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Analytical solution to the problem of heat interaction of fluid flow and solid filling

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Abstract The results of analytical solution to the problem of thermal interaction of a fluid stream with a media of solid particles found on the basis of the ODTPM model [1] are presented. A non-stationary behaviour of the volume-averaged temperature of solid filling, being a measure of the efficiency from heat transfer between fluid and solid phases, and of the heat-carrier temperature at the exit of the layer is studied. Time dependences computed according to the obtained relationships are in good agreement with the results of numerical calculations [1] for different sets of the problem parameters. The found solution expands the possibilities to analyze the efficiency of the "fluid-solid phase" systems and can be a basis to develop methods of experimental evaluation of different parameters of the system.

Analytische Lösung für das Problem der thermischen Wechselwirkung zwischen einer Fluidströmung und darin enthaltenen Feststoffpartikeln

Zusammenfassung Auf analytischem Wege, unter Zugrundelegung des ODTPM-Modells, gefundene Ergebnisse für das Problem der thermischen Wechselwirkung zwischen einer Fluidströmung und darin enthaltenen Feststoffpartikeln werden mitgeteilt. Die Untersuchung bezieht sich auf das nichtstationäre Verhalten der volumengemittelten Feststofftemperatur – welche ein Maß für die Effektivität des Wärmeübergangs zwischen fluider und fester Phase ist – und das der Wärmeträgertemperatur am Austritt der Schicht. Diese, aus den analytisch gefundenen Beziehungen für verschiedene Kombinationen der Systemparameter errechneten Zeitabhängigkeiten stimmen gut mit den rein numerisch eruierten Ergebnissen nach [1] überein. Die mitgeteilte Lösung erweitert die Möglichkeit zur genaueren Untersuchung des Zweiphasensystems Fluid/Feststoff und kann ferner bei der Entwicklung von Methoden zur experimentellen Bestimmung von Systemparametern dienlich sein.

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Nomenclature

ϑ	dimensionless temperature
τ	dimensionless time
x	dimensionless co-ordinate
$\alpha_{1,2,3,4}$	dimensionless parameters of the problem [1]

Subscripts

f	fluid (heat carrier)
p	solid phase
b	medium quantity
0	initial values ($\tau=0$)
l	values at the entrance of the layer ($x=0$)
k	values at the exit from the layer ($x=1$)
∞	stationary conditions ($\tau\to\infty$)

1 Introduction

Various mathematical models are proposed to simulate the interaction between a fluid stream and particles of a solid filling including the case of quasi-liquefaction. In particular a one-dimensional two-phase model (ODTPM) is developed in Ref. [1] taking into account the main features of the physical process. This includes equations to the energy transfer in the fluid and solid phases, reduced to the following dimensionless form

$$\frac{\partial \vartheta_f}{\partial \tau}+\alpha_1(\vartheta_f-\vartheta_p)+\alpha_2\vartheta_f+\alpha_5\frac{\partial \vartheta_f}{\partial x}=0 \tag{1}$$

$$\frac{\partial \vartheta_p}{\partial \tau}-\alpha_3(\vartheta_f-\vartheta_p)-\alpha_4\frac{\partial^2 \vartheta_p}{\partial x^2}=0, \tag{2}$$

supplemented by initial and boundary conditions:

$$\vartheta_f(x,\tau=0)=\vartheta_{f0}, \qquad \vartheta_f(x=0,\tau)=\vartheta_{fl} \tag{3}$$

$$\vartheta_p(x,\tau=0)=\vartheta_{p0}, \qquad \frac{\partial \vartheta_p(x=0,\tau)}{\partial x}=\frac{\partial \vartheta_p(x=1,\tau)}{\partial \tau}=0 \tag{4}$$

Equations (1)–(4) have been integrated numerically in Ref. [1] by the method of finite differences. The time-spatial distributions of the temperatures, $\vartheta_f(x,\tau)$ and $\vartheta_p(x,\tau)$ calculated, were used to compute the integral average temperature of the solid phase

$$\vartheta_{pb}(\tau)=\int_0^1 \vartheta_p(x,\tau)\,\mathrm{d}x,$$

being the measure of the thermal interaction between solid particles and the fluid.

In the present paper exact relationships for $\vartheta_{pb}(\tau)$ and the fluid temperature $\vartheta_{fk}(\tau)=\vartheta_f(1,\tau)$ at the layer exit are obtained on the basis of an analytical solution to Eqs. (1)–(4); both $\vartheta_{pb}(\tau)$ and $\vartheta_{fk}(\tau)$ are of practical interest.

2 Solution

2.1 Determination of temperatures $\vartheta_{pb}(\tau)$ and $\vartheta_{fk}(\tau)$

Integrating Eqs. (1) and (2) with respect to x over the interval $0 \le x \le 1$ and taking into account boundary conditions (3), (4) one obtains

$$\frac{d\vartheta_{fb}}{d\tau}+\alpha_1(\vartheta_{fb}-\vartheta_{pb})+\alpha_2\vartheta_{fb}+\alpha_5(\vartheta_{fk}-\vartheta_{fb})=0 \tag{5}$$

$$\frac{d\vartheta_{pb}}{d\tau}-\alpha_3(\vartheta_{fb}-\vartheta_{pb})=0 \tag{6}$$

with

$$\vartheta_{fb}(\tau)=\int_0^1 \vartheta_f(x,\tau)\,dx$$

There are three unknown variables ϑ_{pb}, ϑ_{fb} and ϑ_{fk} in (5), (6) and they are not a closed system of equations. To exclude one of the variables we introduce the function $\varphi(x)$ satisfying the conditions $\varphi(0)=0$, $\varphi(1)=1$ and

$$\vartheta_f=\vartheta_{fl}-(\vartheta_{fl}-\vartheta_{fk})\varphi(x) \tag{7}$$

Taking (7) into account one has for the averaged temperature

$$\vartheta_{fb}=\vartheta_{fl}-(\vartheta_{fl}-\vartheta_{fk})m \tag{8}$$

with $m=\int_0^1\varphi(x)\,dx$ being a certain number from the range $0<m<1$.

Substitution of ϑ_{fb} according to (8) into Eqs. (5) and (6) results in

$$\frac{d\vartheta_{fk}}{d\tau}=a_{11}\vartheta_{fk}+a_{12}\vartheta_{pb}+f_1(\tau) \tag{9}$$

$$\frac{d\vartheta_{pb}}{d\tau}=a_{21}\vartheta_{fk}+a_{22}\vartheta_{pb}+f_2(\tau) \tag{10}$$

where $a_{11}=-(\alpha_1+\alpha_2+\alpha_5/m)$, $a_{12}=\alpha_1/m$, $a_{21}=\alpha_3 m$, $a_{22}=-\alpha_3$, $f_1=[(m-1)\,d\vartheta_{fl}/d\tau+(m-1)\vartheta_{fl}(\alpha_1+\alpha_2)+\alpha_5\vartheta_{fl}]/m$, $f_2=\alpha_3(1-m)\vartheta_{fl}$.

Equations (9) and (10) to functions ϑ_{pb} and ϑ_{fk} can be solved analytically by e.g. the method of d'Alembert (Ref. [2]):

$$\vartheta_{fk}+k_1\vartheta_{pb}=[\vartheta_{fk0}+k_1\vartheta_{pb0}+I_1(\tau)]\exp[(a_{11}+k_1a_{21})\tau] \tag{11}$$

$$\vartheta_{fk}+k_2\vartheta_{pb}=[\vartheta_{fk0}+k_2\vartheta_{pb0}+I_2(\tau)]\exp[(a_{11}+k_2a_{21})\tau] \tag{12}$$

Here ϑ_{fk0} and ϑ_{pb0} are the initial magnitudes of the temperatures of the fluid and the solid filling; $k_{1,2}$ are the roots of the characteristic equation

$$k_{1,2}=-\frac{a_{11}-a_{22}}{2a_{21}}\pm\sqrt{\left(\frac{a_{11}-a_{22}}{2a_{21}}\right)^2+\frac{a_{12}}{a_{21}}} \tag{13}$$

$$I_{1,2}(\tau)=\int_0^\tau[f_1(\tau_*)+k_{1,2}f_2(\tau_*)]\exp[-(a_{11}+k_{1,2}a_{21})\tau_*]\,d\tau_* \tag{14}$$

For the typical particular case of zero boundary conditions and a stationary temperature of the heat carrier ($\vartheta_{fl}=1$, $d\vartheta_{fl}/d\tau=0$; $f_1(\tau)=\text{const}\equiv f_1$, $f_2(\tau)=\text{const}\equiv f_2$) Eqs. (11)–(14) are of the form

$$\vartheta_{pb}=\frac{b_1(e^{c_1\tau}-1)-b_2(e^{c_2\tau}-1)}{k_1-k_2} \tag{15}$$

$$\vartheta_{fk}=\frac{k_1b_2(e^{c_2\tau}-1)-k_2b_1(e^{c_1\tau}-1)}{k_1-k_2} \tag{16}$$

where $b_{1,2}=(f_1+k_{1,2}f_2)/c_{1,2}$ and $c_{1,2}=a_{11}+k_{1,2}a_{21}$.

Thus the problem formulated has an exact analytical solution if the value m is known. Let us discuss a possibility to determine it.

2.2 Determination of the value m

If the temperature ϑ_{fl} of the heat carrier at the entrance of the layer is time independent or has a finite limit i.e.

$$\lim_{\tau\to\infty}\vartheta_{fl}(\tau)=\text{const}=1,$$

there is a stationary heat state of the system liquid–solid phase, governed by the Eqs. (1), (2) with zero time derivatives

$$\frac{d\vartheta_{f\infty}}{dx}+(\beta_1+\beta_2)\vartheta_{f\infty}-\beta_1\vartheta_{p\infty}=0 \tag{17}$$

$$\frac{d^2\vartheta_{p\infty}}{dx^2}+\beta_3(\vartheta_{f\infty}-\vartheta_{p\infty})=0 \tag{18}$$

where $\beta_1=\alpha_1/\alpha_5$, $\beta_2=\alpha_2/\alpha_5$ and $\beta_3=\alpha_3/\alpha_4$.

Let us average Eqs. (17) and (18) over x. According to the condition of thermal isolation of the whole system, (4), averaging of Eq. (18) results in equality of the averaged temperatures of the liquid and solid phases: $\vartheta_{fb\infty}=\vartheta_{pb\infty}$. Taking this and boundary condition (3) into account one obtains from integration of Eq. (17):

$$\vartheta_{fk\infty}=1-\beta_2\vartheta_{pb\infty} \tag{19}$$

On the other hand (8) holds also in a steady state. In this case it looks as follows:

$$\vartheta_{fb\infty}=1-(1-\vartheta_{fk\infty})m. \tag{20}$$

Resolving the last equation respect to m and taking (19) into account one obtains

$$m=\frac{1}{\beta_2}\left(\frac{1}{\vartheta_{pb\infty}}-1\right) \tag{21}$$

Thus for determination of the parameter m it is necessary to know the steady-state temperature $\vartheta_{pb\infty}$ i.e. the solution of Eqs. (17), (18).

2.3 Solution to the steady-state problem

Performing the Laplace transformation of Eqs. (17), (18) (Ref. [3]) we have

$$s\bar{\vartheta}_f - 1 + (\beta_1 + \beta_2)\,\bar{\vartheta}_f + \beta_1\,\bar{\vartheta}_p = 0 \tag{22}$$

$$s^2\,\bar{\vartheta}_p - s\vartheta_{pl} + \beta_3(\bar{\vartheta}_f - \bar{\vartheta}_p) = 0 \tag{23}$$

Here ϑ_f and ϑ_p are Laplace images of functions $\vartheta_f(x)$ and $\vartheta_p(x)$ (the subscript ∞ is omitted to simplify notations), s is the parameter of the Laplace transformation, ϑ_{pl} is the mean temperature of the solid phase at $x=0$. Resolving Eqs. (22) and (23) we obtain:

$$\bar{\vartheta}_p = \frac{s^2\,\vartheta_{pl} + s\vartheta_{pl}(\beta_1+\beta_2) - \beta_3}{s^3 + s^2(\beta_1+\beta_2) - (s+\beta_2)\,\beta_3} \tag{24}$$

$$\bar{\vartheta}_f = \frac{1+\beta_1\,\bar{\vartheta}_p}{s+\beta_1+\beta_2} \tag{25}$$

The image $\vartheta_p(s)$ can be reduced to a tabulated one if the denominator of (24) is reduced to the form

$$\psi(s) = (s-s_1)\,(s-s_2)\,(s-s_3) \tag{26}$$

where $s_{1,2,3}$ are the roots of the characteristic equation

$$\psi(s) = 0. \tag{27}$$

This is a cubic equation and Kardano's formulas or a certain numerical procedure (Ref. [2]) can be used to solve it.

Inverse transformation to the original $\vartheta_p(x)$ results in

$$\vartheta_p(x) = \vartheta_{pl} - \beta_3(1-\vartheta_{pl})\sum_1^3 \frac{e^{s_i x}}{(s_i-s_p)\,(s_i-s_p)} + \beta_3\beta_2\,\vartheta_{pl}\sum_1^3 \frac{e^{s_i x}-1}{s_i(s_i-s_p)\,(s_i-s_q)} \tag{28}$$

where the subscripts p and q are related to the term number i as follows:

$$p = 0.5i^2 - 2.5i + 4, \qquad q = -0.5i^2 + 1.5i + 2 \tag{29}$$

Performing integration of (28) over x from 0 to 1, one finds ϑ_{pb} needed to calculate the value m according to (21):

$$\vartheta_{pb}(x) = \vartheta_{pl} - \beta_3(1-\vartheta_{pl})\sum_1^3 \frac{e^{s_i}-1}{s_i\,(s_i-s_p)\,(s_i-s_q)} + \beta_3\beta_2\,\vartheta_{pl}\sum_1^3 \frac{e^{s_i}-1-s_i}{s_i^2\,(s_i-s_p)\,(s_i-s_q)} \tag{30}$$

Formulas (28) and (30) can be used immediately if all roots of Eq. (27) are real. As an analysis shows in certain cases it has only one real root $s_1 = a$ and two complex ones $s_{2,3} = b \pm j\cdot c$ with $j = (-1)^{1/2}$. In such a case one has

$$\psi(s) = (s-a)\,[\,(s-b)^2 + c^2], \tag{31}$$

and the original $\vartheta_p(x)$ is as follows:

$$\begin{aligned}\vartheta_p(x) = {} & \vartheta_{pl} - \frac{\beta_3(1-\vartheta_{pl})}{(a-b)^2+c^2}\left(e^{ax} + e^{bx}\cos cx + \frac{b-a}{c}\,e^{bx}\sin cx\right)\\ & + \frac{\beta_3\beta_2\,\vartheta_{pl}}{(a-b)^2+c^2}\left[-\frac{(a-b)^2+c^2}{a\,(b^2+c^2)} + \frac{e^{ax}}{a} - \frac{2b-a}{b^2+c^2}\right.\\ & \left.\times\, e^{bx}\cos cx + \frac{b(b-a)-c^2}{c(b^2+c^2)}\,e^{bx}\sin cx\right].\end{aligned} \tag{32}$$

Averaging (32) one obtains

$$\begin{aligned}\vartheta_p(x) = {} & \vartheta_{pl} - \frac{\beta_3(1-\vartheta_{pl})}{(a-b)^2+c^2}\left[\frac{2b-a}{b^2+c^2}\,(1-e^b\cos c) + \frac{e^{ax}-1}{a}\right.\\ & \left. + \frac{b(b-a)-c^2}{c(b^2+c^2)}\,e^b\sin c\right] + \frac{\beta_3\beta_2\,\vartheta_{pl}}{(a-b)^2+c^2}\\ & \times \left\langle -\frac{(a-b)^2+c^2}{a(b^2+c^2)} + \frac{e^a-1}{a^2} - \frac{2b-c}{(b^2+c^2)^2}\right.\\ & \times [e^b(b\cos c + c\sin c) - b] + \frac{b(b-a)-c^2}{c(b^2+c^2)^2}\\ & \left. \times [e^b(b\sin c - c\cos c) + c]\right\rangle.\end{aligned} \tag{33}$$

Relationships (30) and (33) contain an unknown temperature of the solid phase ϑ_{pl} at the "hot edge" of the layer. Considering ϑ_{pl} as an arbitrary constant we determine from the boundary condition (4) at $x=1$. Differentiating $\vartheta_p(x)$ (defined according to (28) or (32)) and assuming $x=1$ we put the derivative equal to zero. The algebraic equation resulted in has only one unknown value ϑ_{pl} and resolving it one finds:

$$\vartheta_{pl} = 1/(1+\beta_2 z), \tag{34}$$

where

$$z = \sum_1^3 (-1)^{i-1}(s_p - s_q)\,e^{s_i} \Big/ \sum_1^3 (-1)^{i-1} s_i (s_p - s_q)\,e^{s_i} \tag{35}$$

in the case of three real roots $s_{1,2,3}$ or

$$z = \frac{1 - e^{b-a}\cos c + [\,(b-a)/c]\;e^{b-a}\sin c}{a - a\,e^{b-a}\cos c + [b\,(b-a)/c - c]\;e^{b-a}\sin c} \tag{36}$$

if two roots are complex.

Thus relationships (30), (33), (35) and (36) allow to determine the parameter m according (21) and consequently to compute the time dependences of the temperatures $\vartheta_{pb}(\tau)$ and $\vartheta_{fk}(\tau)$. The choice between dependences (30) or (33) should be done according to the sign of the determinant of the characteristic equation (27). The original of the function $\vartheta_f(x)$ can be found from (25) according to the theorem of the transform product

$$\vartheta_f(x) = e^{-(\beta_1+\beta_2)x} + \beta_1\,e^{-(\beta_1+\beta_2)x}\int_0^x e^{(\beta_1+\beta_2)x_*}\,\vartheta_p(x_*)\,dx_*. \tag{37}$$

3
Results and discussion

Using Eqs. (15), (21), (30), (34) and (35) we have calculated the averaged temperature ϑ_{pb} of the solid phase as a function of the dimensionless time τ for different sets of the problem parameters α_i. These results are displayed in Fig. 1. The data of Ref. [1] found by numerical integration of Eqs. (1) and (2) are also presented in the figure for comparison. There is practical coincidence of the results of computations by different approaches.

Certain peculiarities of computations by Eq. (33) should be noted. As an analysis shows practically for all magnitudes of the parameters α_i the coefficients β_i satisfy the relations: $\beta_2 < \beta_3 < \beta_1$ and $\beta_2 \ll \beta_1$. As a result the root a of Eq. (27) approaches to zero. In such a case the Taylor series

$$\frac{e^a - 1}{a} = \sum_{n=0}^{\infty} \frac{a^n}{(n+1)!} \tag{38}$$

should be used in calculations according to formulae (30) and (33).

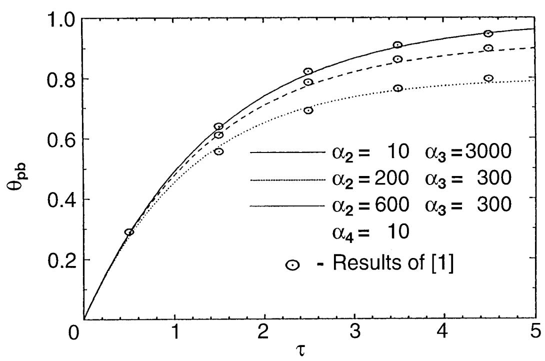

Fig. 1. The time dependence of the average temperature of the solid filling for different sets of parameters

4
Conclusion

An exact analytical solution of the problem of thermal interaction between a fluid flow and solid particles is found in the framework of the model ODTPM. The relationships for the volume average temperature of the solid phase and the temperature of the heat carrier at the layer exit are found.

The approach developed gives a possibility to analyze the efficiency of the fluid-solid phase systems in a more simple way than it can be done on the basis of numerical calculations (Ref. [1]).

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