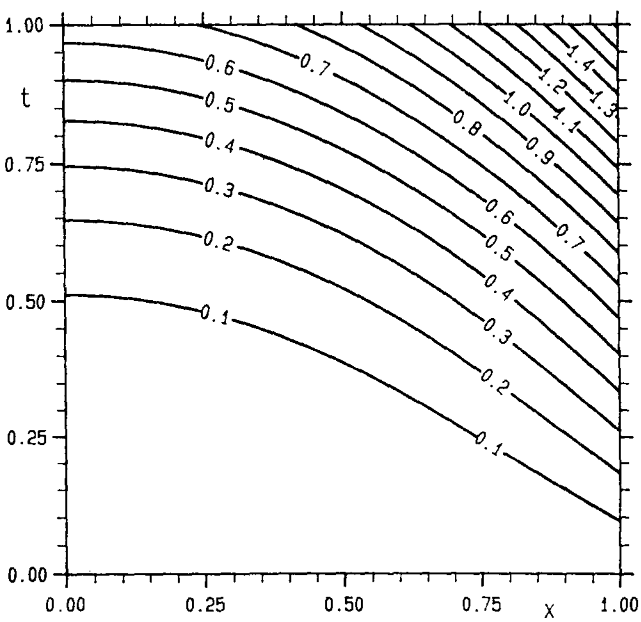

FIG. 2c

Lines of constant temperature for $x_o = 1$.

equations by a modification of the Powell hybrid method, see Powell [7], has been used. The results for the lines of constant temperature, $T(x,t)$ = constant, i.e. the isotherms, are shown in figures 2 for various space locations $x_0 \in \{0, 0.5, 1\}$. From these figures it can be seen that as x_0 increases the time taken for the temperature to reach a certain value increases and, also the temperature at a specified position and time decreases. Finally, for any value of x_0, the temperature on the boundaries of the slab is monotonic increasing with time, with higher temperature values on the boundary which is kept insulted, $x = 0$.

Conclusion

In conclusion, an inverse heat conduction exact solution has been developed for determining the temperature dependent thermal conductivity and the temperature in a slab geometry with prescribed heat flux boundary conditions. The material considered is assumed to have constant thermal diffusivity and the use of the Kirchhoff transformation reduces the nonlinear heat conduction problem to a linear form which posseses an analytical solution. Further, additional time measurements of the temperature at an arbitrary space location are imposed in order to render a unique solution and then the dependence of the thermal conductivity with temperature is obtained in an explicit form as the series (16), whilst the temperature solution is obtained implicitly and is calculated numerically by inverting equation (17). Finally, figures 1 and 2 may serve as an optimal criterion to decide where the sensor recording temperature measurements is to be located such that specific practical requirements are satisfied.

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