Generalized Reynolds number for the flow of power law fluids in cylindrical ducts of arbitrary cross-section

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Abstract

The pressure drop prediction requires the generalization of the Reynolds number for laminar flow of pseudoplastic fluids in cylindrical ducts. This generalized Reynolds number is now well established for very simple cross-sections, such as circular tubes or infinite parallel plane plates. To our knowledge, there is no generalization available for other cross-sections due to the complexity of the velocity profiles.

The aim of this paper is to propose a generalized Reynolds number for cylindrical ducts of arbitrary cross-section. This new Reynolds number involves only one geometrical parameter, which can be easily determined theoretically or experimentally. This Reynolds number was successfully compared with definitions found in the literature. Moreover, this generalized Reynolds number seems to be in good agreement with experimental studies carried out in plate heat exchangers.

1. Introduction

Sizing of heat transfer equipment requires the knowledge of both the heat transfer area and the pressure drop. For heat exchangers, performance is described by empirical correlations between dimensionless numbers, such as the friction factor $f/2$, Reynolds number Re, Prandtl number Pr, Nusselt number Nu, etc.

Isothermal laminar flow of Newtonian fluids in relatively simple geometries (circular, rectangular, triangular, etc.) has been studied extensively. Shah and London [1] gave a complete review of the analytical solutions obtained by applied mathematics.

Laminar flow of non-Newtonian fluids has been the subject of extensive study for many years. The most widely encountered non-Newtonian behaviour is pseudoplastic behaviour. The power law model is often sufficient to describe the rheological behaviour of such shear-thinning fluids for engineering or industrial purposes.

The dimensionless numbers containing viscosity (Re, Pr) must be generalized for shear-thinning non-Newtonian fluids. This generalization is often based on the well-known concept of Metzner and Reed [2]: the dimensionless numbers are generalized to obtain identical correlations for both Newtonian and non-Newtonian fluids in the laminar flow regime. This method has been used successfully for momentum transfer in simple geometries, such as circular ducts or infinite parallel plates. This type of generalization theoretically requires the analytical expression of the velocity profile. For other geometries, no simple analytical velocity profile exists, even for rectangular or triangular ducts. Therefore, for these cross-sections, no simple expression of the generalized Reynolds number exists.

In practice, different methods are used to generalize the Reynolds number for complex flow systems.

(1) Theoretical expressions: when the velocity profile is known analytically (circular ducts, infinite parallel plane plates), a theoretical expression of the generalized Reynolds number can be found by using the method of Metzner and Reed [2].

(2) Semi-theoretical expressions: Kozicki *et al.* [3] proposed a generalized Reynolds number for the flow of power law fluids in ducts of arbitrary cross-section (rectangular, triangular, annular, elliptical) based on a numerical determination of the velocity profiles.

(3) Experimental methods: Leuliet [4] and René *et al.* [5] proposed an experimental method for plate heat exchangers. These high performance heat exchangers present very complex flow passages.

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Therefore no theoretical expression of the generalized Reynolds number could be established. The Reynolds number was determined through pressure drop and flow rate experimental measurements for isothermal flow of pseudoplastic fluids.

This brief review of the different approaches used for the determination of the generalized Reynolds number shows an increase in complexity of the method with an increase in complexity of the flow passage geometry. The purpose of this work is to define a generalized Reynolds number regardless of the geometry characteristics of the duct cross-section.

2. State of the art

2.1. Newtonian fluids

The relationship between the friction factor and the Reynolds number for laminar isothermal flow of Newtonian fluids in cylindrical ducts is

$$\frac{f}{2}\,\mathrm{Re}=\xi \qquad (1)$$

In this equation, $f/2$ is the friction factor

$$\frac{f}{2}=\frac{\bar{\tau}_{\mathrm{w}}}{\rho U_{\mathrm{avg}}^{2}}=\frac{\Delta P D_{\mathrm{H}}}{4L\rho U_{\mathrm{avg}}^{2}} \qquad (2)$$

and Re is the classical Reynolds number

$$\mathrm{Re}=\frac{\rho U_{\mathrm{avg}} D_{\mathrm{H}}}{\eta} \qquad (3)$$

2.2. Non-Newtonian fluids

The power law model is widely used to describe the stress–strain relationship of pseudoplastic fluids

$$\tau=k\dot{\gamma}^{n} \qquad (4)$$

In this relation, $\dot{\gamma}$ is the shear rate, τ is the shear stress, k is the consistency index and n is the flow behaviour index. The power law model is only valid over a limited range of shear rate. A local description of the flow curve is often used for more accurate evaluation [6].

Using this rheological model for simple ducts (circular or infinite parallel plates), it is possible to solve the momentum equation and to obtain the generalized Reynolds number defined by Metzner and Reed [2]

$$\mathrm{Re}_{\mathrm{g}}=\frac{\rho U_{\mathrm{avg}}^{2-n} D_{\mathrm{H}}^{n}}{k\phi(n)^{n}\xi^{n-1}} \qquad (5)$$

where ξ is the product of the friction factor and the Reynolds number for a Newtonian fluid under laminar flow conditions. The function $\phi(n)$ is a simple hyperbolic function of the flow behaviour index. For circular ducts

$$\phi(n)=\frac{3n+1}{4n}$$

For infinite parallel plates

$$\phi(n)=\frac{2n+1}{3n}$$

Consequently, a single friction curve equation is available for both Newtonian and pseudoplastic fluids

$$\frac{f}{2}\,\mathrm{Re}_{\mathrm{g}}=\xi \qquad (6)$$

This method, developed under laminar isothermal conditions, can be used successfully in different situations: turbulent flow regime despite the lack of theoretical justification [7–9]; non-isothermal situations (heat transfer); generalization of other dimensionless numbers, such as the Prandtl number for heat transfer applications.

Due to the complexity of the quasi-elliptic partial differential equation describing the flow of pseudoplastic fluids in other cylindrical ducts, there is no general expression for the generalized Reynolds number.

A simple technique for estimating the flow behaviour of non-Newtonian fluids in ducts of unusual cross-section was proposed by Miller [10]. He considered that, for the range of flow behaviour index $n=0.4$–1, function $\phi(n)$ was approximately equal to unity. In fact, this method leads to important errors when n is less than 0.7 as shown by Midoux [11].

The most practical theory was developed by Kozicki *et al.* [3] and used by many workers [12–14]. The general form of the proposed generalized Reynolds number is based on the generalization of the well-known Rabinowich–Mooney equation. This generalized Reynolds number can be written in the same form as that defined by Metzner and Reed [2]. The function $\phi(n)$ is thus described by the following hyperbolic form

$$\phi(n)=\frac{vn+1}{(v+1)n} \qquad (7)$$

where v is a geometrical parameter depending on the cross-section of the cylindrical duct [3].

When Kozicki *et al.* [3] developed this generalized Reynolds number in 1966, Wilson and Thomas's method [6] permitting an accurate description of pseudoplastic fluid flow curves using the power law model, was not available. Kozicki and Tiu [15], being aware of the problem caused by the power law model, proposed different generalized Reynolds numbers using an accurate description of flow curves by complex mathematical models (Metter, Reiner–Rilvin).

Whatever model is used for the description of the flow curve, two main parameters are needed to define a generalized Reynolds number.

(1) The parameter ξ, which is the product of the friction factor and the Reynolds number for the isothermal laminar flow of Newtonian fluids. As previously mentioned this geometrical parameter may be theoretical (simple geometries), semi-theoretical (Kozicki *et al.* [3]) or experimental (Leuliet [4]).

(2) The parameter v must be identified to calculate function ϕ. This function is theoretically known for circular ducts and infinite parallel plates [2]. For other simple geometries (rectangular, isoceles triangular, annular, elliptical), Kozicki *et al.* [3] proposed a numerical determination of this value. Results from this approach were in agreement with experimental data in the literature. However, the complexity of the velocity profile in such geometries always resulted in a complex mathematical form for parameter v.

The only available method for complex geometries, such as plate heat exchangers, involves pressure drop and flow rate measurements under laminar isothermal conditions with different pseudoplastic fluids in order to obtain different values of the flow behaviour index n [4, 5].

In conclusion, the determination of a generalized Reynolds number is a complex problem which can only be solved simply for circular ducts and infinite parallel plates. For complex flow passage, such as that encountered in high performance heat exchangers, no simple method is available. The experimental method proposed by Leuliet [4] is a good example of the complexity of this problem.

3. Proposed generalized Reynolds number

The following form of the generalized Reynolds number is proposed for any cylindrical duct of arbitrary cross-section in order to avoid the difficulties mentioned above

$$\mathrm{Re_g} = \frac{\rho U_{\mathrm{avg}}{}^{2-n} D_{\mathrm{H}}{}^{n}}{k\left\{(24n+\xi)/(24+\xi)n\right\}^{n} \xi^{n-1}} \quad (8)$$

This definition is identical to the equation (5) where the parameter v (eqn. (7)) is defined by

$$v = \frac{24}{\xi} \quad (9)$$

The great advantage of this definition is that only one geometrical parameter, ξ, is necessary to define the generalized Reynolds number. Moreover, this parameter is known (theoretically or experimentally) for a very large number of ducts.

Obviously, eqn. (8) reduces to Metzner and Reed's [2] definitions for circular ducts ($\xi = 8$) and infinite parallel plates ($\xi = 12$).

The function $\phi(n)$ proposed by Kozicki *et al.* [3], which is in agreement with data available in the literature, was compared with our function. The different values of the parameters ξ and v are summarized in Table 1.

For the different geometries presented in Table 1, the evolution of function ϕ with the flow behaviour index ($0.05 \leqslant n \leqslant 1$) has been studied and compared with the results of Kozicki *et al.* [3].

(1) For rectangular ducts, for the four aspect ratios, the difference between the ϕ functions never exceeds 3%. However, we can estimate that, for the rectangular duct of aspect ratio 0.5, the v value proposed by Kozicki *et al.* [3] is underestimated: we can expect a value greater than 3 (circular duct) when the ξ parameter is smaller than 8.

(2) Concentric circular ducts are widely used in many fluid flow and heat transfer devices. For an inner diameter to outer diameter ratio from 0.05

TABLE 1. Comparison of parameter v found by the two methods

Geometry	Aspect ratio	ξ [1]	v [3]	$v = 24/\xi$
Circular ducts	–	8	3	3
Infinite parallel plates	–	12	2	2
Rectangular ducts	$E = 1.00$	7.113	3.190	3.374
	$E = 0.75$	7.238	3.152	3.316
	$E = 0.50$	7.774	2.982	3.087
	$E = 0.25$	9.116	2.547	2.633
Concentric annuli	$K = 0.90$	11.998	2.001	2.000
	$K = 0.50$	11.906	2.015	2.016
	$K = 0.10$	11.171	2.135	2.148
	$K = 0.05$	10.783	2.196	2.226
Isoceles triangular ducts	$A = 10°$	6.237	4.058	3.848
	$A = 40°$	6.611	3.490	3.630
	$A = 60°$	6.667	3.446	3.600
	$A = 90°$	6.576	3.494	3.650
Elliptical ducts	$\beta = 0.90$	8.011	2.999	2.996
	$\beta = 0.50$	8.411	3.000	2.853
	$\beta = 0.30$	8.948	3.000	2.682
	$\beta = 0.10$	9.657	3.000	2.485

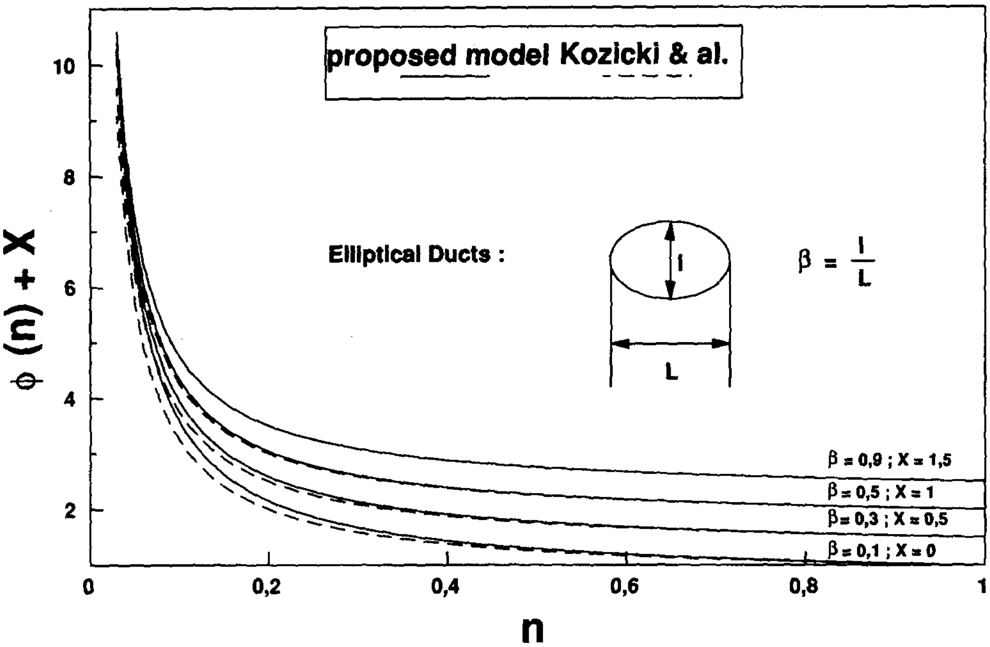

Fig. 1. Comparison of function ϕ for elliptical ducts.

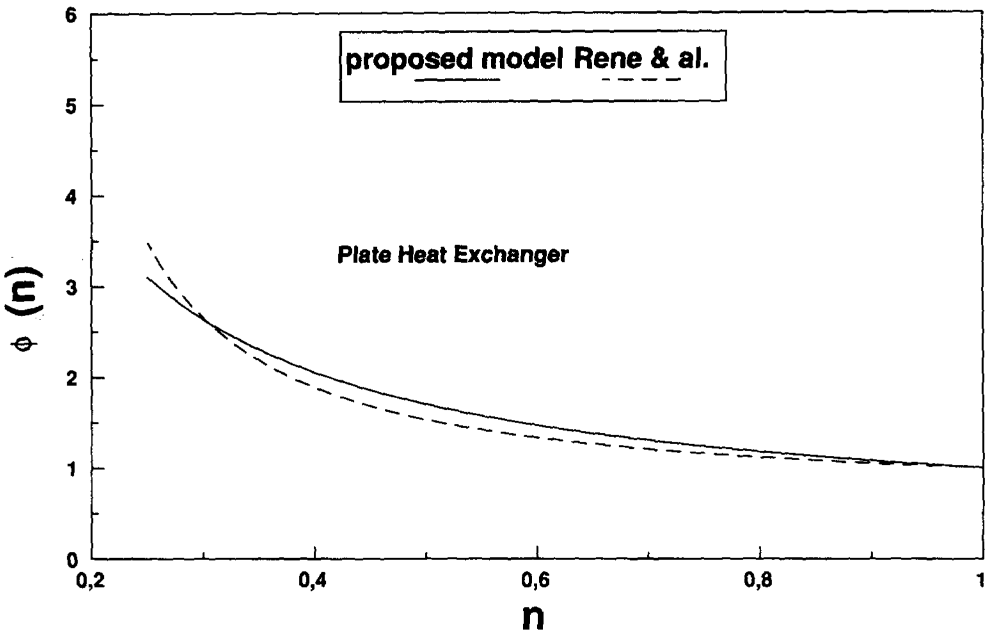

Fig. 2. Comparison of function ϕ for a plate heat exchanger.

to 0.9, the difference between the two ϕ function values does not exceed 1%, which is quite acceptable.

(3) Differences between $\phi(n)$ functions fall within a range of 1% for the flow of a power law fluid through isoceles triangular ducts. The highest differences are observed when the flow behaviour indices are below 0.2.

(4) The highest disagreement between the two functions is found for elliptical ducts (Fig. 1), especially for values of n below 0.4. However, parameter v should not be considered as a constant equal to 3 when the ξ parameter varies in the range 8.01–9.66 (see Table 1).

Finally, in order to test our approach for complex geometry, the results published by René *et al.* [5] on the flow of power law fluids in complex flow arrangement plate heat exchangers were investigated. The exchanger equipped with straight corrugated plates (model V7, Vicarb Co., France) was studied experimentally by Leuliet [4] in order to determine friction curves for both Newtonian and pseudoplastic fluids.

The experimental values found for parameter ξ and function $\phi(n)$ in this complex flow system were

$$\xi = 56.6 \text{ and } \phi(n) = \left(\frac{2n+1}{3n}\right)\left(\frac{1}{n^{0.1}}\right)^{1/n} \tag{10}$$

In Fig. 2, the experimental function ϕ (eqn. (10)) is compared with the proposed form (eqn. (11))

$$\phi(n) = \frac{24n + 56.6}{(24 + 56.6)n} \tag{11}$$

This comparison was performed for the range of flow behaviour index used in the experimental work. Agreement between the two curves is quite satisfactory: the highest deviation between the two functions produces a variation in the generalized Reynolds number of less than 3%.

4. Concluding remarks

In this paper, we have proposed a generalized Reynolds number for cylindrical ducts of arbitrary cross-section. Such Reynolds numbers exist in the literature, but usually require two parameters: v and ξ.

Despite the lack of theoretical justification, we have shown that only one geometrical parameter can be used to generalize the Reynolds number. Moreover, this geometrical parameter (ξ parameter) can easily be obtained analytically or experimentally: ξ is the product of the friction factor and the Reynolds number for the isothermal laminar flow of Newtonian fluids.

We have shown that the use of this generalized Reynolds number is sufficient for engineering purposes, *e.g.* for pressure drop prediction, and even for very complex flow passages such as those encountered in plate heat exchangers.

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Appendix A: Nomenclature

A	opening angle of the isoceles triangular duct (deg or rad)
D_H	hydraulic diameter (m)
E	ratio of minor to major side of the rectangular duct (dimensionless)
f	Fanning number (dimensionless)
K	ratio of inner to outer radius of the annulus (dimensionless)
k	consistency index (Pa s^n)
L	duct length (m)
n	flow behaviour index for Ostwald de Waele fluids (dimensionless)
ΔP	pressure drop (Pa)
U_{avg}	mean velocity of flow (m s^{-1})
v	parameter (dimensionless)

Greek letters

β	ratio of minor to major axis of the ellipse (dimensionless)
$\dot{\gamma}$	shear rate (s^{-1})
η	Newtonian viscosity (Pa s)
ξ	geometrical parameter (dimensionless)
ρ	density (kg m^{-3})
τ	shear stress (Pa)
$\bar{\tau}_w$	average shear stress (Pa)
ϕ	function of the flow behaviour index (dimensionless)