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UNIVERSALITY AND COMPLEXITY IN CELLULAR AUTOMATA

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Cellular automata are discrete dynamical systems with simple construction but complex self-organizing behaviour. Evidence is presented that all one-dimensional cellular automata fall into four distinct universality classes. Characterizations of the structures generated in these classes are discussed. Three classes exhibit behaviour analogous to limit points, limit cycles and chaotic attractors. The fourth class is probably capable of universal computation, so that properties of its infinite time behaviour are undecidable.

1. Introduction

Cellular automata are mathematical models for complex natural systems containing large numbers of simple identical components with local interactions. They consist of a lattice of sites, each with a finite set of possible values. The value of the sites evolve synchronously in discrete time steps according to identical rules. The value of a particular site is determined by the previous values of a neighbourhood of sites around it.

The behaviour of a simple set of cellular automata were discussed in ref. 1, where extensive references were given. It was shown that despite their simple construction, some cellular automata are capable of complex behaviour. This paper discusses the nature of this complex behaviour, its characterization, and classification. Based on investigation of a large sample of cellular automata, it suggests that many (perhaps all) cellular automata fall into four basic behaviour classes. Cellular automata within each class exhibit qualitatively similar behaviour. The small number of classes implies considerable university in the qualitative behaviour of cellular automata. This universality implies that many details of the construction of a cellular automaton are irrelevant in determining its qualitative behaviour. Thus complex physical and biological systems may lie in the same universality classes as the idealized mathematical models provided by cellular automata. Knowledge of cellular automaton behaviour may then yield rather general results on the behaviour of complex natural systems.

Cellular automata may be considered as discrete dynamical systems. In almost all cases, cellular automaton evolution is irreversible. Trajectories in the configuration space for cellular automata therefore merge with time, and after many time steps, trajectories starting from almost all initial states become concentrated onto "attractors". These attractors typically contain only a very small fraction of possible states. Evolution to attractors from arbitrary initial states allows for "self-organizing" behaviour, in which structure may evolve at large times from structureless initial states. The nature of the attractors determines the form and extent of such structures.

The four classes mentioned above characterize the attractors in cellular automaton evolution. The attractors in classes 1, 2 and 3 are roughly anal-

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ogous respectively to the limit points, limit cycles and chaotic ("strange") attractors found in continuous dynamical systems. Cellular automata of the fourth class behave in a more complicated manner, and are conjectured to be capable of universal computation, so that their evolution may implement any finite algorithm.

The different classes of cellular automaton behaviour allow different levels of prediction of the outcome of cellular automaton evolution from particular initial states. In the first class, the outcome of the evolution is determined (with probability 1), independent of the initial state. In the second class, the value of a particular site at large times is determined by the initial values of sites in a limited region. In the third class, a particular site value depends on the values of an ever-increasing number of initial sites. Random initial values then lead to chaotic behaviour. Nevertheless, given the necessary set of initial values, it is conjectured that the value of a site in a class 3 cellular automaton may be determined by a simple algorithm. On the other hand, in class 4 cellular automata, a particular site value may depend on many initial site values, and may apparently be determined only by an algorithm equivalent in complexity to explicit simulation of the cellular automaton evolution. For these cellular automata, no effective prediction is possible; their behaviour may be determined only by explicit simulation.

This paper describes some preliminary steps towards a general theory of cellular automaton behaviour. Section 2 below introduces notation and formalism for cellular automata. Section 3 discusses general qualitative features of cellular automaton evolution illustrating the four behaviour classes mentioned above. Section 4 introduces entropies and dimensions which characterize global features of cellular automaton evolution. Successive sections consider each of the four classes of cellular automata in turn. The last section discusses some tentative conclusions.

This paper covers a broad area, and includes many conjectures and tentative results. It is not intended as a rigorous mathematical treatment.

2. Notation and formalism

$a_i^{(t)}$ is taken to denote the value of site i in a one-dimensional cellular automaton at time step t. Each site value is specified as an integer in the range 0 through $k-1$. The site values evolve by iteration of the mapping

$$a_i^{(t)} = \mathbf{F}[a_{i-r}^{(t-1)}, a_{i-r+1}^{(t-1)}, \ldots, a_i^{(t-1)}, \ldots, a_{i+r}^{(t-1)}]. \quad (2.1)$$

F is an arbitrary function which specifies the cellular automaton rule.

The parameter r in eq. (2.1) determines the "range" of the rule: the value of a given site depends on the last values of a neighbourhood of at most $2r+1$ sites. The region affected by a given site grows by at most r sites in each direction at every time step; propagating features generated in cellular automaton evolution may therefore travel at most r sites per time step. After t time steps, a region of at most $1+2rt$ sites may therefore be affected by a given initial site value.

The "elementary" cellular automata considered in ref. 1 have $k=2$ and $r=1$, corresponding to nearest-neighbour interactions.

An alternative form of eq. (2.1) is

$$a_i^{(t)} = \mathbf{f}\left[\sum_{j=-r}^{j=r} \alpha_j a_{i+j}^{(t-1)}\right], \quad (2.2)$$

where the α_j are integer constants, and the function **f** takes a single integer argument. Rules specified according to (2.1) may be reproduced directly by taking $\alpha_j = k^{r-j}$.

The special class of additive cellular automaton rules considered in ref. 2 correspond to the case in which **f** is a linear function of its argument modulo k. Such rules satisfy a special additive superposition principle. This allows the evolution of any initial configuration to be determined by superposition of results obtained with a few basis configurations, and makes possible the algebraic analysis of ref. 2.

"Totalistic" rules defined in ref. 1, and used in several examples below, are obtained by taking

$$\alpha_j = 1 \tag{2.3}$$

in eq. (2.2). Such rules give equal weight to all sites in a neighbourhood, and imply that the value of a site depends only on the total of all preceding neighbourhood site values. The results of section 3 suggest that totalistic rules exhibit behaviour characteristic of all cellular automata.

Cellular automaton rules may be combined by composition. The set of cellular automaton rules is closed under composition, although composition increases the number of sites in the neighbourhood. Composition of a rule with itself yields patterns corresponding to alternate time steps in time evolution according to the rule. Compositions of distinct results do not in general commute. However, if a composition $\mathbf{F}_1\mathbf{F}_2$ of rules generates a sequence of configurations with period π, then the rule $\mathbf{F}_2\mathbf{F}_1$ must also allow a sequence of configurations with period π. As discussed below, this implies that the rules $\mathbf{F}_1\mathbf{F}_2$ and $\mathbf{F}_2\mathbf{F}_1$ must yield behaviour of the same class.

The configuration $a_i = 0$ may be considered as a special "null" configuration ("ground state"). The requirement that this configuration remain invariant under time evolution implies

$$\mathbf{F}[0, 0, \ldots, 0] = 0 \tag{2.4a}$$

and

$$\mathbf{f}[0] = 0\,. \tag{2.4b}$$

All rules satisfy this requirement if iterated at most k times, at least up to a relabelling of the k possible values.

It is convenient to consider symmetric rules, for which

$$\mathbf{F}[a_{i-r}, \ldots, a_{i+r}] = \mathbf{F}[a_{i+r}, \ldots, a_{i-r}]\,. \tag{2.5}$$

Once a cellular automaton with symmetric rules has evolved to a symmetric state (in which $a_{n+i} = a_{n-i}$ for some n and all i), it may subsequently generate only symmetric states (assuming symmetric boundary conditions), since the operation of space reflection commutes with time evolution in this case.

Rules satisfying the conditions (2.4) and (2.5) will be termed "legal".

The cellular automaton rules (2.1) and (2.2) may be considered as discrete analogues of partial differential equations of order at most $2r + 1$ in space, and first order in time. Cellular automata of higher order in time may be constructed by allowing a particular site value to depend on values of a neighbourhood of sites on a number s of previous time steps. Consideration of "effective" site values $\Sigma_{n=0}^{s-1} m^n a_i^{(t-n)}$ always allows equivalent first-order rules with $k = m^s - 1$ to be constructed.

The form of the function $\mathbf{F}$ in the time evolution rule (2.1) may be specified by a "rule number" [1]

$$\mathbf{R}_{\mathbf{F}} = \sum_{\{a_{i-r}, a_{i+r}\}} \mathbf{F}[a_{i-r}, \ldots, a_{i+r}] k^{\Sigma_{i=-r}^{r} k^{r-j} a_{i+j}}\,. \tag{2.6}$$

The function $\mathbf{f}$ in eq. (2.2) may similarly be specified by a numerical "code"

$$\mathbf{C}_{\mathbf{f}} = \sum_{n=0}^{(2r+1)(k-1)} k^n \mathbf{f}[n]\,. \tag{2.7}$$

The condition (2.4) implies that both $\mathbf{R}_F$ and $\mathbf{C}_f$ are multiples of k.

In general, there are a total of $'k^{k^{(2r+1)}}$ possible cellular automaton rules of the form (2.1) or (2.2). Of these, $k^{k^{r+1}(k^r+1)/2-1}$ are legal. The rapid growth of the number of possible rules with r implies that an exponentially small fraction of rules may be obtained by composition of rules with smaller r.

A few cellular automaton rules are "reducible" in the sense that the evolution of sites with particular values, or on a particular grid of positions and times, are independent of other site values. Such cellular automata will usually be excluded from the classification described below.

Very little information on the behaviour of a cellular automaton can be deduced directly from simple properties of its rule. A few simple results are nevertheless clear.

First, necessary (but not sufficient) conditions for a rule to yield unbounded growth are

$$\mathbf{F}[a_{i-r}, a_{i-r+1}, \ldots, a_{i-1}, 0, 0, \ldots, 0] \neq 0\,,$$
$$\mathbf{F}[0, \ldots, 0, 0, a_{i+1}, \ldots, a_{i+r}] \neq 0\,, \qquad (2.8)$$

for some set of a_i. If these conditions are not fulfilled then regions containing nonzero sites surrounded by zero sites can never grow, and the cellular automaton must exhibit behaviour of class 1 or 2. For totalistic rules, the condition (2.8) becomes

$$\mathbf{f}[n] \neq 0 \qquad (2.9)$$

for some $n < r$.

Second, totalistic rules for which

$$\mathbf{f}[n_1] \geq \mathbf{f}[n_2] \qquad (2.10)$$

for all $n_1 > n_2$ exhibit no "growth inhibition" and must therefore similarly be of class 1 or 2.

One may consider cellular automata both finite and infinite in extent.

When finite cellular automata are discussed below, they are taken to consist of N sites arranged around a circle (periodic boundary conditions). Such cellular automata have a finite number k^N of possible states. Their evolution may be represented by finite state transition diagrams (cf. [2]), in which nodes representing each possible configuration are joined by directed arcs, with a single arc leading from a particular node to its successor after evolution for one time step. After a sufficiently long time (less than k^N), any finite cellular automaton must enter a cycle, in which a sequence of configurations is visited repeatedly. These cycles represent attractors for the cellular automaton evolution, and correspond to cycles in the state transition graph. At nodes in the cycles may be rooted trees representing transients. The transients are irreversible in the sense that nodes in the tree have a single successor, but may have several predecessors. In the course of time evolution, all states corresponding to nodes in the trees ultimately evolve through the configurations represented by the roots of the trees to the cycles on which the roots lie. Configurations corresponding to nodes on the periphery of the state transition diagram (terminals or leaves of the transient trees) are never reached in the evolution: they may occur only as initial states. The fraction of configurations which may be reached after one time step in cellular automaton evolution, and which are therefore not on the periphery of the state transition diagram, gives a simple measure of irreversibility.

The configurations of infinite cellular automata are specified by (doubly) infinite sequences of site values. Such sequences are naturally identified as elements of a Cantor set (e.g. [3]). (They differ from real numbers through the inequivalence of configurations such as .111111 . . . and 1.0000 . . .). Cellular automaton rules define mappings from this Cantor set to itself. The mappings are invariant under shifts by virtue of the identical treatment of each site in eqs. (2.1) and (2.2). With natural measures of distance in the Cantor set, the mappings are also continuous. The typical irreversibility of cellular automaton evolution is manifest in the fact that the mapping is usually not injective, as discussed in section 4.

Eqs. (2.1) and (2.2) may be generalized to several dimensions. For $r = 1$, there are at least two possible symmetric forms of neighbourhood, containing $2d + 1$ (type I) and 3^d (type II) sites respectively; for larger r other "unit cells" are possible.

3. Qualitative characterization of cellular automaton behaviour

This section discusses some qualitative features of cellular automaton evolution, and gives empirical evidence for the existence of four basic classes of behaviour in cellular automata. Section 4 introduces some methods for quantitative analysis of cellular automata. Later sections use these methods to suggest fundamental characterizations of the four cellular automaton classes.

Fig. 1 shows the pattern of configurations generated by evolution according to each of the 32 possible legal totalistic rules with $k = 2$ and $r = 2$, starting from a "disordered" initial configuration (in which each site value is independently chosen as 0 or 1 with probability $\frac{1}{2}$). Even with such a structureless initial state, many of the rules are seen to generate patterns with evident structure. While the patterns obtained with different rules all differ in detail, they appear to fall into four qualitative classes:

1) Evolution leads to a homogeneous state (realized for codes 0, 4, 16, 32, 36, 48, 54, 60 and 62).

2) Evolution leads to a set of separated simple stable or periodic structures (codes 8, 24, 40, 56 and 58).

3) Evolution leads to a chaotic pattern (codes 2, 6, 10, 12, 14, 18, 22, 26, 28, 30, 34, 38, 42, 44, 46 and 50).

4) Evolution leads to complex localized structures, sometimes long-lived (codes 20 and 52).

Some patterns (e.g. code 12) assigned to class 3 contain many triangular "clearings" and appear more regular than others (e.g. code 10). The degree of regularity is related to the degree of irreversibility of the rules, as discussed in section 7.

Fig. 2 shows patterns generated from several different initial states according to a few of the cellular automaton rules of fig. 1. Patterns obtained with different initial states are seen to differ in their details, but to exhibit the same characteristic qualitative features. (Expectional initial states giving rise to different behaviour may exist with low or zero probability.) Fig. 3 shows the differences between patterns generated by various cellular automaton rules from initial states differing in the value of a single site.

Figs. 4, 5 and 6 show examples of various sets of totalistic cellular automata. Fig. 4 shows some $k = 2$, $r = 3$ rules, fig. 5 some $k = 3$, $r = 1$ rules, and fig. 6 some $k = 5$, $r = 1$ rules. The patterns generated are all seen to be qualitatively similar to those of fig. 1, and to lie in the same four classes.

Patterns generated by all possible $k = 2$, $r = 1$ cellular automata were given in ref. 1, and are found to lie in classes 1, 2 and 3. Totalistic $k = 2$, $r = 1$ rules are found to give patterns typical of all $k = 2$, $r = 1$ rules. In general, totalistic rules appear to exhibit no special simplifications, and give rise to behaviour typical of all cellular automaton rules with given k and r.

An extensive sampling of many other cellular automaton rules supports the general conjecture that the four classes introduced above cover all one-dimensional cellular automata*.

Table I gives the fractions of various sets of cellular automata in each of the four classes. With increasing k and r, class 3 becomes overwhelmingly the most common. Classes 1 and 2 are decreasingly common. Class 4 is comparatively rare, but becomes more common for larger k and r.

"Reducible" cellular automata (mentioned in section 2) may generate patterns which contain features from several classes. In a typical case, fixed or propagating "membranes" consisting of sites with a particular value may separate regions containing patterns from classes 3 or 4 formed from sites with other values.

This paper concerns one-dimensional cellular automata. Two-dimensional cellular automata also appear to exhibit a few distinct classes of behaviour. Superficial investigations [5] suggest

Table I
Approximate fractions of legal totalistic cellular automaton rules in each of the four basic classes

Class	$k = 2$ $r = 1$	$k = 2$ $r = 2$	$k = 2$ $r = 3$	$k = 3$ $r = 1$
1	0.50	0.25	0.09	0.12
2	0.25	0.16	0.11	0.19
3	0.25	0.53	0.73	0.60
4	0	0.06	0.06	0.07

*This sampling and many other investigations reported in this paper were performed using the C language computer program[4]. Requests for copies of this program should be directed to the author.

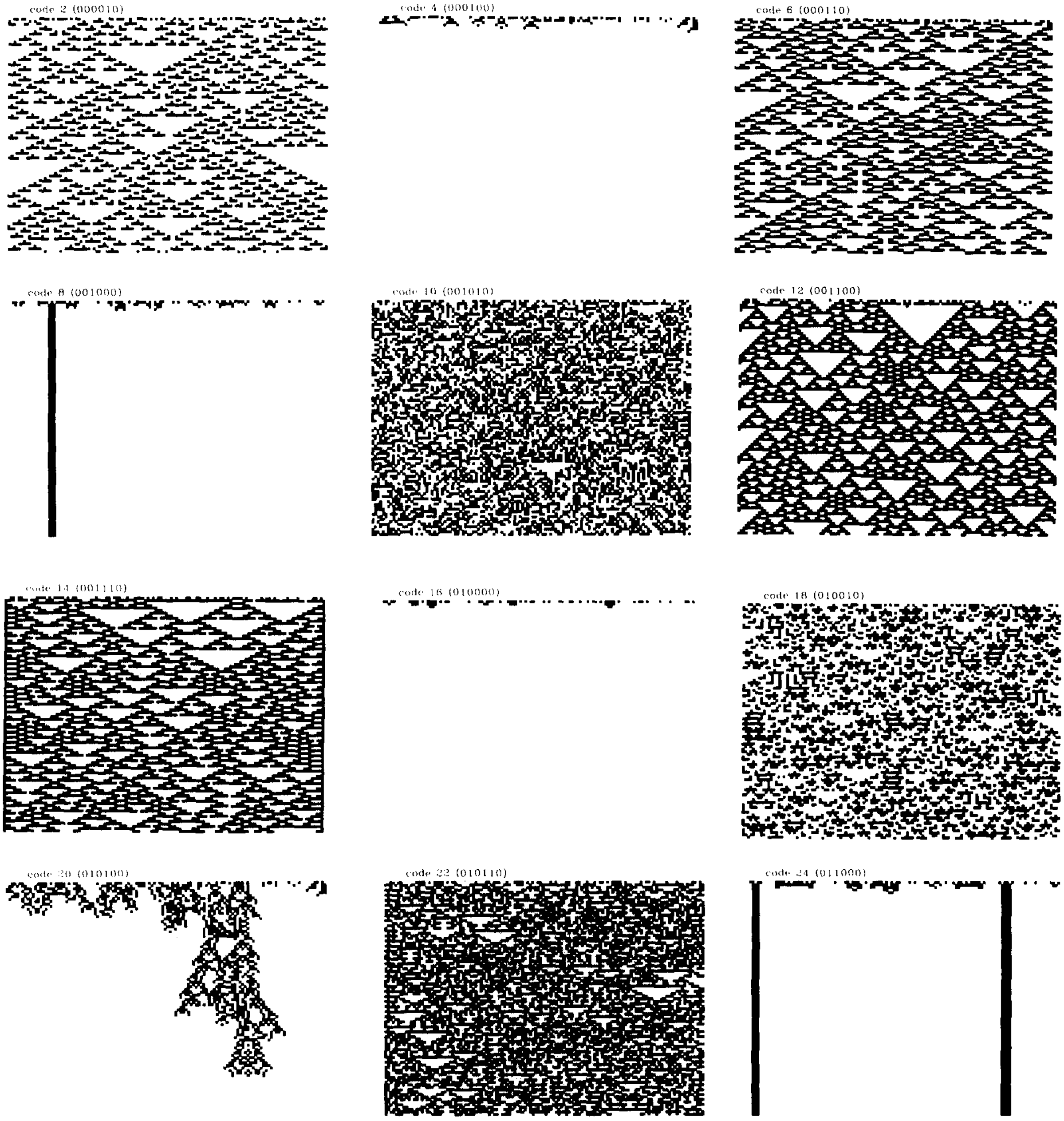

code 2 (000010)
code 4 (000100)
code 6 (000110)
code 8 (001000)
code 10 (001010)
code 12 (001100)
code 14 (001110)
code 16 (010000)
code 18 (010010)
code 20 (010100)
code 22 (010110)
code 24 (011000)

Fig. 1a.

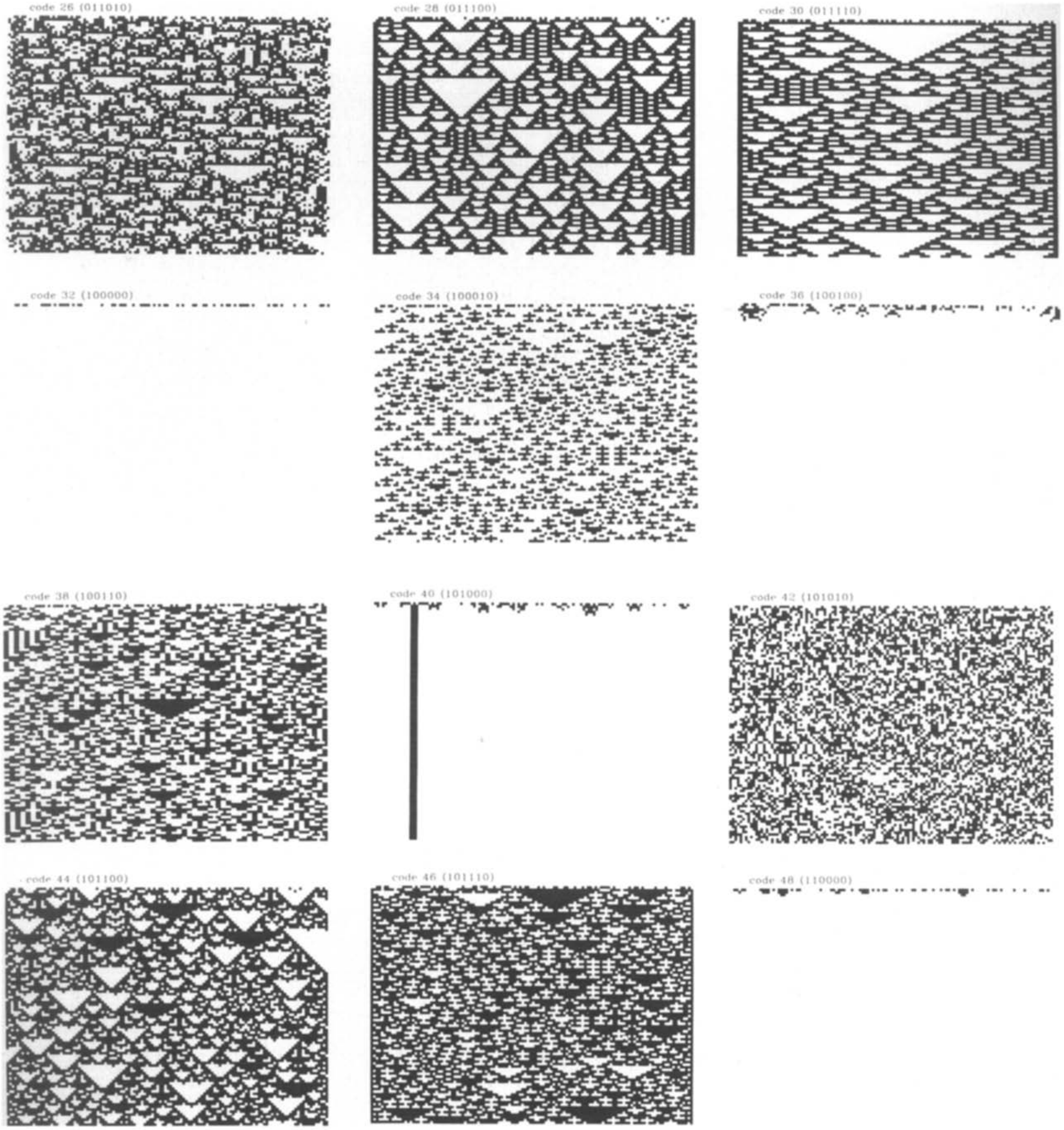

code 26 (011010)
code 28 (011100)
code 30 (011110)
code 32 (100000)
code 34 (100010)
code 36 (100100)
code 38 (100110)
code 40 (101000)
code 42 (101010)
code 44 (101100)
code 46 (101110)
code 48 (110000)

Fig. 1b.

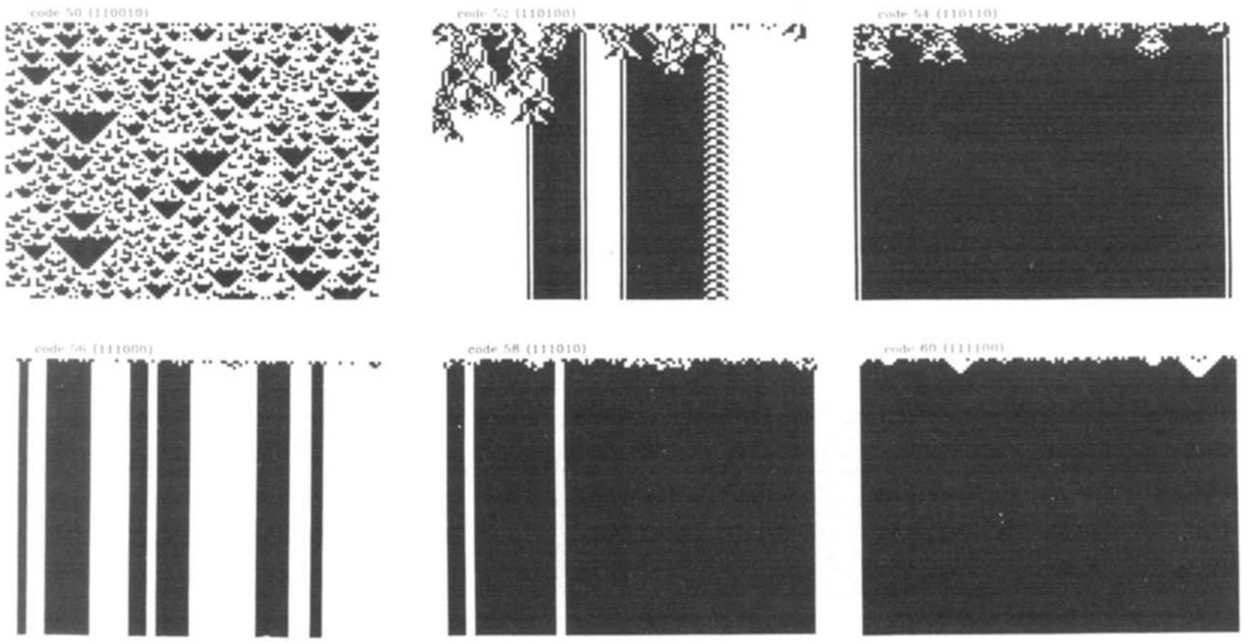


Fig. 1c.

Fig. 1a–c. Evolution of all possible legal one-dimensional totalistic cellular automata with $k = 2$ and $r = 2$. k gives the number of possible values for each site, and r gives the range of the cellular automaton rules. A range $r = 2$ allows the nearest and next-nearest neighbours of a site to affect its value on the next time step. Time evolution for totalistic cellular automata is defined by eqns. (2.2) and (2.7). The initial state is taken disordered, each site having values 0 and 1 with independent equal probabilities. Configurations obtained at successive time steps in the cellular automaton evolution are shown on successive horizontal lines. Black squares represent sites with value 1; white squares sites with value 0. All the cellular automaton rules illustrated are seen to exhibit one of four qualitative classes of behaviour.

that these classes may in fact be identical to the four found in one-dimensional cellular automata.

4. Quantitative characterizations of cellular automaton behaviour

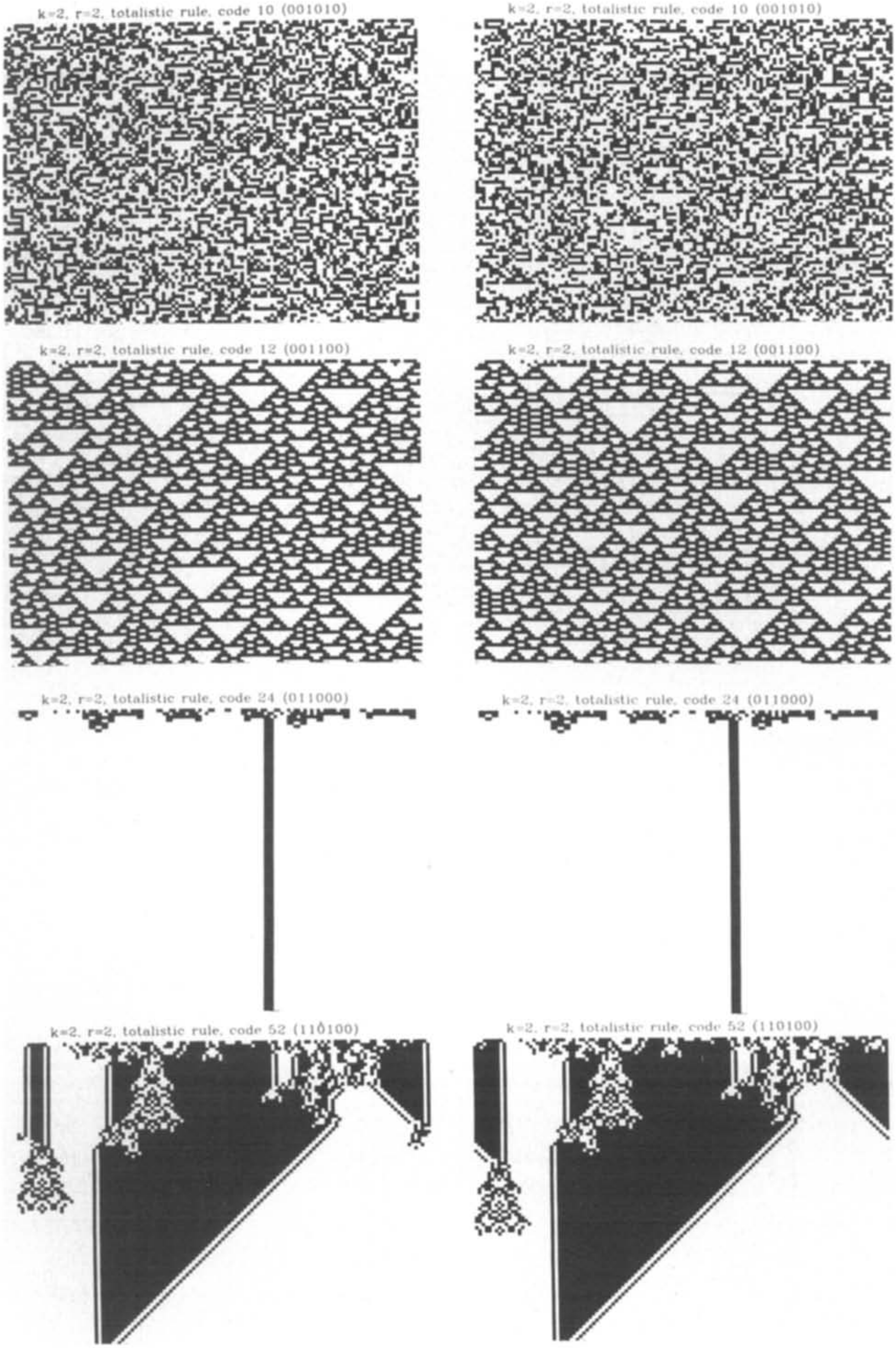
This section describes quantitative statistical measures of order and chaos in patterns generated by cellular automaton evolution. These measures may be used to distinguish the four classes of behaviour identified qualitatively above.

Consider first the statistical properties of configurations generated at a particular time step in cellular automaton evolution. A disordered initial state, in which each site takes on its k possible values with equal independent probabilities, is statistically random. Irreversible cellular automaton evolution generates deviations from statistical randomness. In a random sequence, all k^X possible subsequences ("blocks") of length X must occur with equal probabilities. Deviations from randomness imply unequal probabilities for different subsequences. With probabilities $p_i^{(x)}$ for the k^X possible sequences of site values in a length X block, one may define a specific "spatial set entropy"

$$s^{(x)}(X) = \frac{1}{X}\log_k\left(\sum_{j=1}^{k^X}\theta(p_j^{(x)})\right), \qquad (4.1)$$

where $\theta(p) = 1$ for $p > 0$ and $\theta(0) = 0$, and a specific "spatial measure entropy"

$$s_\mu^{(x)}(X) = -\frac{1}{X}\sum_{j=1}^{k^X} p_j^{(x)}\log_k p_j^{(x)}. \qquad (4.2)$$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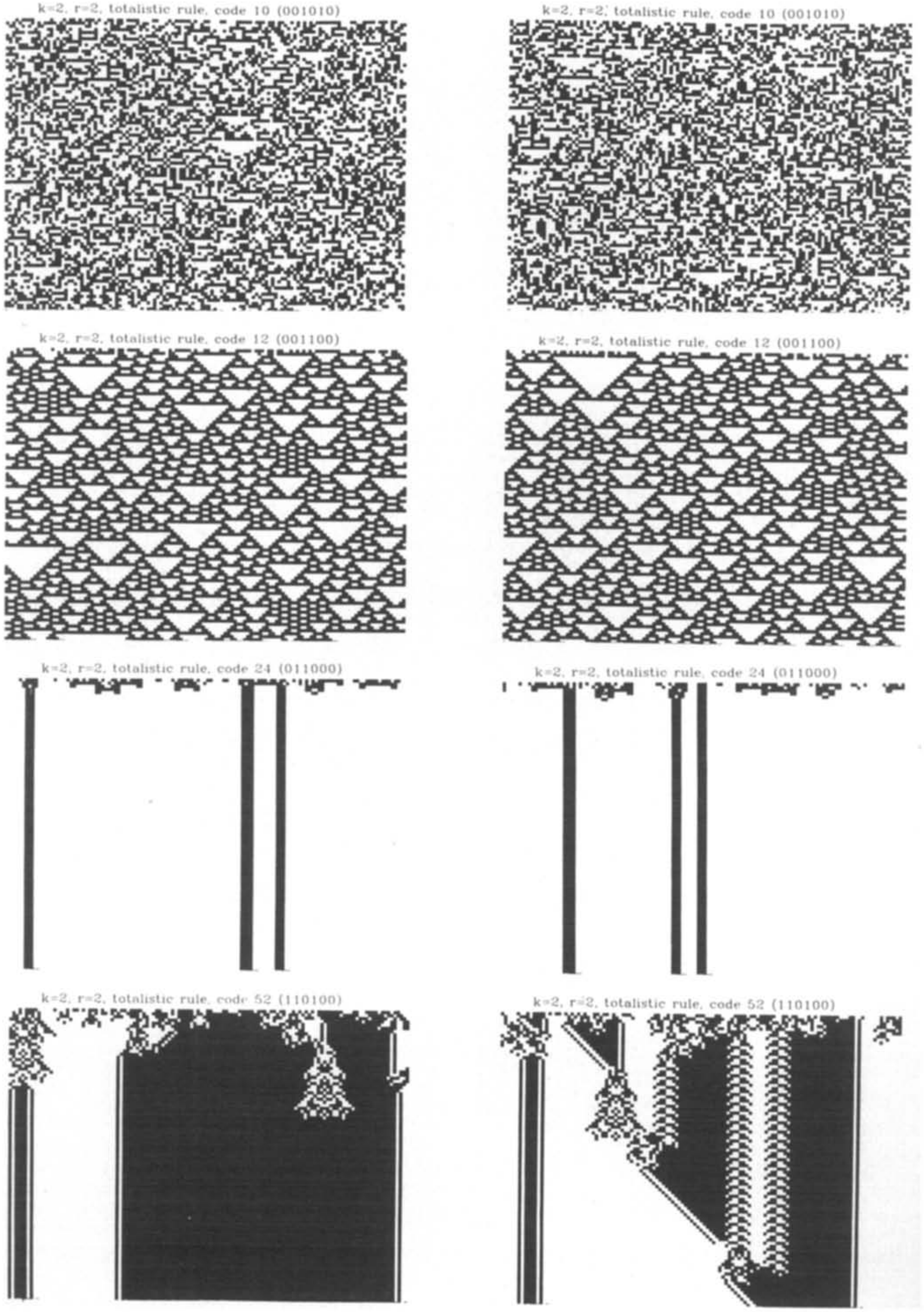
k=2, r=2, totalistic rule, code 10 (001010)
k=2, r=2, totalistic rule, code 12 (001100)
k=2, r=2, totalistic rule, code 12 (001100)
k=2, r=2, totalistic rule, code 24 (011000)
k=2, r=2, totalistic rule, code 24 (011000)
k=2, r=2, totalistic rule, code 52 (110100)
k=2, r=2, totalistic rule, code 52 (110100)

Fig. 2a.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k=2, r=2, totalistic rule, code 10 (001010)
k=2, r=2, totalistic rule, code 12 (001100)
k=2, r=2, totalistic rule, code 12 (001100)
k=2, r=2, totalistic rule, code 24 (011000)
k=2, r=2, totalistic rule, code 24 (011000)
k=2, r=2, totalistic rule, code 52 (110100)
k=2, r=2, totalistic rule, code 52 (110100)

Fig. 2b.

Fig. 2. Evolution of some cellular automata illustrated in fig. 1 from several disordered states. The first two initial states shown differ by a change in the values of two sites, the next by a change in the values of ten sites. The last state is completely different.

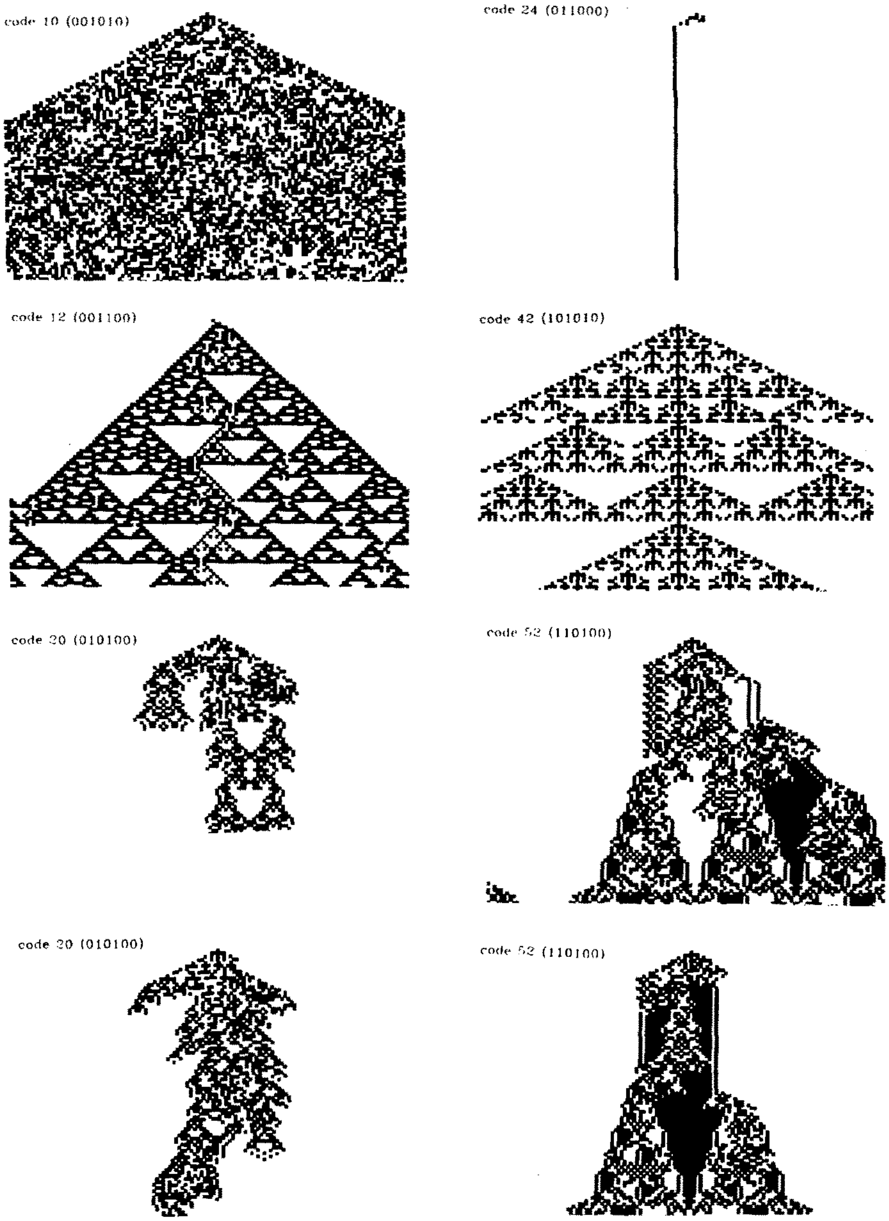

Fig. 3. Differences modulo two between patterns generated by the time evolution of several cellular automata illustrated in fig. 1 with disordered states differing by a change in the value of a single site.

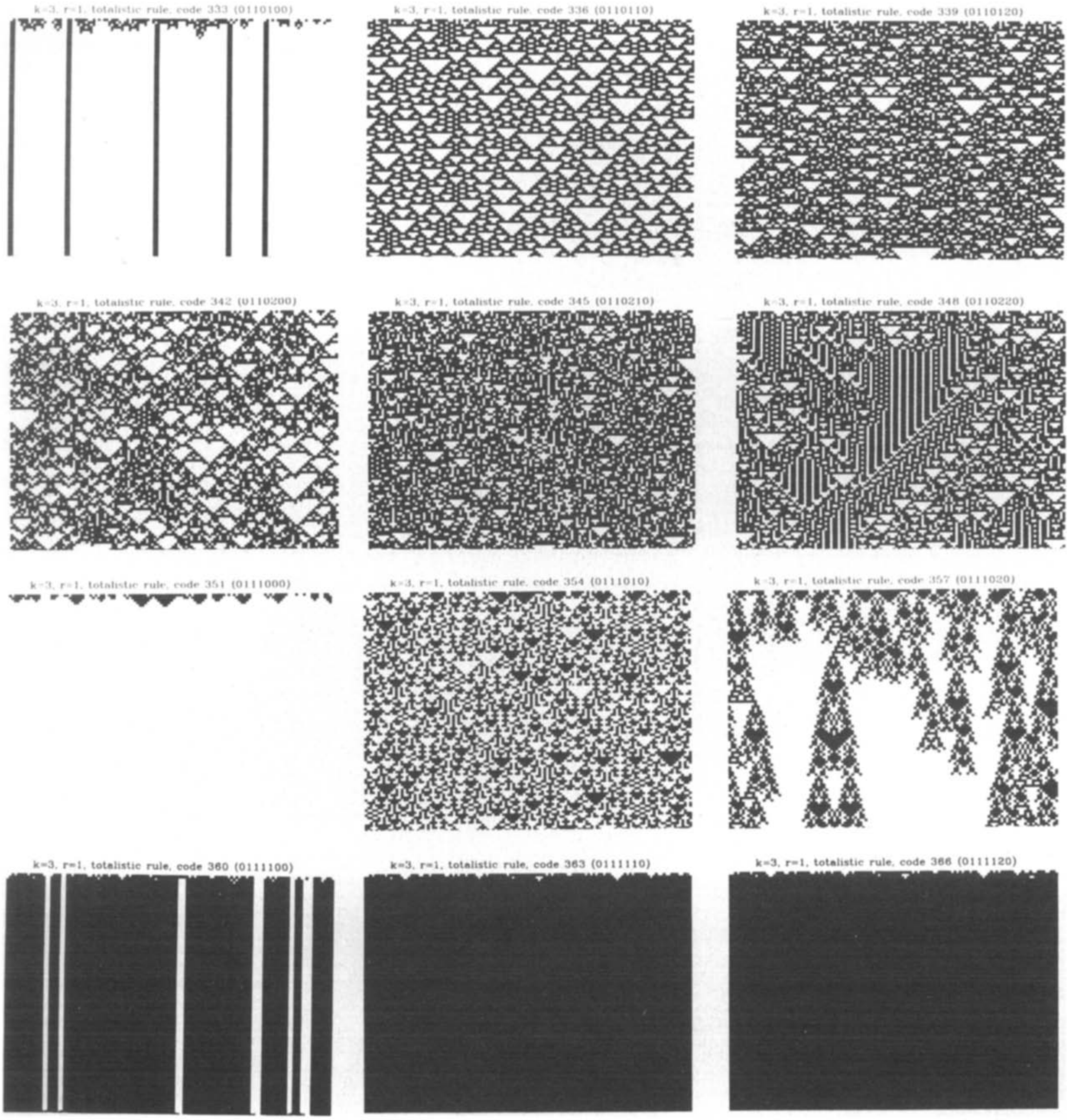
k=3, r=1, totalistic rule, code 333 (0110100)
k=3, r=1, totalistic rule, code 336 (0110110)
k=3, r=1, totalistic rule, code 339 (0110120)
k=3, r=1, totalistic rule, code 342 (0110200)
k=3, r=1, totalistic rule, code 345 (0110210)
k=3, r=1, totalistic rule, code 348 (0110220)
k=3, r=1, totalistic rule, code 351 (0111000)
k=3, r=1, totalistic rule, code 354 (0111010)
k=3, r=1, totalistic rule, code 357 (0111020)
k=3, r=1, totalistic rule, code 360 (0111100)
k=3, r=1, totalistic rule, code 363 (0111110)
k=3, r=1, totalistic rule, code 366 (0111120)

Fig. 4. Examples of the evolution of typical cellular automata with $k = 3$ (three possible site values) and $r = 1$ (only nearest neighbours included in time evolution rules). White squares represent value 0, grey squares value 1, and black squares value 2. The initial state is taken disordered, with each site having values 0, 1 and 2 with equal independent probabilities.

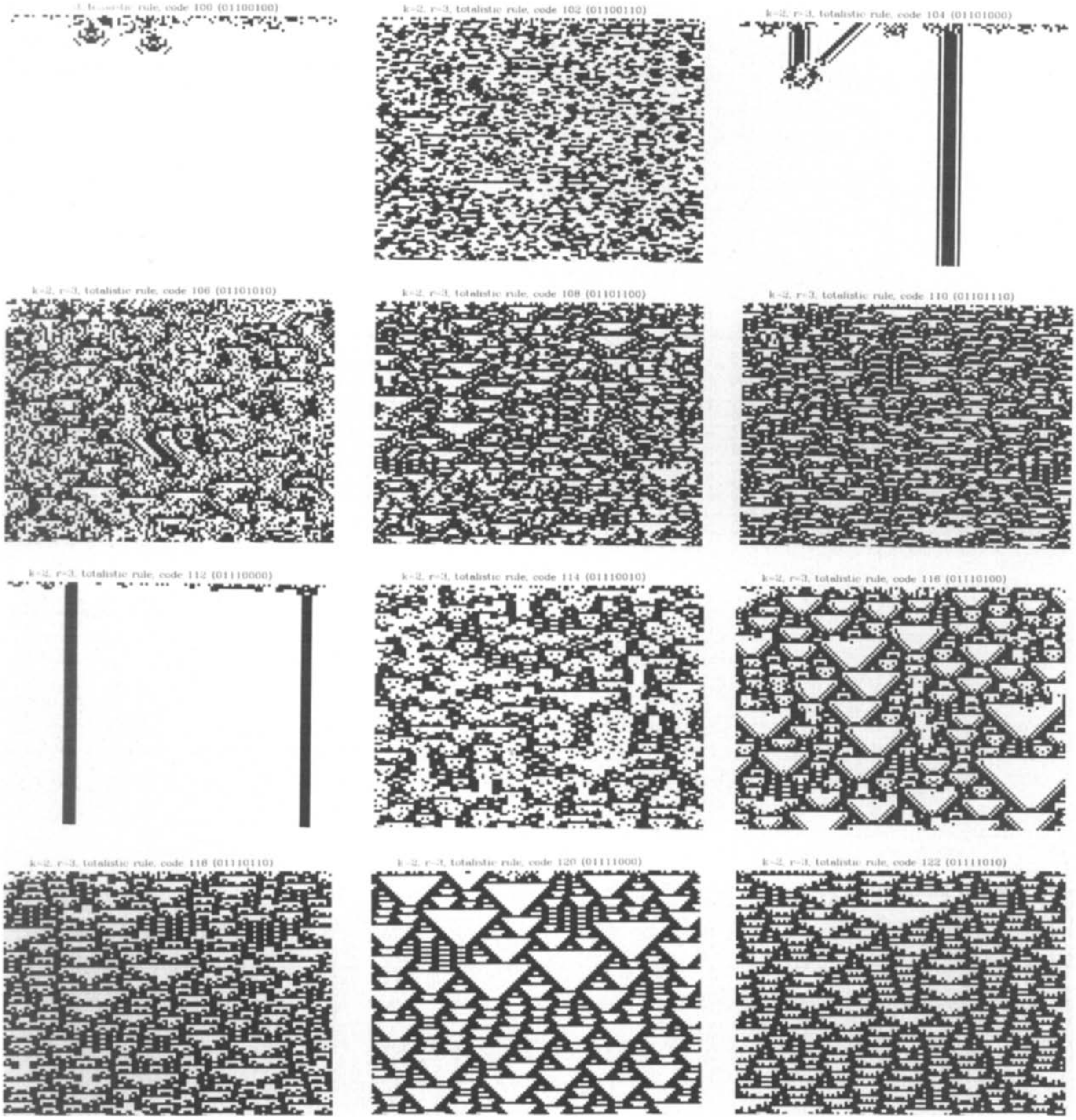
k=2, r=3, totalistic rule, code 100 (01100100)
k=2, r=3, totalistic rule, code 102 (01100110)
k=2, r=3, totalistic rule, code 104 (01101000)
k=2, r=3, totalistic rule, code 106 (01101010)
k=2, r=3, totalistic rule, code 108 (01101100)
k=2, r=3, totalistic rule, code 110 (01101110)
k=2, r=3, totalistic rule, code 112 (01110000)
k=2, r=3, totalistic rule, code 114 (01110010)
k=2, r=3, totalistic rule, code 116 (01110100)
k=2, r=3, totalistic rule, code 118 (01110110)
k=2, r=3, totalistic rule, code 120 (01111000)
k=2, r=3, totalistic rule, code 122 (01111010)

Fig. 5. Examples of the evolution of typical $k = 2$, $r = 3$ cellular automata from a disordered initial state.

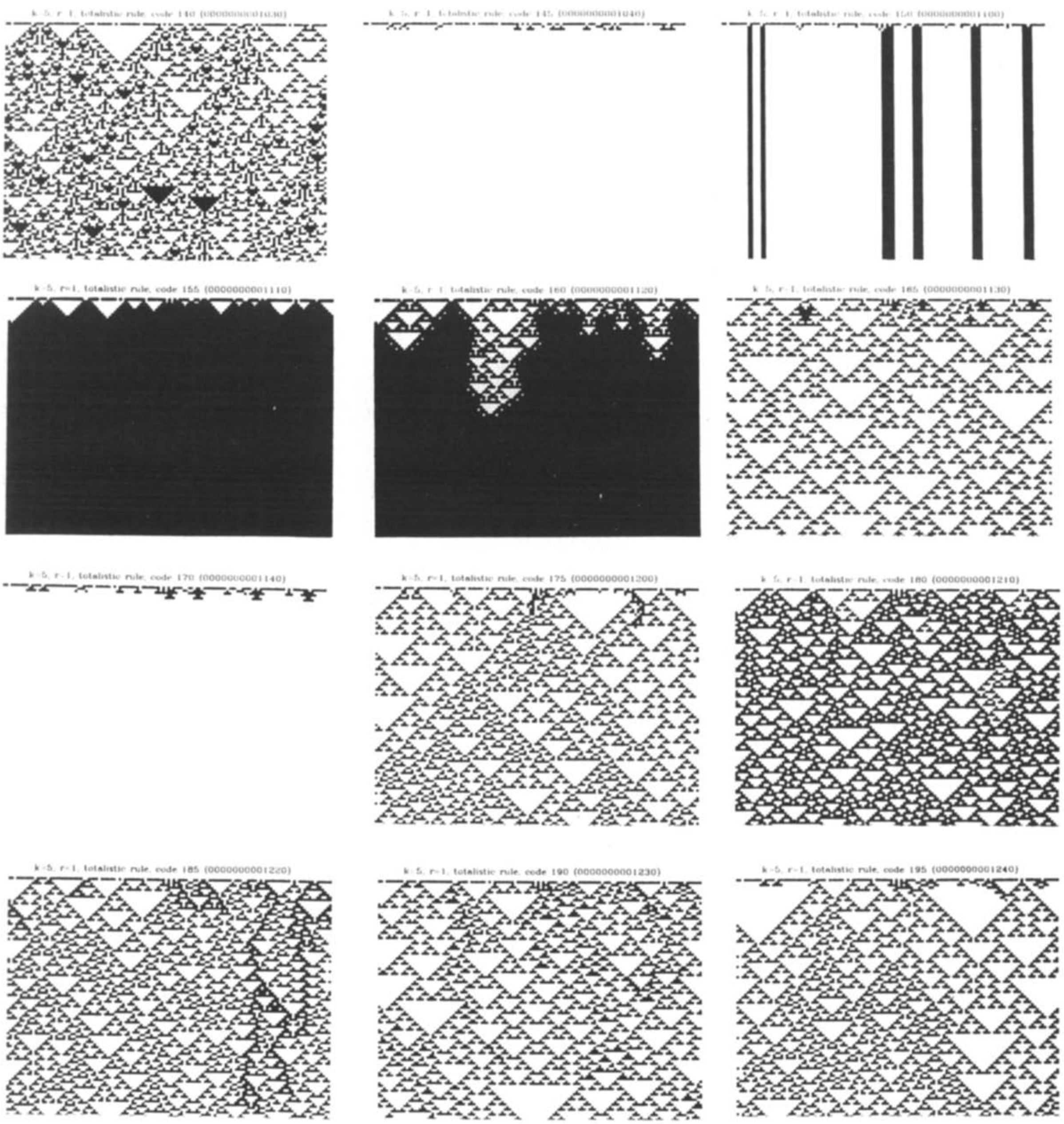
k=5, r=1, totalistic rule, code 140 (0000000001030)
k=5, r=1, totalistic rule, code 145 (0000000001040)
k=5, r=1, totalistic rule, code 150 (0000000001100)
k=5, r=1, totalistic rule, code 155 (0000000001110)
k=5, r=1, totalistic rule, code 160 (0000000001120)
k=5, r=1, totalistic rule, code 165 (0000000001130)
k=5, r=1, totalistic rule, code 170 (0000000001140)
k=5, r=1, totalistic rule, code 175 (0000000001200)
k=5, r=1, totalistic rule, code 180 (0000000001210)
k=5, r=1, totalistic rule, code 185 (0000000001220)
k=5, r=1, totalistic rule, code 190 (0000000001230)
k=5, r=1, totalistic rule, code 195 (0000000001240)

Fig. 6. Examples of the evolution of typical $k = 5$, $r = 1$ cellular automata from a disordered initial state. Darker squares represent sites with larger values.

In both cases, the superscript (x) indicates that "spatial" sequences (obtained at a particular time step) are considered. The "set entropy" (4.1) is determined directly by the total number $N^{(x)}(X)$ of length X blocks generated (with any nonzero probability) in cellular automaton evolution, according to

$$s^{(x)}(X) = \frac{1}{X} \log_k N^{(x)}(X) . \tag{4.3}$$

In the "measure entropy" (4.2) each block is weighted with its probability, so that the result depends explicitly on the probability measure for different cellular automaton configurations, as indicated by the subscript μ. Set entropy is often called "topological entropy"; measure entropy is sometimes referred to as "metric entropy"* (e.g. [6]). For blocks of length 1, the measure entropy $s_\mu^{(x)}(1)$ is related to the densities ρ_i, of sites with each of the k possible values i. $s_\mu^{(x)}(2)$ is related to the densities of "digrams" (blocks of length 2), and so on. In general, the measure entropy gives the average "information content" per site computed by allowing for correlations in blocks of sites up to length X. Note that the entropies (4.1) and (4.2) may be considered to have units of (k-ary) bits per unit distance.

In the equation below, $s_{(\mu)}^{(x)}$ stands for either set entropy $s^{(x)}$ or for measure entropy $s_\mu^{(x)}$.

The definitions (4.1) and (4.2) yield immediately

$$s_\mu^{(x)}(X) \le s^{(x)}(X) \le 1 . \tag{4.4}$$

The first inequality is saturated (equality holds) only for "equidistributed" systems, in which all nonzero block probabilities $p_i^{(x)}$ are equal. The second inequality is saturated if all possible length X blocks of site values occur, but perhaps with unequal probabilities. $s_\mu(X) = 1$ only for "X-random" sequences [7], in which all k^X possible sequences of X site values occur with equal probabilities. In addition to (4.4), the definitions (4.1) and (4.2) imply

$$0 \le s_\mu^{(x)}(X) \le s^{(x)}(X) . \tag{4.5}$$

$s_\mu^{(x)}(X) = 0$ if and only if just one length X block occurs with nonzero probability, so that $s^{(x)}(X) = 0$ also. As discussed below, the inequality (4.5) is saturated for class 1 cellular automata.

Both set and measure entropies satisfy the subadditivity condition

$$(X_1 + X_2) s_{(\mu)}^{(x)}(X_1 + X_2) \le X_1 s_{(\mu)}^{(x)}(X_1) + X_2 s_{(\mu)}^{(x)}(X_2) . \tag{4.6}$$

The inequality is saturated if successive blocks of sites are statistically uncorrelated. In general, it implies some decrease in $s_{(\mu)}^{(x)}(X)$ with X (for example, $s_{(\mu)}^{(x)}(2X) \le s_{(\mu)}^{(x)}(X)$). For cellular automata with translation invariant initial probability measures, stronger constraints may be obtained (analogous to those for "stationary" processes in communication theory [8]). First, note that bounds on $s_{(\mu)}^{(x)}(X)$ valid for any set of probabilities $p_i^{(x)}$ also apply to $s^{(x)}(X)$, since $s^{(x)}(X)$ may formally be reproduced from the definition (4.2) for $s_\mu^{(x)}(X)$ by a suitable (extreme) choice of the $p_i^{(x)}$. The probability $p^{(x)}[a_1, \ldots, a_X]$ for the sequence of site values $a_1, \ldots, a_X$ is given in general by

$$p^{(x)}[a_1, \ldots, a_X] = p^{(x)}[a_1, \ldots, a_{X-1}] p^{(x)}[a_X | a_1, \ldots, a_{X-1}] , \tag{4.7}$$

where $p^{(x)}[a_X | a_1, \ldots, a_{X-1}]$ denotes the conditional probability for a site value a_X, preceded by site values $a_1, \ldots, a_{X-1}$. Defining a total entropy

$$S_\mu^{(x)}[a_1, \ldots, a_X] = -\sum p^{(x)}[a_1, \ldots, a_X] \log_k p^{(x)}[a_1, \ldots, a_X] , \tag{4.8}$$

and corresponding conditional total entropy

*The terms "set" and "measure" entropy, together with "set" and "measure" dimension, are introduced here to rationalize nomenclature.

$$S^{(x)}_{\mu}[a_X|a_1,\ldots,a_{X-1}]$$
$$= -\sum p^{(x)}[a_1,\ldots,a_X]\log_k p^{(x)}[a_X|a_1,\ldots,a_{X-1}]$$
$$\leq S^{(x)}_{\mu}[a_1,\ldots,a_X]\,, \tag{4.9}$$

one obtains

$$X s^{(x)}_{\mu}(X) = S^{(x)}_{\mu}(X) \leq \frac{X-1}{X} S^{(x)}_{\mu}(X-1) + \frac{1}{X} S^{(x)}_{\mu}(X)\,. \tag{4.10}$$

Hence,

$$s^{(x)}_{(\mu)}(X) \leq s^{(x)}_{(\mu)}(X-1)\,, \tag{4.11}$$

so that the set and measure entropies for a translationally invariant system decrease monotonically with the block size X. One finds in addition in this case that

$$\Delta^2_X(X s^{(x)}_{(\mu)}(X)) = (X+1)s^{(x)}_{(\mu)}(X+1) - 2X s^{(x)}_{(\mu)}(X) + (X-1)s^{(x)}_{(\mu)}(X-1) \leq 0\,, \tag{4.12}$$

so that $X s^{(x)}_{(\mu)}(X)$ is a convex function of X.

With the definition $s^{(x)}(0)=1$, this implies that there exists a critical block size X_c, such that

$$\begin{aligned} s^{(x)}(X) &= 1, &&\text{for } X < X_c\,,\\ s^{(x)}(X) &< 1, &&\text{for } X \geq X_c\,. \end{aligned} \tag{4.13}$$

The significance and values of the critical block size X_c will be discussed in section 7 below.

The entropies $s^{(x)}$ and $s^{(x)}_{\mu}$ may be evaluated either for many blocks in a single cellular automaton configuration, or for blocks in an ensemble of different configurations. For smooth probability measures on the ensemble of possible initial configurations, the results obtained in these two ways are almost always the same. (A probability measure will be considered "smooth" if changes in the values of a few sites in an infinite configuration lead only to infinitesimal changes in the probability for the configuration.) The set entropy $s^{(x)}$ is typically independent of the probability measure on the ensemble, for any smooth measure. The measure entropy $s^{(x)}_{\mu}$ in general depends on the probability measure for initial configurations, although for class 3 cellular automata, it is typically the same for at least a large classes of smooth measures. Notice that with smooth measures, the values of $s^{(x)}(X)$ and $s^{(x)}_{\mu}(X)$ are the same whether the length X blocks used in their computation are taken disjoint or overlapping.

The entropies (4.1) and (4.2) are defined for infinite cellular automata. A corresponding definition may be given for finite cellular automata, with a maximum block length given by the total number of sites N the cellular automaton. The entropies $s^{(x)}(N)$ and $s^{(x)}_{\mu}(N)$ are related to global properties of the state transition diagram for the finite cellular automaton. The value of $s^{(x)}(N)$ at a particular time is determined by the fraction of possible configurations which may be reached at that time by evolution from any initial configuration. The limiting value of $s^{(x)}(N)$ at large times is determined by the fraction of configuration on cycles in the state transition graph. Starting from an initial ensemble in which all kN configurations occur with equal probabilities, the limiting value of $s^{(x)}_{\mu}(N)$ is equal to the limiting value of $s^{(x)}(N)$ if all transient trees in the state transition graph for the finite cellular automaton are identical, so that all configurations with nonzero probabilities are generated with the same probability (cf. [2]).

As mentioned in section 2, the configurations of an infinite cellular automaton may be considered as elements of a Cantor set. For an ensemble of disordered configurations (in which each site takes on its k possible values with equal independent probabilities), this Cantor set has fractal dimension 1. Irreversible cellular automaton evolution may lead to an ensemble of configurations corresponding to elements of a Cantor set with dimension less than one. The limiting value of $s^{(x)}(X)$ as $X \to \infty$ gives the fractal or "set" dimension of this set.

Relations between entropy and dimension may be derived in many ways (e.g. [6, 9]). Consider a set of numbers in the interval [0, 1] of the real line. Divide this interval into k^b bins of width k^{-b}, and let the fraction of bins containing numbers in the set be $N(b)$. For large b (small bin width), this number grows as k^{db}. The exponent d is the Kolmogorov dimension (or "capacity" (cf. [8])) of the set. If the set contains all real numbers in the interval [0, 1], then $N(b) = k^b$, and $d = 1$, as expected. If the set contains only a finite number of points, then $N(b)$ must tend to a constant for large b, yielding $d = 0$. The classic Cantor set consists of real numbers in the interval [0, 1], whose ternary decomposition contains only the digits 0 and 2. Dividing the interval into 3^b equal bins, it is clear that 2^b of these bins contain points in the set. The dimension of the set is thus $\log_3 2$. This dimension may also be found by an explicit recursive geometrical construction, using the fact that the set is "self-similar", in the sense that with appropriate magnification, its parts are identical to the whole.

The example above suggests that one may define a "set dimension" d according to

$$d = \lim_{b\to\infty} \frac{1}{b} \log_k N(b)\,, \tag{4.14}$$

where $N(b)$ is the number of bins which contain elements of the set. The bins are of equal size, and their total number is taken as k^b. Except in particularly pathological examples*, the dimension obtained with this definition is equal to the more usual Hausdorff (or "fractal") dimension (e.g. [11]) obtained by considering the number of patches at arbitrary positions required to cover the set (rather than the number of fixed bins containing elements of the set).

The definition (4.14) may be applied directly to cellular automaton configurations. The k^b "bins" may be taken to consist of cellular automaton configurations in which a block of b sites has a particular sequence of values. The definition (4.3) of set entropy then shows that the set dimension is given by

$$d^{(x)} = \lim_{X\to\infty} s^{(x)}(X)\,. \tag{4.15}$$

A disordered cellular automaton configuration, in which all possible sequences of site values occur with nonzero probability (or an ensemble of such configurations), gives $d^{(x)} = 1$, as expected. Similarly, a homogeneous configuration, such as the null configuration, gives $d^{(x)} = 0$.

The set of configurations which appear at large times in the evolution of a cellular automaton constitute the attractors for the cellular automaton. The set dimension of these attractors is given in terms of the entropies for configurations appearing at large times by eq. (4.15).

Accurate direct evaluation of the set entropy $s^{(x)}(X)$ from cellular automaton configurations typically requires sampling of many more than k^X length X blocks. Inadequate samples yield systematic underestimates of $s^{(x)}(X)$. Direct estimates are most accurate when all nonzero probabilities for length X blocks are equal. In this case, a sample of k^b blocks yields an entropy underestimated on average by approximately

$$\log_k(1 - \exp(-k^{b - Xs(X)}))\,. \tag{4.16}$$

Unequal probabilities increase the magnitude of this error, and typically prevent the generation of satisfactory estimates of $d^{(x)}$ from direct simulations of cellular automaton evolution. (If the probabilities follow a log normal distribution, as in many continuous chaotic dynamical systems [12], then the exponential in eq. (4.16) is apparently replaced by a power [13].)

The dimension (4.15) is given as the limiting exponent with which $N^{(x)}(X)$ increases for large X. In the formula (4.15), this exponent is obtained as the limit of $\log_k[N(X)^{1/X}]$ for large X. If $N^{(x)}(X)$ indeed increases roughly exponentially with X,

* Such as the set formed from the end points of the intervals at each stage in the geometrical construction of the classic Cantor set. This set has zero Hausdorff dimension, but Kolmogorov dimension $\log_3 2$ [9].

Caption to color plates

Patterns generated by the evolution of typical one-dimensional cellular automata from disordered initial states. Successive time steps in the evolution are shown on successive horizontal lines. Each site takes on k possible values; value zero is represented by black, 1 by red, 2 by green, 3 by blue, and 4 by yellow. The cellular automata in the first column have $k = 4$ while those in the second column have $k = 5$. In both cases, the range r of the cellular automaton rule is taken to be one. (I am grateful to R. Pike and J. Condon of Bell Laboratories for their help in preparing these figures.)

and a specific temporal measure entropy in analogy with eq. (4.2) by

$$s_\mu^{(t)}(T) = -\frac{1}{T}\sum_{j=1}^{kT} p_j^{(t)} \log_k p_j^{(t)} . \tag{4.21}$$

These entropies satisfy relations directly analogous to these given in eqs. (4.3) through (4.6) for spatial entropies. They obey relations analogous to (4.11) and (4.12) only for cellular automata in "equilibrium", statistically independent of time. The temporal entropies (4.20) and (4.21) may be considered to have units of (k-ary) bits per unit time.

Sequences of values in particular cellular automaton configurations typically have little similarity with the "time series" of values attained by a particular site under cellular automaton evolution. The spatial and temporal entropies for a cellular automaton are therefore in general quite different. Notice that the spatial entropy of a cellular automaton configuration may be considered as the temporal entropy of a pure shift mapping applied to the cellular automaton configuration.

Just as dimensions may be assigned to the set of spatial configurations generated in cellular automaton evolution, so also one may assign dimensions to the set of temporal sequences generated by the evolution. The temporal set dimension may be defined in analogy with eq. (4.15) by

$$d^{(t)} = \lim_{T\to\infty} s^{(t)}(T) , \tag{4.22}$$

and the temporal measure dimension may be defined by

$$d_\mu^{(t)} = \lim_{T\to\infty} s_\mu^{(t)}(T) . \tag{4.23}$$

If the evolution of a cellular automaton is periodic, so that each site takes on a fixed cycle of values, then

$$d^{(t)} = d_\mu^{(t)} = 0 . \tag{4.24}$$

As discussed in section 6 below, class 2 cellular automata yield periodic structures at large times, so that the correspondingly temporal entropies vanish.

As a generalization of the spatial and temporal entropies introduced above, one may consider entropies associated with space-time "patches" in the patterns generated by cellular automaton evolution, as illustrated in fig. 7. With probabilities $p_i^{(t,x)}$ for the k^{XT} possible patches of spatial width X and temporal extent T, one may define a set entropy

$$s^{(t;x)}(T;X) = \frac{1}{T}\log_k\left(\sum_{j=1}^{k^{XT}} \theta(p_j^{(t,x)})\right) , \tag{4.25}$$

and a measure entropy

$$s_\mu^{(t;x)}(T;X) = -\frac{1}{T}\sum_{j=1}^{k^{XT}} p_j^{(t,x)} \log_k p_j^{(t,x)} . \tag{4.26}$$

Clearly,

$$s_{(\mu)}^{(t)}(T) = s_{(\mu)}^{(t;x)}(T;1) , \tag{4.27}$$

$$s_{(\mu)}^{(x)}(X) = \frac{1}{X} s_{(\mu)}^{(t;x)}(1;X) .$$

If no relation existed between configurations at successive time steps then the entropies (4.25) and (4.26) would be bounded simply by

$$s_\mu^{(t;x)}(T;X) \le s^{(t;x)}(T;X) \le X . \tag{4.28}$$

The cellular automaton rules introduce definite relations between successive configurations and tighten this bound. In fact, the values of all sites in a $T \times X$ space-time patch are determined according to the cellular automaton rules by the values in the "rind" of the patch, as indicated in fig. 7. The rind contains only $X + 2r(T-1)$ sites (where r is the "range" of the cellular automaton rule, defined in section 2), so that

$$s_\mu^{(t;x)}(T;X) \le s^{(t;x)}(T;X) \le [X + 2r(T-1)]/T . \tag{4.29}$$

For large T (and fixed X), therefore

$$s_\mu^{(t;x)}(T;X) \leq s^{(t;x)}(T;X) \leq 2r\,. \tag{4.30}$$

If both X and T tend to infinity with T/X fixed, eq. (4.30) implies that the "information per site" $s_\mu^{(t;x)}(T;X)/X$ in a $T \times X$ patch must tend to zero. The evolution of cellular automata can therefore never generate random space-time patterns.

With $T \to \infty$, X fixed, the length X horizontal section of the rind makes a negligible contribution to the entropies. The entropy is maximal if the $2r$ vertical columns in the rind are statistically independent, so that

$$s_{(\mu)}^{(t;x)}(\infty;X) \leq 2rs_{(\mu)}^{(t)}(\infty) = 2rd_{(\mu)}^{(t)}\,. \tag{4.31}$$

In addition,

$$s_{(\mu)}^{(t;x)}(\infty;X) \leq s_{(\mu)}^{(t;x)}(\infty;X+1)\,, \tag{4.32}$$

where the bounds are saturated for large X if the time series associated with different sets of sites are statistically uncorrelated.

The limiting set entropy

$$\mathbf{h} = \lim_{\substack{T\to\infty\\ X\to\infty\\ T/X\to\infty}} s^{(t;x)}(T;X) \tag{4.33}$$

for temporally-extended patches is a fundamental quantity equivalent to the set (or topological) entropy of the cellular automaton mapping in symbolic dynamics. $\mathbf{h}$ may be considered as a dimension for the mapping. It specifies the asymptotic rate at which the number of possible histories for the cellular automaton increases with time. The limiting measure entropy

$$\mathbf{h}_\mu = \lim_{\substack{T\to\infty\\ X\to\infty\\ T/X\to\infty}} s_\mu^{(t;x)}(T;X) \tag{4.34}$$

gives the average amount of "new information" contained in each cellular automaton configuration, and not already determined from previous configurations. Eqs. (4.31) and (4.32) show that

$$d_{(\mu)}^{(t)} \leq \mathbf{h}_{(\mu)} \leq 2rd_{(\mu)}^{(t)}\,. \tag{4.35}$$

In addition,

$$\mathbf{h}_{(\mu)} \leq 2rd_{(\mu)}^{(x)}\,. \tag{4.36}$$

The basic cellular automaton time evolution rule (2.1) implies that the value a_i of a site i at a particular time step depends on sites a maximum distance r away on the previous time step according to the function $\mathbf{F}[a_{i-r}, \ldots, a_{i+r}]$. After T time steps, the values of the site could depend on sites at distances up to rT, so that features in patterns generated by cellular automaton evolution could propagate at "speeds" up to r sites per time step. For many rules, however, the value of a site after many time steps depends on fewer initial site values, and features may propagate only at lower speeds. In general, let $\|\mathbf{F}^T\|$ denote the minimum R for which the value of site i depends only on the initial values of sites $i-R, \ldots, i+R$. Then the maximum propagation speed associated with the cellular automaton rule $\mathbf{F}$ may be defined as

$$\lambda_+ = \overline{\lim_{T\to\infty}}\, \|\mathbf{F}^T\|/T\,. \tag{4.37}$$

(The rule is assumed symmetric; for nonsymmetric rules, distinct lèft and right propagation speeds may be defined.) Clearly,

$$\lambda_+ \leq r\,. \tag{4.38}$$

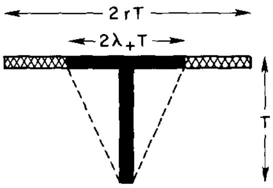

Fig. 8. Pattern of dependence of temporal sequences on spatial sequences, used in the proof of inequalities between spatial and temporal entropies.

When $\lambda_+ = 0$, finite regions of the cellular automaton must ultimately become isolated, so that

$$d^{(t)}_{(\mu)} = \mathbf{h}^{(t)}_{(\mu)} = 0\,. \tag{4.39}$$

The construction of fig. 8 shows that for any T,

$$s^{(t)}_{(\mu)}(T) \le 2rs^{(x)}_{(\mu)}(2rT)\,. \tag{4.40}$$

In the limit $T \to \infty$, the construction implies

$$d^{(t)}_{(\mu)} \le 2\lambda_+ d^{(x)}_{(\mu)}\,, \tag{4.41}$$

The ratio of temporal to spatial entropy is thus bounded by the maximum propagation speed in the cellular automaton. The relation is consistent with the assignment of units to the spatial and temporal entropies mentioned above.

The corresponding inequalities for mapping entropies are:

$$\begin{aligned} &d^{(t)}_{(\mu)} \le \mathbf{h}_{(\mu)} \le 2\lambda_+ d^{(x)}_{(\mu)}\,,\\ &\mathbf{h}_{(\mu)} \le 2rd^{(t)}_{(\mu)}\,. \end{aligned} \tag{4.42}$$

The quantity λ_+ defined by eq. (4.37) gives the maximum speed with which any feature in a cellular automaton may propagate. With many cellular automaton rules, however, almost all "features" propagate much more slowly. To define an appropriate maximum average propagation speed, consider the effect after many time steps of changes in the initial state. Let $G(|x - x'|; t)$ denote the probability that the value of a site at position x' is changed when the value of a site at position x is changed t time steps before. The form of $G(|x - x'|; t)$ for various cellular automaton rules is suggested by fig. 3. $G(|x - x'|; t)$ may be considered as a Green function for the cellular automaton evolution. For large t, $G(|x - x'|; t)$ typically vanishes outside a "cone" defined by $|x - x'| = \bar{\lambda}_+ t$. $\bar{\lambda}_+$ may then be considered as a maximum average propagation speed. In analogy with eqs. (4.41) and (4.42), one expects

$$d^{(t)}_{(\mu)} \le \mathbf{h}_{(\mu)} \le 2\bar{\lambda}_+ d^{(t)}_{(\mu)}\,. \tag{4.43}$$

Mapping and temporal entropies thus vanish for cellular automata with zero maximum average propagation speed. Cellular automata in class 2 have this property.

The maximum average propagation speed $\bar{\lambda}_+$ specifies a cone outside which $G(|x - x'|; t)$ almost always vanishes. One may also define a minimum average propagation speed $\bar{\lambda}_-$, such that $G(|x - x'|; t) > 0$ for almost any $|x - x'| < \bar{\lambda}_-$.

The Green function $G(|x - x'|; t)$ gives the probability that a particular site is affected by changes in a previous configuration. The total effect of changes may be measured by the "Hamming distance" $H(t)$ between configurations before and after the changes, defined as the total number of site values which differ between the configurations after t time steps. ($H(t)$ is analogous to Lyapunov exponents for continuous dynamical systems.) Changing the values of initial sites in a small region, $H(t)$ may be given as a space integral of the Green function, and for large t obeys the inequality

$$H(t)/t \le 2\bar{\lambda}_+\,, \tag{4.44}$$

to be compared with the result (4.43) obtained above.

The definitions and properties of dimension given above suggests that the behaviour these quantities determines the degree of "chaotic" behaviour associated with cellular automaton evolution. "Spatial chaos" occurs when $d^{(x)}_{(\mu)} > 0$, and "temporal chaos" when $d^{(t)}_{(\mu)} > 0$. Temporal chaos requires a nonzero maximum average propagation speed for features in cellular automaton patterns, and implies that small changes in initial conditions lead to effects ever-increasing with time.

5. Class 1 cellular automata

Class 1 cellular automata evolve after a finite number of time steps from almost all initial states to a unique homogeneous state, in which all sites have the same value. Such cellular automata may be considered to evolve to simple "limit points" in phase space; their evolution completely destroys any information on the initial state. The spatial and temporal dimensions for such attractors are zero.

Rules for class 1 cellular automata typically take the function $\mathbf{F}$ of eq. (2.1) to have the same value for almost all of its $k^{(2r+1)}$ possible sets of arguments.

Some exceptional configurations in finite class 1 cellular automata may not evolve to a homogeneous state, but may in fact enter non-trivial cycles. The fraction of such exceptional configurations appears to decrease very rapidly with the size N, suggesting that for infinite class 1 cellular automata the set of exceptional configurations is always of measure zero in the set of all possible configurations. For (legal) class 1 cellular automata whose usual final state has $a_i = n$, $n \neq 0$ (such as code 60 in fig. 1), the null configuration is exceptional for any size N, and yields $a_i = 0$.

6. Class 2 cellular automata

Class 2 cellular automata serve as "filters" which generate separated simple structures from particular (typically short) initial site value sequences*. The density of appropriate sequences in a particular initial state therefore determines the statistical properties of the final state into which it evolves. (There is therefore no unique large-time (invariant) probability measure on the set of possible configurations.) Changes of site values in the initial state almost always affect final site values only within a finite range, typically of order r. The maximum average propagation speed $\bar{\lambda}_+$ defined in section 4 thus vanishes for class 2 cellular automata. The temporal and mapping (but not spatial) dimensions for such automata therefore also vanish.

Although $\bar{\lambda} = 0$ for all class 2 cellular automata, λ is often nonzero. Thus exceptional initial state may exist, from which, for example, unbounded growth may occur. Such initial states apparently occur with probability zero for ensembles of (spatially infinite) cellular automata with smooth probability measures.

The simple structures generated by class 2 cellular automata are either stable, or are periodic, typically with small periods. The class 2 rules with codes 8, 24, 40 and 56 illustrated in fig. 1 all apparently exhibit only stable perisistent structures. Examples of class 2 cellular automata which yield periodic, rather than stable, persistent structures include the $k = 2$, $r = 1$ cellular automaton with rule number 108 [1], and the $k = 3$, $r = 1$ totalistic cellular automaton with code 198. The periods of persistent structures generated in the evolution of class 2 cellular automata are usually less than $k!$. However, examples have been found with larger periods. One is the $k = 2$, $r = 3$ totalistic cellular automata with code 228, in which a persistent structure with period 3 is generated.

The finiteness of the periods obtained at large times in class 2 cellular automata implies that such systems have $d^{(t)}_{(\mu)} = \mathbf{h}_{(\mu)} = 0$, as deduced above from the vanishing of $\bar{\lambda}_+$. The evolution of class 2 cellular automata to zero (temporal) dimension attractors is analogous to the evolution of some continuous dynamical systems to limit cycles.

The set of persistent structures generated by a given class 2 cellular automaton is typically quite simple. For some rules, there are only a finite number of persistent structures. For example, for the code 8 and code 40 rules of fig. 1, only the sequence 111 (surrounded by 0 sites) appears to be persistent. For code 24, 111 and 1111 are both persistent. Other rules yield an infinite sequence of peristent structures, typically constructed by a simple process. For example, with code 56 in fig. 1, any sequence of two or more consecutive 1 sites is persistent.

In general, it appears that the set of persistent structures generated by any class 2 cellular automaton corresponds to the set of words generated

*They are thus of direct significance for digital image processing.

by a regular grammar. A regular grammar [15–18] (or "sofic system" [19]) specifies a regular language, whose legal works may be recognized by a finite automaton, represented by a finite state transition graph. A sequence of symbols (site values) specifies a particular traversal of the state transition graph. The traversal begins at a special "start" node; the symbol sequence represents a legal word only if the traversal does not end at an absorbing "stop" node. Each successive symbol in the sequence causes the automaton to make a transition from one state (node) to one of k others, as specified by the state transition graph. At each step, the next state of the automaton depends only on its current state, and the current symbol read, but not on its previous history.

The set of configurations (symbol sequences) generated from all possible initial configurations by one time step of cellular automaton evolution may always be specified by a regular grammar. To determine whether a particular configuration $a^{(1)}$ may be generated after one time step of cellular automaton evolution, one may attempt to construct an explicit predecessor $a^{(0)}$ for it. Assume that a predecessor configuration has been found which reproduces all site values up to position i. Definite values $a_j^{(0)}$ for all $j \leq i - r$ are then determined. Several of the total of k^{2r} sequences of values $a_{i-r+1}^{(0)}, \ldots, a_{i+r+1}^{(0)}$ may be possible. Each sequence may be specified by an integer $q = \Sigma_{j=0}^{2r} k^j a_{i-r+j+1}^{(0)}$. An integer ψ_i between 0 and $2^{k^{2r}}$ may then be defined, with the qth binary bit in ψ_i equal to one if sequence q is allowed, and 0 otherwise. Each possible value of ψ may be considered to correspond to a state in a finite automaton. $\psi = 0$ corresponds to a "stop" state, which is reached if and only if $a^{(1)}$ has no predecessors. Possible values for $a_{i+r+1}^{(0)}$ are then found from ψ_i and the value of $a_{i+1}^{(0)}$. These possible values then determine the value of ψ_{i+1}. A finite state transition graph, determined by the cellular automaton rules, gives the possible transitions $\psi_i \rightarrow \psi_{i+1}$. Configurations reached after one time step of cellular automaton evolution may thus be recognized by a finite automaton with at most $2^{k^{2r}}$ states. The set of such configurations is thus specified by a regular grammar.

In general, if the value of a given site after t steps of cellular automaton evolution depends on m initial site values, then the set of configurations generated by this evolution may be recognized by a finite automaton with at most 2^{k^m} states. The value of m may increase as $2rt$, potentially requiring an infinite number of states in the recognizing automaton, and preventing the specification of the set of possible configurations by a regular grammar. However, as discussed above, the value of m for a class 2 cellular automaton apparently remains finite as $t \rightarrow \infty$. Thus the set of configurations which may persist in such a cellular automaton may be recognized by a finite automaton, and are therefore specified by a regular grammar. The complexity of this grammar (measured by the minimum number of states required in the state transition graph for the recognizing automaton) may be used to characterize the complexity of the large time behaviour of the cellular automaton.

Finite class 2 cellular automata usually evolve to short period cycles containing the same persistent structures as are found in the infinite case. The fraction of exceptional initial states yielding other structures decreases rapidly to zero as N increases.

7. Class 3 cellular automata

Evolution of infinite class 3 cellular automata from almost all possible initial states leads to aperiodic ("chaotic") patterns. After sufficiently many time steps, the statistical properties of these patterns are typically the same for almost all initial states. In particular, the density of nonzero sites typically tends to a fixed nonzero value (often close to $1/k$). In infinite cellular automata, "equilibrium" values of statistical quantities are approached roughly exponentially with time, and are typically attained to high accuracy after a very few time steps. For a few rules (such as the $k = 2$, $r = 1$ rule with rule number 18 [20]), however,

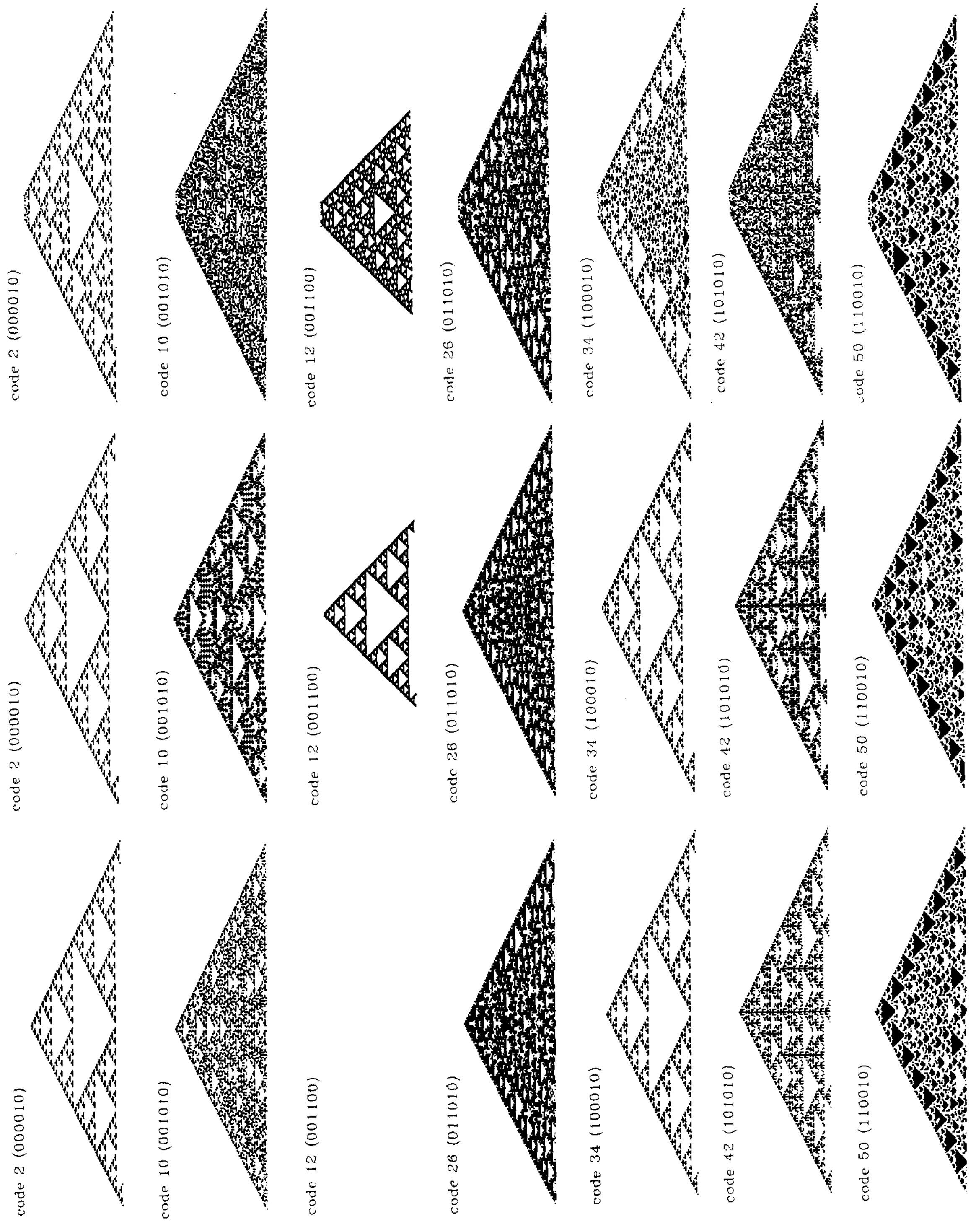

Fig. 9. Evolution of some class 3 totalistic cellular automata with $k = 2$ and $r = 2$ (as illustrated in fig. 1) from initial states containing one or a few nonzero sites. Some cases yield asymptotically self-similar patterns, while others are seen to give irregular patterns.

"defects" consisting of small groups of sites may exist, and may execute approximate random walks, until annihilating, usually in pairs. Such processes lead to transients which decrease with time only as $t^{-1/2}$.

Fig. 1 showed examples of the patterns generated by evolution of some typical class 3 cellular automata from disordered initial states. The patterns range from highly irregular (as for code 10), to rather regular (as for code 12). The most obvious regularity is the appearance of large triangular "clearings" in which all sites have the same value. These clearings occur when a "fluctuation" in which a sequence of consequence of consecutive sites have the same value, is progressively destroyed by the effects of other sites. The rate at which "information" from other sites may "flow" into the fluctuation, and thus the slope of the boundaries of the clearing, may range from $1/k$ to r sites per time step. The qualitative regularity of patterns generated by some class 3 rules arises from the high density of long sequences of correlated site values, and thus of triangular clearings. In general, however, it appears that the density of clearings decreases with their size n roughly as σ^{-n}. Different cellular automata appear to yield a continuous range of σ values. Those with larger σ yield more regular patterns, while those with smaller σ yield more irregular patterns. No sharp distinction appears to exist between class 3 cellular automata yielding regular and irregular patterns.

The first column in fig. 9 shows patterns obtained by evolution with typical class 3 cellular automaton rules from initial states containing a single nonzero site. Unbounded growth, leading to an asymptotically infinite number of nonzero sites, is evident in all cases. Some rules are seen to give highly regular patterns, others lead to irregular patterns.

The regular patterns obtained with rules such as code 2 are asymptotically self-similar fractal curves (cf. [11]). Their form is identical when viewed at different magnifications, down to length scales of order r sites. The total number of nonzero sites in such patterns after t time steps approaches t^d, where d gives the fractal dimension of the pattern. Many class 3 $k = 2$ rules generate a similar pattern, illustrated by codes 2 and 34 in fig. 9, with $d = \log_2 3 \approx 1.59$. Some rules yield self-similar patterns with other fractal dimensions (for example, code 38 yields $d \approx 1.75$), but all self-similar patterns have $d < 2$, and lead to an asymptotic density of sites which tends to zero as t^{d-2}.

Rule such as code 10 are seen to generate irregular patterns by evolution even from a single site initial state. The density of nonzero sites in such patterns is found to tend asymptotically to a nonzero value; in some, but not all, cases the value is the same as would be obtained by evolution from a disordered initial state. The patterns appear to exhibit no large-scale structure.

Cellular automata contain no intrinsic scale beyond the size of neighbourhood which appears in their rules. A configuration containing a single nonzero site is also scale invariant, and any pattern obtained by evolution from it with cellular automaton rules must be scale invariant. The regular patterns in fig. 9 achieve this scale invariance by their self-similarity. The irregular patterns presumably exhibit correlations only over a finite range, and are therefore effectively uniform and scale invariant at large distances.

The second and third columns in fig. 11 shows the evolution of several typical class 3 cellular automata from initial states with nonzero sites in a small region. In some cases (such as code 12), the regular fractal patterns obtained with single nonzero sites are stable under addition of further nonzero initial sites. In other cases (such as code 2) they are seen to be unstable. The numbers of rules yielding stable and unstable fractal patterns are found to be roughly comparable.

Many but not all rules which evolve to regular fractal patterns from simple initial states generate more regular patterns in evolution from disordered initial states. Similarly, many but not all rules which produce stable fractal patterns yield more regular patterns from disordered initial states. For example, code 42 in figs. 1 and 9 generates stable fractal patterns from simple initial state, but

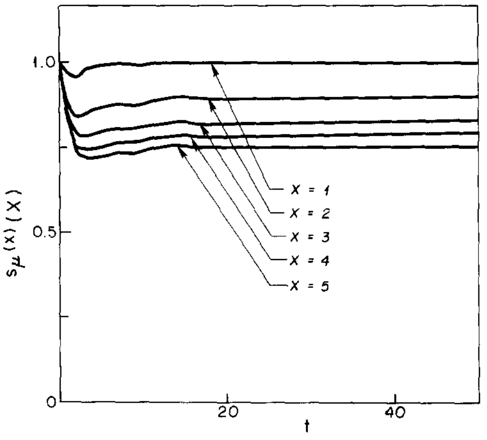

Fig. 10. Evolution of spatial measure entropies $s_\mu^{(x)}(X)$ as a function of time for evolution of the class 3 cellular automaton with code 12 illustrated in fig. 1 from a disordered initial state. The irreversibility of cellular automaton evolution results in a decrease of the entropies with time. Rapid relaxation to an "equilibrium" state is nevertheless seen.

leads to an irregular patterns under evolution from a disordered state. (Although not necessary for such behaviour, this rule possesses the additivity property mentioned in section 2.)

The methods of section 4 may be used to analyse the general behaviour of class 3 cellular automata evolving from typical initial states, in which all sites have nonzero values with nonzero probability. Class 3 cellular automata apparently always exhibit a nonzero minimum average propagation speed $\bar{\lambda}_-$. Small changes in initial states thus almost always lead to increasingly large changes in later states. This suggests that both spatial and temporal dimensions $d_{(\mu)}^{(x)}$ and $d_{(\mu)}^{(t)}$ should be nonzero for all class 3 cellular automata. These dimensions are determined according to eqs. (4.15), (4.18), (4.22) and (4.23) by the limiting values of spatial and temporal entropies.

A disordered or statistically random initial state, in which each site takes on its k possible values with equal independent probabilities, has maximal spatial entropy $s_{(\mu)}^{(x)}(X) = 1$ for all block lengths X. Fig. 10 shows the behaviour of $s_\mu^{(x)}(X)$ as a function of time for several block lengths X in the evolution of a typical class 3 cellular automaton from a disordered (maximal entropy) initial state. The entropies are seen to decrease for a few time steps, and then to reach "equilibrium" values. The "equilibrium" values of $s_\mu^{(x)}(X)$ for class 3 cellular automata are typically independent of the probability measure on the ensemble of possible initial states, at least for "smooth" measures. The decrease in

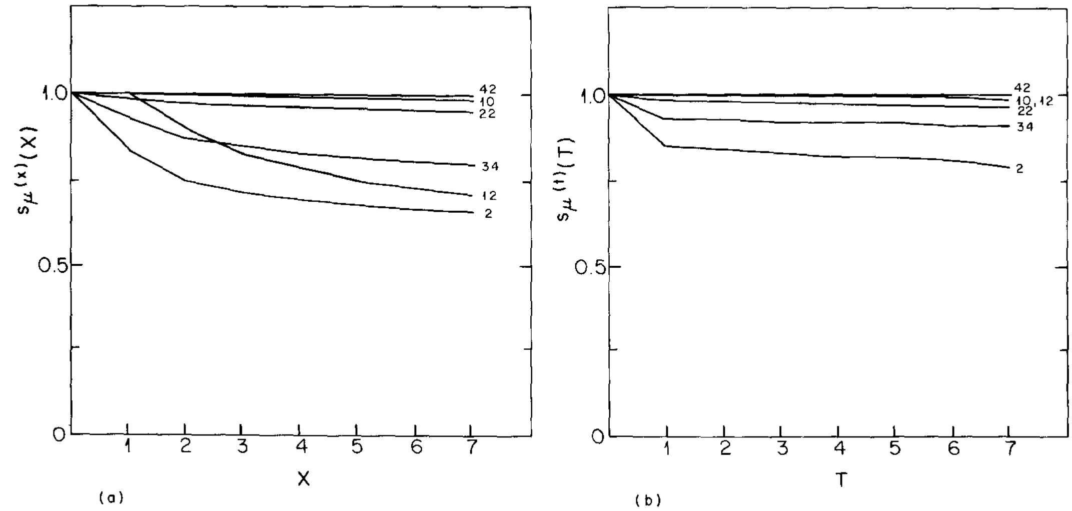

Fig. 11. Evolution of (a) spatial and (b) temporal measure entropies $s_\mu^{(x)}(X)$ and $s_\mu^{(t)}(T)$ obtained at equilibrium by evolution of several class 3 cellular automata illustrated in fig. 1, as a function of the spatial and temporal block lengths X and T. The entropies are evaluated for the region indicated in figs. 7(a) and 7(b). The limit of $s_\mu^{(x)}(X)$ as $X \to \infty$ is the spatial measure dimension of the attractor for the system; the limit of $s_\mu^{(t)}(T)$ as $T \to \infty$ is the temporal measure dimension.

entropy with time manifests the irreversible nature of the cellular automaton evolution. The decrease is found to be much greater for class 3 cellular automata which generate regular patterns (with many triangular clearings) than for those which yield irregular patterns. The more regular patterns require a higher degree of self-organization, with correspondingly greater irreversibility, and larger entropy decrease.

As discussed in section 4, the dependence of $s^{(x)}_{(\mu)}(X)$ on X measures spatial correlations in cellular automaton configurations. $s^{(x)}_{(\mu)}(X)$ therefore tends to a constant if X is larger than the range of any correlations between site values. In the presence of correlations, $s^{(x)}_{(\mu)}(X)$ always decreases with X. Available data from simulations provide reliable accurate estimates for $s^{(x)}_{(\mu)}(X)$ only for $0 \leq X \lesssim 8$. Fig. 11 shows the behaviour of the equilibrium value of $s^{(x)}_{\mu}(X)$ as a function of X over this range for several typical class 3 cellular automata. For rules which yield irregular patterns the equilibrium value of $s^{(x)}_{\mu}(X)$ typically remains $\gtrsim 0.9$ for $X \lesssim 8$. $s^{(x)}_{\mu}(X)$ at equilibrium typically decreases much more rapidly for class 3 cellular automata which generate more regular patterns. At least for small X, $s^{(x)}_{\mu}(X)$ for such cellular automata typically decreases roughly as $X^{-\eta}$ with $\eta \approx 0.1$.

The values of the spatial set entropy $s^{(x)}(X)$ provide upper bounds on the spatial measure entropy $s^{(x)}_{\mu}(X)$. The distribution of nonzero probabilities $p^{(x)}_{i}$ for possible length X blocks is typically quite broad, yielding an $s^{(x)}_{\mu}(X)$ significantly smaller than $s^{(x)}(X)$. Nevertheless, the general behaviour of $s^{(x)}_{\mu}(X)$ with X usually roughly follows $s^{(x)}(X)$, but with a slight X delay.

As discussed in section 4, the set entropy $s^{(x)}(X)$ attains its maximum value of 1 if and only if all k^X sequences of length X appear (with nonzero probability) in evolution from some initial state. Notice that if $s^{(x)}(X) = 1$ after one time step, then $s^{(x)}(X) = 1$ at any time. In general, $s^{(x)}(X)$ takes on value 1 for blocks up to some critical length X_c (perhaps infinite), as defined in eq. (4.13).

Since a block of length X is completely determined by a sequence of length $X + 2r$ in the previous configuration, any predecessors for the block may in principle be found by an exhaustive search of all k^{X+2r} possible length $X + 2r$ sequences. The procedure for progressive construction of predecessors outlined in section 6 provides a more efficient procedure [21]. The critical block length X_c is determined by the minimum number of nodes in the finite automaton state transition graph visited on any path from the "start" to "stop" node. The state transition graph is determined by the set of transition rules $\Psi_i \to \Psi_{i+1}$. Starting with length 1 blocks, these transition rules may be found by considering construction of all possible progressively longer blocks, but ignoring blocks associated with values Ψ_i for which the transition rules have already been found. If X_c is finite, the "stop" node $\Psi = 0$ is reached in the construction of length X_c blocks. Alternatively, the state transition graph may be found to consist of closed cycles, not including $\Psi = 0$. In this case, X_c is determined to be infinite. Since the state transition graph contains at most $2^{k^{2r}}$ nodes, the value of X_c may be found after at most this many tests. The procedure thus provides a finite algorithm for determining whether all possible arbitrarily long sequences of site values may be generated by evolution with a particular cellular automaton rule.

Table II gives the critical block lengths X_c for the cellular automata illustrated in fig. 1. Class 3 cellular automata with smaller X_c tend to generate more regular patterns. Those with larger X_c presumably give systematically larger entropies and their evolution is correspondingly less irreversible.

For additive cellular automata (such as code 42 in fig. 1 and table II), all possible blocks of any length X may be reached, and have exactly k^{2r} predecessors of length $X + 2r$. In this case, therefore, evolution from a disordered initial state gives $s^{(x)}(X) = 1$ for all X (hence $X_c = \infty$). The equality of the number of predecessors for each block implies in addition in this case that $s^{(x)}_{\mu}(X) = 1$, at least for evolution from disordered initial states. Hence for additive cellular automata

$$d^{(x)} = d^{(x)}_{\mu} = 1 \,. \tag{7.1}$$

Table II
Values of critical block length X_c for legal totalistic $k = 2$, $r = 2$ cellular automata as illustrated in fig. 1. For $X < X_c$, all k^X possible blocks of X site values appear with nonzero probability in configurations generated after any number of time steps in evolution from disordered initial states, while for $X \geq X_c$, some blocks are absent, so that the spatial set entropy $s^{(x)}(X) < 1$

Code	X_c	Code	X_c
2	5	32	3
4	12	34	5
6	7	36	12
8	12	38	7
10	36	40	12
12	5	42	∞
14	5	44	5
16	5	46	5
18	5	48	5
20	36	50	5
22	12	52	22
24	7	54	12
26	12	56	7
28	5	58	12
30	3	60	5

The configurations generated by additive cellular automata are thus maximally chaotic.

In general cellular automata evolving according to eq. (2.1) yield $s^{(x)}(X) = 1$ for all X, so that $d^{(x)} = 1$, if **F** is an injective (one-to-one) function of either its first or last argument (or can be obtained by composition of functions with such a property). This may be proved by induction. Assume that all the blocks of length X are reachable, with predecessors of lengths $X + 2r$. Then form a block of length $X + 1$ by adding a site at one end. To obtain all possible length $X + 1$ blocks, the value a' of this additional site must range over k possibilities. Any predecessors for length $X + 1$ blocks must be obtained by adding a $(X + 2r + 1)$-th site (with value a) at one end. For all length $X + 1$ blocks to be reachable, all values of a' must be generated when a runs over its k possible values, and the result follows. Notice that not all length $X + 1$ blocks need have the same (nonzero) number of predecessors,· so that the measure entropy $s_\mu^{(x)}(X)$ may be less than the set entropy $s^{(x)}(X)$.

While injectivity of the rule function **F** for a cellular automaton in its first or last arguments is sufficient to give $d^{(x)} = 1$, it is apparently not necessary. A necessary condition is not known.

In section 6 it was shown that the set of configurations obtained by cellular automaton evolution for a finite number of time steps from any initial state could be specified by a regular grammar. In general the complexity of the grammar may increase rapidly with the number of time steps, potentially leading at infinite time to a set not specifiable by a regular grammar. Such behaviour may generically be expected in class 3 cellular, for which the average minimum propagation speed $\bar{\lambda} > 0$.

As discussed in section 4, one may consider the statistics of temporal as well as spatial sequences of site values. The temporal aperiodicity of the patterns generated by evolution of class 3 cellular automata from almost all initial states suggests that these systems should have nonvanishing temporal entropies $s_{(\mu)}^{(t)}(T)$ and nonvanishing temporal dimensions $d_{(\mu)}^{(t)}$. Once again, the temporal entropies for blocks starting at progressively later times quickly relax to equilibrium values. Notice that the dimension $d_{(\mu)}^{(t)}$ obtained from the large T limit of the $s_{(\mu)}^{(t)}(T)$ is always independent of the starting times for the blocks. This is to be contrasted with the spatial dimensions $d_{(\mu)}^{(x)}$, which depend on the time at which they are evaluated. Just as for spatial entropies, it found that the equilibrium temporal entropies are essentially independent of probability measure for initial configurations.

The temporal entropies $s_{(\mu)}^{(t)}(T)$ decrease slowly with T. In fact, it appears that in all cases

$$s_{(\mu)}^{(t)}(Z) \geq s_{(\mu)}^{(x)}(Z) . \tag{7.2}$$

The ratio $s_{(\mu)}^{(t)}(Z)/s_{(\mu)}^{(x)}(Z)$ is, however, typically much smaller than its maximum value (4.38) equal to the maximum propagation speed λ_+. Notice that the value of λ_+ determines the slopes of the edges of triangular clearings in the patterns generated by cellular automaton evolution.

At least for the class 3 cellular automata in fig. 1 which generate irregular patterns, the equi-

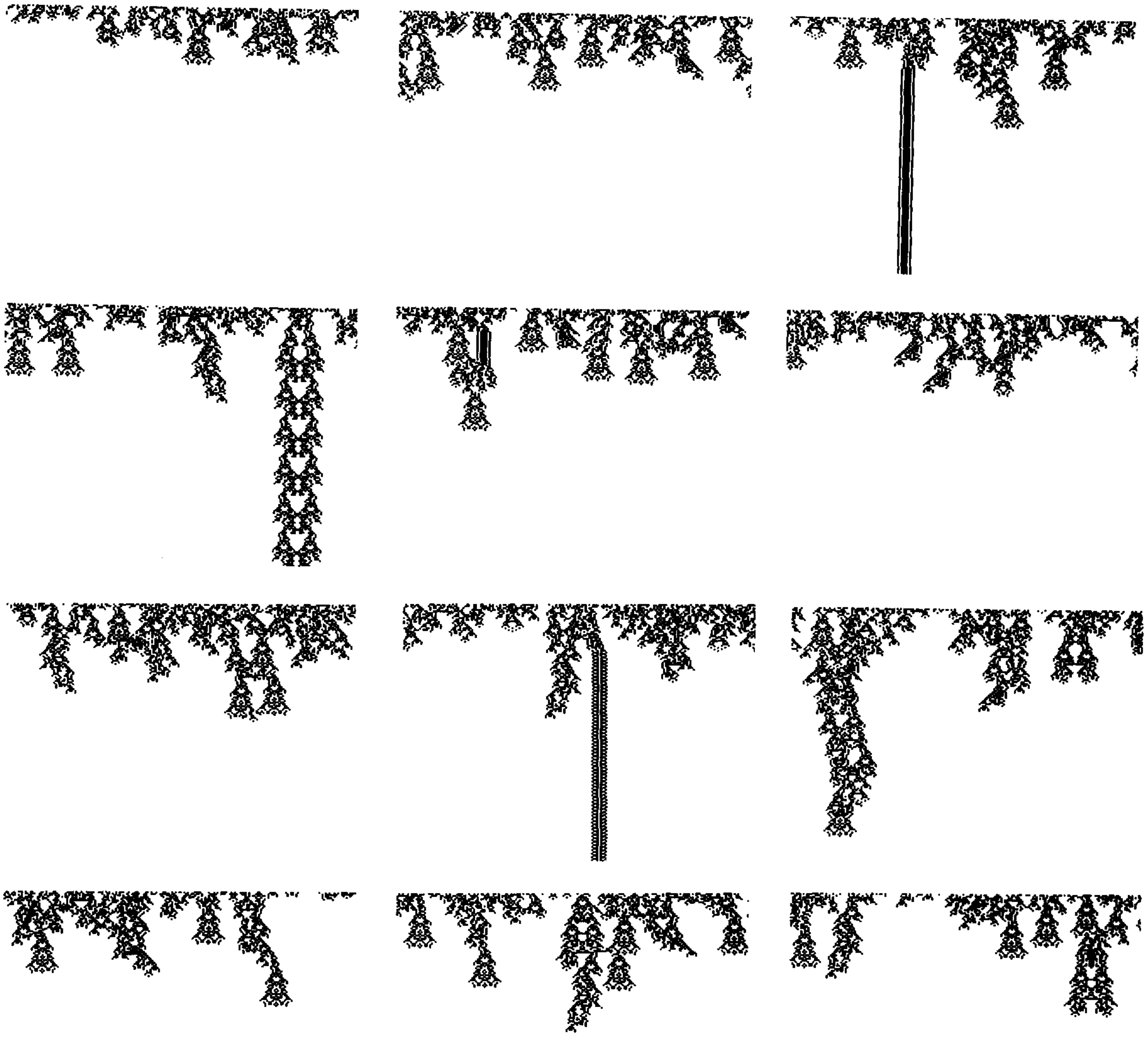

Fig. 12. Examples of the evolution of a class 4 cellular automaton (totalistic code 20 $k = 2$, $r = 2$ rule) from several disordered initial states. Persistent structures are seen to be generated in a few cases. The evolution is truncated after 120 time steps.

librium set entropy $s^{(t)}(T) = 1$ for all $T \lesssim 8$ for which data are available. Note that the result $s^{(t)}(T) = 1$ holds for all T for any additive cellular automaton rule. One may speculate that class 3 cellular automata which generate apparently irregular patterns form a special subclass, characterized by temporal dimension $d^{(t)} = 1$.

For class 3 cellular automata which generate more regular patterns, $s_{(t)}(T)$ appears to decrease, albeit slowly, with T. Just as for spatial sequences, one may consider whether the temporal sequences which appear form a set described by a regular grammar. For the particular case of the $k = 2$, $r = 1$ cellular automaton with rule number 18, there is some evidence [21] that all possible temporal sequences which contain no 11 subsequences may appear, so that $N^{(t)}(T) = F_T$ where F_T is the Tth Fibonacci number ($F_T = F_{T-1} + F_{T-2}$, $F_0 = F_1 = 1$). This implies that $N_{(t)}(T) \sim \phi^T$ ($\phi = (\sqrt{5}+1)/2 \simeq 1.618$) for large T, suggesting a temporal set dimension $d^{(t)} = \log_2 \phi \approx 0.694$. In general, however, the set of possible temporal sequences is not expected to be described by a regular grammar.

The nonvanishing value of the average minimum propagation speed $\bar{\lambda}_-$ for class 3 cellular automata, suggests that in all cases the value of a particular site depends on an ever-increasing number of initial site values. However, the complexity of the dependence is not known. The value of a site after t time steps can always be specified by a table with an entry for each of $k^{2\lambda+t}$ relevant initial sequences. Nevertheless, it is possible that a finite state automaton, specified by a finite state transition graph, could determine the value of sites at any time

The behaviour of finite class 3 cellular automata with additive rules was analysed in some detail in ref. 2. It was shown there that the maximal cycle length for additive cellular automata grows on average exponentially with the size N of the cellular automaton. Most cycles were found to have maximal length, and the number of distinct cycles was found also to grow on average exponentially with N. The lengths of transients leading to cycles was found to grow at most linearly with N. The fraction of states on cycles was found on average to tend a finite limit.

For most class 3 cellular automata, the average cycle length grows quite slowly with N, although in some cases, the absolute maximum cycle length appears to grow rapidly. The lengths of transients are typically short for cellular automata which generate more regular patterns, but often become very long as N increases for cellular automata which generate more irregular patterns. The fractions of states on cycles are typically much larger for finite class 3 cellular automata which generate irregular patterns than for those which generate more regular patterns. This is presumably a reflection of the lower irreversibility and larger attractor dimension found for the former case in the infinite size limit.

8. Class 4 cellular automata

Fig. 12 shows the evolution of the class 4 cellular automaton with $k = 2$, $r = 2$ and code number 20, from several disordered initial configurations. In most cases, all sites are seen to "die" (attain value zero) after a finite time. However, in a few cases, stable or periodic structures which persist for an infinite time are formed. In addition, in some cases, propagating structures are formed. Fig. 13 shows the persistent structures generated by this cellular automaton from all initial configurations whose nonzero sites lie in a region of length 20 (reflected versions of the last three structures are also found). Table III gives some characteristics of these structures. An important feature, shared by other class 4 cellular automata, is the presence of propagating structures. By arranging for suitable reflections of these propagating structures, final states with any cycle lengths may be obtained.

The behaviour of the cellular automata illustrated in fig. 13, and the structures shown in fig. 14 are strongly reminiscent of the two-dimensional (essentially totalistic) cellular automaton known as the "Game of Life"* (for references see [1]). The Game of Life has been shown to have the important property of computational universality. Cellular automata may be viewed as computers, in which data represented by initial configurations is processed by time evolution. Computational universality (e.g. [15–18]) implies that suitable initial configurations can specify arbitrary algorithm procedures. The system can thus serve as a general purpose computer, capable of evaluating any (computable) function. Given a suitable encoding, the system may therefore in principle simulate any other system, and in this sense may be considered capable of arbitrarily complicated behaviour.

The proof of computational universality for the Game of Life [22] uses the existence of cellular

*Each site in this cellular automaton can take on one of two possible values; the time evolution rule involves nine site (type II) neighbourhoods. If the values of less than 2 or more than 3 of the eight neighbours of a particular site are nonzero then the site takes on value 0 at the next time step; if 2 neighbouring sites are nonzero the site takes the same value as on the previous time steps; if exactly 3 neighbouring sites are nonzero, the site takes on value 1.

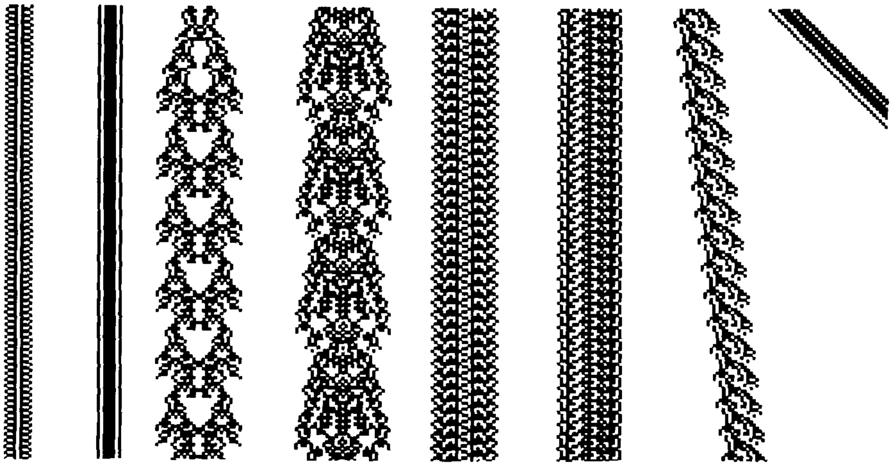

Fig. 13. Persistent structures found in the evolution of the class 4 cellular automaton illustrated in fig. 12 from initial states with nonzero sites in a region of 20 or less sites. Reflected versions of the last three structures are also found. Some properties of the structures are given in table III. These structures are almost sufficient to provide components necessary to demonstrate a universal computation capability for this cellular automaton.

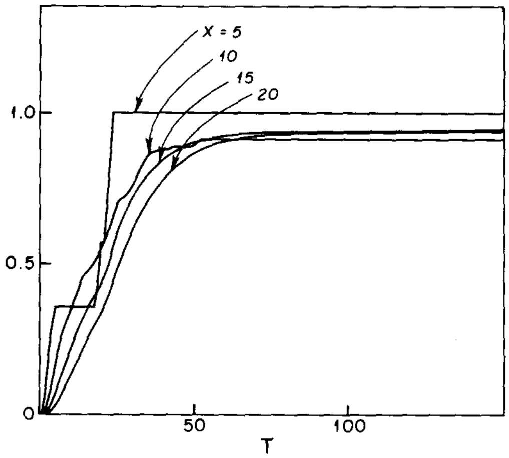

Fig. 14. Fraction of configurations in the class 4 cellular automaton of figs. 12 and 13 which evolve to the null configuration after T time steps, from initial states with nonzero sites in a region of length less than X (translates of configurations are not included). The asymptotic "halting probability" is around 0.93; 7% of initial configurations generate the persistent structures of fig. 13 and never evolve to the null configuration.

automaton structures which emulate components (such as "wires" and "NAND gates") of a standard digital computer. The structures shown in fig. 14 represent a significant fraction of those necessary. A major missing element is a configuration (dubbed the "glider gun" in the Game of Life) which acts like a clock, and generates an infinite sequence of propagating structures. Such a configuration would involve a finite number of initial nonzero sites, but would lead to unbounded growth, and an asymptotically infinite number of nonzero sites. There are however indications that the required initial configuration is quite large, and is very difficult to find.

These analogies lead to the speculation that class 4 cellular automata are characterized by the capability for universal computation. $k = 2$, $r = 1$ cellular automata are too simple to support universal computation; the existence of class 4 cellular automata with $k = 2$, $r = 2$ (cf. figs. 12 and 13) and $k = 3$, $r = 1$ suggests that with suitable time evolution rules even such apparently simple systems may be capable of universal computation.

There are important limitations on predictions which may be made for the behaviour of systems capable of universal computation. The behaviour of such systems may in general be determined in detail essentially only by explicit simulation of their time evolution. It may in general be predicted using other systems only by procedures ultimately equivalent to explicit simulation. No finite algo-

Table III
Persistent structures arising from initial configurations with length less than 20 sites in the class 4 totalistic cellular automaton with $k = 2$, $r = 2$ and code number 20, illustrated in figs. 12, 13 and 14. $\phi(X)$ gives the fraction of initial configurations with nonzero sites in a region less than X sites in length which generate a particular structure. When an initial configuration yields multiple structures, each is included in this fraction.

Period	Minimal predecessor	$\phi(10)$	$\phi(20)$
2	10010111 (151)	0.027	0.024
9R	10111011 (187)	0.012	0.0061
1	10111101 (189)	0.014	0.0075
22	11000011 (195)	0.018	0.017
9L	11011101 (221)	0.012	0.0061
1R	1001111011 (635)	0.0020	0.00066
1L	1101111001 (889)	0.0020	0.00066
38	11110100100101111 (125231)	0	2.9×10^{-5}
4	10010001011011110111 (595703)	0	7.6×10^{-6}
4	10010101001010110111 (610999)	0	7.6×10^{-6}
4	10011000011111101111 (624623)	0	7.6×10^{-6}

rithm or procedure may be devised capable of predicting detailed behaviour in a computationally universal system. Hence, for example, no general finite algorithm can predict whether a particular initial configuration in a computationally universal cellular automaton will evolve to the null configuration after a finite time, or will generate persistent structures, so that sites with nonzero values will exist at arbitrarily large times. (This is analogous to the insolubility of the halting problem for universal Turing machines (e.g. [15–18]).) Thus if the cellular automaton of figs. 12 and 13 is indeed computationally universal, no finite algorithm could predict whether a particular initial state would ultimately "die", or whether it would ultimately give rise to one of the persistent structures of fig. 13. The result could not be determined by explicit simulation, since an arbitrarily large time might elapse before one of the required states was reached. Another universal computer could also in general determine the result effectively only by simulation, with the same obstruction.

If class 4 cellular automata are indeed capable of universal computation, then their evolution involves an element of unpredictability presumably not present in other classes of cellular automata. Not only does the value of a particular site after many time steps potentially depend on the values of an increasing number of initial site values; in addition, the value cannot in general be determined by any "short-cut" procedure much simpler than explicit simulation of the evolution. The behaviour of a class 4 cellular automaton is thus essentially unpredictable, even given complete initial information: the behaviour of the system may essentially be found only by explicitly running it.

Only infinite cellular automata may be capable of universal computation; finite cellular automata involve only a finite number of internal states, and may therefore evaluate only a subset of all computable functions (the "space-bounded" ones).

The computational universality of a system implies that certain classes of general predictions for its behaviour cannot be made with finite algorithms. Specific predictions may nevertheless often be made, just as specific cases of generally noncomputable function may often be evaluated. Hence, for example, the behaviour of all configurations with nonzero sites in a region of length 20 or less evolving according to the cellular automaton rules illustrated in figs. 12 and 13 has been completely determined. Fig. 14 shows the fraction of initial configurations which evolve to the null state within T time steps, as a function of T, for various sizes X of the region of nonzero sites. For large X and large T, it appears that the fraction of configurations which generate no persistent structures (essentially the "halting probability") is approximately 0.93. It is noteworthy that the curves in fig. 14 as a function of T appear to approach a fixed form at large X. One may speculate that some aspects of the form of such curves may be universal to all systems capable of universal computation.

The sets of persistent structures generated by class 4 cellular automata typically exhibit no simple patterns, and do not appear to be specified, for example, by regular grammars. Specification of persistent structures by a finite procedure is necessarily impossible if class 4 cellular automata are indeed capable of universal computation. Strong support of the conjecture that class 4 cellular automata are capable of universal computation would be provided by a demonstration of the equivalence of systematic enumeration of all persistent structures in particular class 4 cellular automata to the systematic enumeration of solutions to generally insoluble Diophantine equations or word problems.

Although one may determine by explicit construction that specific cellular automata are capable of universal computation, it is impossible to determine in general whether a particular cellular automaton is capable of universal computation. This is a consequence of the fact that the structures necessary to implement universal computation may be arbitrarily complicated. Thus, for example, the smallest propagating structure might involve an arbitrarily long sequence of site values.

For class 1, 2 and 3 cellular automata, fluctuations in statistical quantities are typically found to become progressively smaller as larger numbers of sites are considered. Such systems therefore exhibit definite properties in the "infinite volume" limit. For class 4 cellular automata, it seems likely that fluctuations do not decrease as larger number of sites are considered, and no simple smooth infinite volume limit exists. Important qualitative effects can arise from special sequences appearing with arbitrarily low probabilities in the initial state. Consider for example the class 4 cellular automaton illustrated in figs. 12 and 13. The evolution of the finite sequences in this cellular automaton shown in fig. 12 (and many thousands of other finite sequences tested) suggests that the average density of nonzero sites in configurations of this cellular automaton should tend to a constant at large times. However, in a sufficiently long finite initial sequence, there should exist a subsequence from which a "glider gun" structure evolves. This structure would generate an increasing number of nonzero sites at large times, and its presence would completely change the average large time density. As a more extreme example, it seems likely that a sufficiently long (but finite) initial sequence should evolve to behave as a self-reproducing "organism", capable of eventually taking over its environment, and leading to completely different large time behaviour. Very special, and highly improbable, initial sequences may thus presumably result in large changes in large time properties for class 4 cellular automata. These sequences must appear in a truly infinite (typical) initial configuration. Although their density is perhaps arbitrarily low, the sequences may evolve to structures which come to dominate the statistical properties of the system. The possibility of such phenomena suggest that no smooth infinite volume exists for class 4 cellular automata.

Some statistical results may be obtained from large finite class 4 cellular automata, although the results are expected to be irrelevant in the truly infinite volume limit. The evolution of most class 4 cellular automata appears to be highly irreversible*. This irreversibility is reflected in the small set of persistent structures usually generated as end-products of the evolution. Changes in small regions of the initial state may affect many sites at

*This feature allows practical simulation of such cellular automata to be made more efficient by storing information on the evolution of the specific sequences of sites which occur with larger probabilities (cf. [23]).

large times. There are however very large fluctuations in the propagation speed, and no meaningful averages may be obtained. It should be noted that groups of class 4 cellular automata with different rules often yield qualitatively similar behaviour, and similar sets of persistent structures, suggesting further classification.

The frequency with which a particular structure is generated after an infinite time by the evolution of a universal computer from random (disordered) input gives the "algorithmic probability" p_A [24] for that structure. This algorithmic probability has been shown to be invariant (up to constant multiplicative factors) for a wide class of universal computers. In general, one may define an "evolutionary probability" $p_E(t)$ which gives the probability for a structure to evolve after t time steps from a random initial state. Complex structures formed by cellular automata will typically have evolutionary probabilities which are initially small, but later grow. As a simple example, the probability for the sequence which yields a period 9 propagating structure in the cellular automaton of figs. 12 and 13 begins small, but later increases to a sufficiently large value that such structures are almost always generated from disordered states of 2000 or more sites. In a much more complicated example, one may imagine that the probability for a self-reproducing structure begins small, but later increases to a substantial value. Structures whose evolutionary probability becomes significant only after a time $>T$ may be considered to have "logical depth" [25] T.

9. Discussion

Cellular automata are simple in construction, but are capable of very complex behaviour. This paper has suggested that a considerable universality exists in this complex behaviour. Evidence has been presented that all one-dimensional cellular automata fall into four basic classes. In the first class, evolution from almost all initial states leads ultimately to a unique homogeneous state. The second class evolves to simple separated structures. Evolution of the third class of cellular automata leads to chaotic patterns, with varying degrees of structure. The behaviours of these three classes of cellular automata are analogous to the limit points, limit cycles and chaotic ("strange") attractors found in continuous dynamical systems. The fourth class of cellular automata exhibits still more complicated behaviour, and its members are conjectured to be capable of universal computation.

Even starting from disordered or random initial configurations, cellular automata evolve to generate characteristic patterns. Such self-organizing behaviour occurs by virtue of the irreversibility of cellular automaton evolution. Starting from almost any initial state, the evolution leads to attractors containing a small subset of all possible states. At least for the first three classes of cellular automata, the states in these attractors form a Cantor set, with characteristic fractal and other dimensions. For the first and second classes, the states in the attractor may be specified as sentences with a regular grammar. For the fourth class, the attractors may be arbitrarily complicated, and no simple statistical characterizations appear possible.

The four classes of cellular automata may be distinguished by the level of predictability of their "final" large time behaviour given their initial state. For the first class, all initial states yield the same final state, and complete prediction is trivial. In the second class, each region of the final state depends only on a finite region of the initial state; knowledge of a small region in the initial state thus suffices to predict the form of a region in the final state. In the evolution of the third class of cellular automata, the effects of changes in the initial state almost always propagate forever at a finite speed. A particular region thus depends on a region of the initial state of ever-increasing size. Hence any prediction of the "final" state requires complete knowledge of the initial state. Finally, in the fourth class of cellular automata, regions of the final state again depend on arbitrarily large regions of the initial state. However, if cellular automata in the class are indeed capable of universal computation, then this dependence may be arbitrarily complex, and the behaviour of the system can be found by no procedure significantly simpler than direct simulation. No meaningful prediction is therefore possible for such systems.

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