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A COMPARISON OF STEADY-STATE LAND-DRAINAGE EQUATIONS

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ABSTRACT

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The physical assumptions and mathematical approximations leading to ten steady-state drainage equations for installations of parallel cylindrical drains laid above a horizontal impermeable barrier, are critically examined. Water-table heights were calculated from the equations for a range of depths of impermeable barrier and of rainfall rates, using the drain's radius obtained by the hodograph analysis for infinite depth of soil. A consideration of the available solutions show that the water-table height is known with sufficient accuracy for small depths of impermeable barrier and for large depths, but at intermediate depths the solutions lead to an unsatisfactorily large uncertainty in its position. Of the drainage equations Houghoudt's equivalent depth equation, when used with the optimum drain radius given by the hodograph analysis for infinite soil depth, is the only one that gives results contained mainly within the known bounds that result from a consideration of the combination of equations.

INTRODUCTION

Solutions to Laplace's equation for the hydraulic potential in problems of groundwater flow in drained lands provide a rational basis for the design of drainage installations by giving the water-table heights for known inputs of water from surface precipitation and artesian flow. In particular, these solutions yield "drainage equations" that relate the maximum water-table height to a steady uniform rainfall on the soil surface.

The first drainage equation was derived by Colding (1872). By assuming horizontal flow in the groundwater region that is fed by steady uniform rainfall, he obtained an elliptic equation for the shape of the water table between parallel drain lines, with the maximum water-table height proportional to the square root of the rainfall rate. While the assumption of horizontal flow must make this relationship approximate, only during the

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last few decades have drainage equations been derived that do not make this assumption and allow for a vertical component of flow. However, even today certain physical and mathematical assumptions have generally to be made in the derivation of drainage equations, leading to different mathematical forms for the same drainage situation. The drainage engineer is thus faced with a number of different design equations that are claimed to give correct drain spacings, without any guide to their applicability. Kirkham (1966) reviewed steady-state drainage theories for parallel drain lines and commented on the need for the various equations to be compared by computing numerical values for the same drainage geometry. While Wesseling (1964) and Hammad (1962) have made comparisons of some equations, there has been no comprehensive examination and comparison of the collection of equations that Kirkham reviewed. In this paper we attempt to give this comparison of drainage equations that Kirkham (1966) suggested was required. A critical examination of the physical and mathematical assumptions, used in the derivation of the many different drainage equations that have been proposed, is made, and the equations compared numerically.

THE PHYSICAL PROBLEM

The physical situation that is considered here is the two-dimensional one most commonly analysed in land-drainage theory, shown in Fig. 1. Uniform steady rainfall is incident on the surface of a uniform soil and percolates through the soil to an installation of parallel cylindrical drains laid above a horizontal impermeable boundary. A water table divides the saturated groundwater zone, where the soil-water pressure is positive and the potential ϕ of the soil water is described by Laplace's equation:

$$\nabla^2 \phi = 0 \tag{1}$$

from the unsaturated soil-water zone above, where the soil-water pressure is negative and the flow is described in the steady state by

$$\text{div}\,(K_u \,\text{grad}\, \phi) = 0 \tag{2}$$

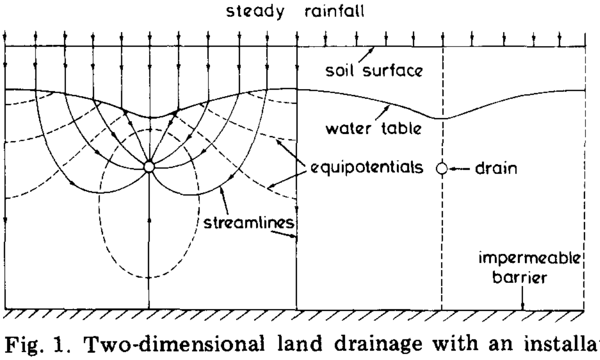

Fig. 1. Two-dimensional land drainage with an installation of parallel cylindrical drains.

where K_u is the hydraulic conductivity, dependent on the water content. The flow region consists of both zones, with boundary conditions given by the flux through the soil surface and by the imposed potential at the surface of the drain. Numerical methods can be used to solve for the potential ϕ, and hence the magnitude and direction of the flux, at any point in the complete flow region, given the soil-water characteristics. However, for analytical solutions concerning the groundwater zone, it is convenient to consider each zone separately. Thus, to obtain the extent of the saturated groundwater zone, we have to consider only the boundary conditions imposed on this zone which is bounded on top by the water table.

All the analytical solutions to the drainage problems have been obtained by making two simplifying assumptions, that have been validated by experiment, concerning the boundary conditions. The first assumption is that water percolates vertically from the soil surface through the unsaturated soil-water zone, so that the uniform flux through the soil surface is also that through the water table. This was shown to be very nearly the case in electric analogue experiments done by Childs (1945) and in a numerical study by Gureghian and Youngs (1976). The boundary condition at the water table is then a uniform flux together with a potential equal to the water-table height. The other assumption concerns the potential condition at the surface of the drain channel. In all analyses the drain pipe is supposed to be equivalent to a line sink at infinitely negative potential of strength $Q/4\pi K$, where Q is the drain flow per unit length of drain and K the hydraulic conductivity of the surrounding soil, with the surface of the drain coinciding with an equipotential surface around this sink. For small radii, the deviation of the equipotentials from a cylindrical shape is small, so that the replacement of the drain by a line sink in the analyses gives negligible error so long

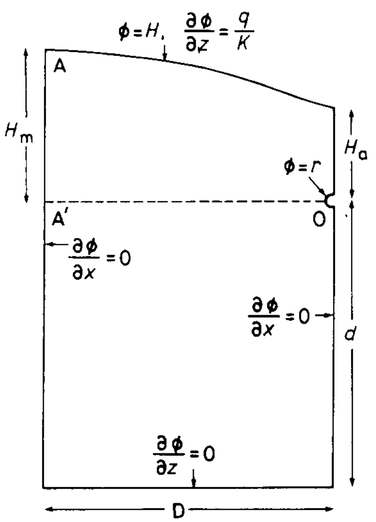

Fig. 2. Groundwater region of the drainage problem.

as the wall of the cylindrical drain is continuously permeable and the drain is running full. However, the common agricultural drain is not a continuously permeable cylinder but is formed by clay-tile pipes butted together or from plastic tubing with small perforations, so that the water flow converges to the openings with the result that the drain wall does not approximate to an equipotential cylinder. Nonetheless, experiments have shown that these non-ideal gappy drains may be considered to act as continuously permeable cylindrical drains with an equivalent drain radius usually much smaller than the actual drain radius (Childs and Youngs, 1958; Youngs, 1974, 1983).

The flow region is shown in Fig. 2 for the drainage problem to which solutions of Equations (1) are sought with boundary conditions:

$$\left.\begin{aligned} \phi &= H \\ K\frac{\partial \phi}{\partial z} &= q \end{aligned}\right\} \text{ at the water table}$$

$$\begin{aligned} \phi &= r && \text{on the drain wall} \\ \frac{\partial \phi}{\partial x} &= 0 && \text{on vertical planes of symmetry} \\ \frac{\partial \phi}{\partial z} &= 0 && \text{at the impermeable barrier} \end{aligned} \tag{3}$$

where the potential ϕ, expressed as a head of water, is measured from a datum level through the drain axis, H is the water-table height above the datum level, q the uniform rainfall rate assumed to be the vertical flux through the water table, and r the equivalent radius of the drain channel. In writing the potential condition $\phi = r$ at the drain wall, the drain is assumed to be running just full with the soil-water pressure zero at the top of the drain. Solutions are of the form:

$$q/K = f(H/D, d/D, r/D) \tag{4}$$

where d is the depth of the impermeable barrier below the datum level and $2D$ is the spacing of the drains.

INFINITE DEPTH OF SOIL ($d/D \rightarrow \infty$)

Hodograph solution

Where there is infinite depth of soil below the drains, solutions to the drainage problems shown in Fig. 2 are possible by means of conformal transformations using the hodograph method (Van Deemter, 1950; Engelund, 1951; Childs, 1959, 1969). The analysis shows that the height and shape of the water table is dependent on the drain radius and that there is an optimum drain radius for which the water is at its lowest level and is drawn down to the top of the drain channel. For drains smaller or larger than this optimum size (assuming the drains to be running full without back pressure) the water table is raised overall and is flatter with less difference between maximum and minimum heights, midway and immediately above the drains respectively.

The water-table heights H_m and H_a, midway and immediately above the drain lines, are given by:

$$\frac{H_m}{D} = \frac{1}{\pi}\left[\ln\left(1+\frac{2}{\lambda}\right) + \frac{2}{\gamma}\ln\left(1+\frac{\lambda}{2}\right)\right] \tag{5}$$

and

$$\frac{H_a}{D} = \frac{H_m}{D} - \frac{2}{\pi\gamma}\ln(1+\lambda) \tag{6}$$

where

$$1+\gamma = K/q \tag{7}$$

and

$$1+\lambda = \frac{(K/q)\tan\theta + [(K/q)^2\tan^2\theta + 1]^{1/2}}{\tan\theta + (1+\tan^2\theta)^{1/2}} \tag{8}$$

In Equation (8), θ is the maximum slope that the water table makes with the horizontal. The hydrostatic pressure head p at a height z immediately above the drain axis is given by:

$$\frac{p}{D} = \frac{1}{\pi}\left[\frac{q}{K}\ln\left(\frac{\delta}{2-\delta}\right) + \ln\left(\frac{\lambda+2-\delta}{\lambda+\delta}\right)\right] \tag{9}$$

where δ is given by

$$\frac{z}{D} = \frac{1}{\pi}\left[\ln\left(\frac{\lambda+\delta}{\lambda}\right) + \left(1+\frac{2}{\gamma}\right)\ln\left(\frac{\lambda+2}{\lambda+2-\delta}\right) + \frac{2}{\gamma}\ln\left(1-\frac{\delta}{2}\right)\right] \tag{10}$$

Since the drain is just running full, $p = 0$ at $z = r$, giving the value of δ from Equation (9) to use in Equation (10) to find the drain radius r. Another equipotential for which $p = 0$ above the drain axis, passes through $z = H_a$, so that another possible drain periphery coincides with this equipotential; that is, the drain with radius $r = H_a$. For optimum conditions, the water table is drawn down to the top of the drain and only one drain radius r_o exists. Then $\theta = \pi/2$ so that $\lambda = \gamma$ and Equations (5) and (6) become:

$$\frac{H_m}{D} = \left[\frac{1}{\pi}\ln\left(1+\frac{2}{\gamma}\right) + \frac{2}{\gamma}\ln\left(1+\frac{\gamma}{2}\right)\right] \tag{11}$$

and

$$\frac{r_o}{D} = \frac{H_m}{D} - \frac{2}{\pi\gamma}\ln(1+\gamma) \tag{12}$$

Equation (11) is an exact solution for the maximum water-table height in the drainage problem posed in Fig. 2 for an infinite depth of soil and for a drain radius given by Equation (12). The physical assumptions made are that the flow is vertical through the unsaturated zone from the soil surface to the water table and that the drain can be considered as a line sink. Since analogue, model and numerical studies show that these give negligible errors, the hodograph solution, of which Equation (11) is a particular case, has to be regarded as the most accurate drainage equation that is available.

Hooghoudt's solution

Hooghoudt (1940) (see also Childs, 1969, pp. 385—388) obtained an approximate solution to the drainage problem with infinite depth of soil. He assumed that the water-table height above the drain was small so that the flow region could be regarded as being bounded on top by a horizontal plane passing through the drain. The uniform surface precipitation q was assumed therefore to cause a uniform flux to pass through the horizontal plane OA′ in Fig. 2. By superimposing a fictitious upward uniform vertical flux q, OA′ is made a streamline. The potential at A′ is then obtained by summing the potentials due to all the sinks of the drainage installation. Hooghoudt then assumed that the potential at A′ was a good approximation to the water-table height; that is, he assumed an infinite vertical hydraulic conductivity component above OA′ so that there is no potential difference between the position A on the water table and A′. Hooghoudt's resulting drainage equation for infinite soil depth is:

$$\frac{H_m}{D} = \frac{r}{D} + \frac{2q}{\pi K}\left(\ln\frac{D}{r} - 0.454\right) \tag{13}$$

This solution, in addition to the approximation concerning the reduction of the flow region, also assumes (as does the hodograph method) that there is vertical flow in the unsaturated soil-water zone above the water table and that the drain tubes can be considered as line sinks.

Effect of drain radius

Both the hodograph analysis and Hooghoudt's solution give drainage equations that show that the height of the water table is dependent on the drain radius. This is illustrated in Fig. 3 where H_m/D is shown plotted against r/D for $q/K = 0.01$. Similar results are found for other values of q/K with a minimum value for H_m/D found for an optimum drain radius r_o for both

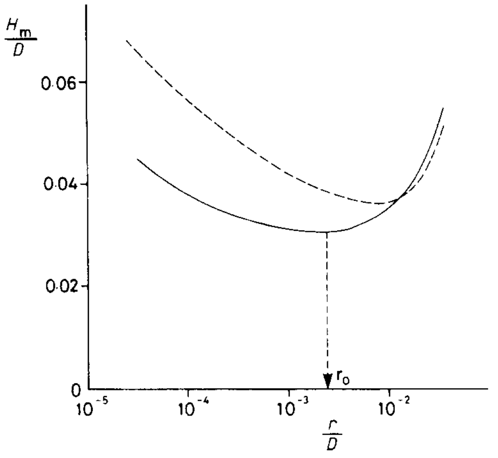

Fig. 3. Variation of the water-table height (expressed as the ratio H_m/D) with the drain radius (expressed as the ratio r/D) for cylindrical drains installed in a uniform soil of infinite depth when $q/K = 0.01$, as calculated from the hodograph analysis (full line) and from Hooghoudt's analysis (dashed line).

solutions. However, the value of r_o obtained with the hodograph analysis was found to be less than that obtained with Hooghoudt's equation for all values of q/K; the difference is presumably the result of the approximations used in obtaining the latter.

In the derivation of these drainage equations, one requirement is that the drains run full without back pressure, as illustrated in Fig. 4a, so that the water pressure is zero at the top of the drain. Physical conditions have to be such as to produce this, for instance by arranging the height of the outfall. If this is not done, in normal practice large drains will not run full but will run with the upper part of the drain being a surface of seepage where the potential is equal to the height above the datum, as shown in Fig. 4c. When this is the case, the water-table height does not increase with increase of drain radius since the drain channels then behave in the same way as open ditches. Van Deemter (1950, p. 37) compared the hodograph solutions for a ditch drain with that for a cylindrical drain channel of optimum radius (Fig. 4b), and found little difference for the water-table heights when $q/K < 0.1$. It may be concluded from this that drains of radius greater than the optimum,

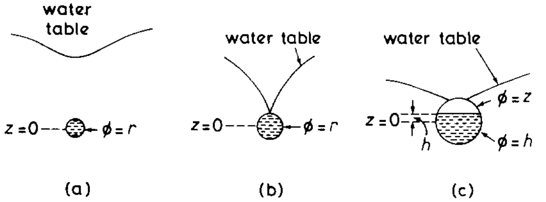

Fig. 4. Physical behaviour at the drain channel: (a) drain running full (sub-optimum conditions); (b) drain running full (optimum conditions); (c) drain running partially full.

when not forced to run full but allowed to empty, will maintain the water table at the same height as if the drain were of optimum size. This was shown to be the case in the experiments reported by Collis-George and Youngs (1958).

So that a comparison can be made between the drainage equations obtained by the hodograph analysis and those derived by Hooghoudt using potential theory with a simplified reduced flow region, values of H_m/D were calculated for a range of q/K values using the optimum r_0/D value found in the hodograph analysis for the particular q/K value. The relationships are shown in Fig. 5. The water-table heights found from the Hooghoudt equation were larger than those found by the hodograph analysis which may be assumed to give accurate values. This is to be expected because of the smaller flow region assumed in Hooghoudt's derivation; this has the effect of increasing the length of the streamlines to the drain from the water table. Hence the resistance to flow is greater, requiring a greater head of water, and hence a higher water table, for a given drain discharge or steady rainfall rate.

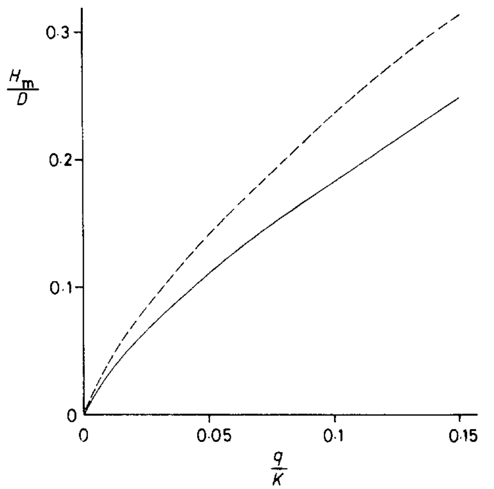

Fig. 5. Relationship between H_m/D and q/K for optimum drain-size conditions with drains installed in soil of infinite depth as calculated from the hodograph analysis (full line) and from Hooghoudt's analysis (dashed line).

DRAIN ON IMPERMEABLE FLOOR ($d/D = 0$)

There is no accurate analytical solution for the drainage problem for a cylindrical drain laid directly on top of an impermeable barrier. Engelund (1951) gives a solution for a trench drain that is level with an impermeable floor, while Youngs (1965) argued that for a ditch:

$$\sqrt{q/K} < H_m/D < \sqrt{(q/K)/(1 - q/K)} \tag{14}$$

The left hand side of Inequality (14) is the same as that obtained using the Dupuit-Forchheimer analysis and obtained by Colding (1872), while the right hand side is Engelund's solution for his trench drain. In view of the equivalence of an installation of ditch drains and of one with cylindrical drains of optimum size, as shown by Van Deemter (1950), where there is infinite soil depth, it may be tentatively assumed that ditches and optimum-sized drains give the same water-table control when there is an impermeable barrier at some finite depth and even zero depth. Thus Inequality (14) is considered to be applicable for cylindrical drains of optimum size laid on top of an impermeable barrier.

IMPERMEABLE BARRIER AT SOME DEPTH BELOW DRAIN LEVEL ($0 < d/D < \infty$)

The general problem shown in Fig. 2 with the drain lines at some height d above an impermeable barrier has no exact analytical solution. Many solutions, however, have been derived by making various simplifying assumptions, both in the physical model and in the mathematical analysis. The drainage equations that result all have the general form of Equation (4) with the drain radius explicitly involved. A comprehensive comparison, involving many drain radii would be long, and, in view of the remarks made earlier concerning the nature of field drains, not very pertinent. In the comparison here, the optimum drain radius that was obtained for the given q/K value by means of the hodograph method for infinite depth of soil, was therefore used. While it cannot be expected that the optimum drain radius is independent of soil depth, the use of one value for all values of d/D for a given q/K value is seen to be acceptable from an analysis due to List (1964). In his exact solution of the drainage problem, the impermeable boundary below the drain was a slightly undulating surface. The optimum drain radius when the water table was drawn down to the top of the drain was found to vary little with the position of the impermeable barrier.

In a series of experiments in a hydraulic model sand tank and with electric analogues, the effect of the depth of an impermeable barrier on the relationship between water-table height and steady rainfall rate was investigated by Collis-George and Youngs (1958). The water-table height was found to decrease as the depth of the impermeable barrier below the drain channels increased from zero until $d/D \sim 0.3$, after which no further change was observed. The effect was much greater for small values of q/K because of the proportionally greater increase in the extent of the flow region as d/D increased from zero. Childs (1969) fitted the results of Collis-George and Youngs (1958) to the empirical equation:

$$\frac{H_m}{D} = \frac{H_\infty}{D}\left[\left(\frac{H_0}{H_\infty} - 1\right) 10^{-5.8d/D} + 1\right] \tag{15}$$

where H_∞ and H_0 are the maximum water-table heights when $d/D \to \infty$ and 0, respectively. Collis-George and Youngs' results are useful in assessing the applicability of the drainage equations derived for finite depth of soil.

Hooghoudt's equivalent depth equation

Hooghoudt (1940) argued that the flow of water was approximately horizontal in the region midway between drains and radial near the drains as the flow converged towards them. To separate the two regions, Hooghoudt chose the vertical plane along which the potential difference between the water table and the impermeable barrier is a minimum. He assumed this plane was very nearly an equipotential, located at a distance $d/\sqrt{2}$ from the drain. Practically, Hooghoudt found it easier to adopt the horizontal flow hypothesis throughout, accounting for the radial flow to the drain by considering the horizontal flow in an equivalent layer of soil of depth d_e. Thus Hooghoudt proposed the drainage equation:

$$\frac{H_m}{D} = -\frac{d_e}{D} + \left[\left(\frac{d_e}{D}\right)^2 + \frac{q}{K}\right]^{1/2} \tag{16}$$

where

$$\frac{d_e}{D} = 1\bigg/\left\{4\left[\frac{(2D - \sqrt{2}d)^2}{16dD} + \frac{1}{\pi}\ln\frac{d}{\sqrt{2}r_o}\right]\right\} \tag{17}$$

Although founded on a physical model, it is not possible to be very specific concerning the effect that the approximations make on the relationship, and Hooghoudt's equivalent depth equation should be regarded as an empirical equation to which one is led by physical considerations.

Childs' hodograph analysis

The hodograph analysis, as given by Van Deemter (1950), of the drainage problem for infinite depth of soil also includes upward artesian flow as well as steady uniform rainfall. The combination of the two flows results in there being a dividing streamline PO separating them, as shown in Fig. 6. This dividing streamline may be replaced by an impermeable barrier without changing the potential and streamline configurations. Childs (1960) calculated the position of the stagnation point P. He then approximated the horizontal plane PP′ passing through P to the dividing plane PO of the upward and downward flows. This gave the drainage equation with optimum drain radius:

$$\frac{H_m}{D} = \frac{1}{\pi}\left[\ln\left(1 + \frac{2}{\gamma}\right) + \frac{2}{\gamma}\ln\left(1 + \frac{\gamma}{2}\right)\right] \tag{18}$$

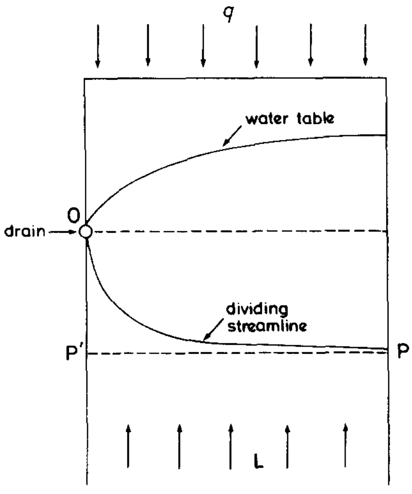

Fig. 6. Shape of the impermeable barrier that emerges in Childs' hodograph analysis.

with the depth of the impermeable barrier given by

$$\frac{d}{D} = \frac{H_{\mathrm{m}}}{D} + \frac{1}{\pi}\left[\frac{2}{\gamma}\ln\left(\frac{1+K/q}{1+\gamma+K/q}\right) + \ln\left(\frac{K/q-1-\gamma}{1+\gamma+K/q}\right)\right] \tag{19}$$

where γ is now $(K-q)/(L+q)$ with L the fictitious upward artesian flux used in order to give the boundary PO. For $d \to \infty$, $L \to 0$ and Equation (18) becomes Equation (11).

When $L > 0$, in order to give the dividing streamline, the flow region that accommodates the steady rainfall is less than that if there were a horizontal impermeable barrier PP′ at depth d below the drain. Thus, for the same rainfall rate, a greater head is required for the barrier PO than for the horizontal barrier PP′; that is, a higher water table emerges in the hodograph solution than would occur if the impermeable barrier were actually horizontal. The difference in water-table heights for the two situations is greatest for small values of q/K when there is proportionally a greater difference in the extents of the two flow regions.

List's hodograph analysis

An attempt to obtain a more accurate drainage equation for drains laid above an impermeable barrier was made by List (1964), again making use of the hodograph method. In his analysis a fictitious upward vertical flow from infinite depth was removed by a lower fictitious drain while the actual drain at a higher level removed all the water originating from the uniform flux through the water table. This arrangement of flows created the dividing streamline PP′, as shown in Fig. 7, that, as in Childs' analysis, can be regarded as if it were an impermeable barrier. The boundary is undulating

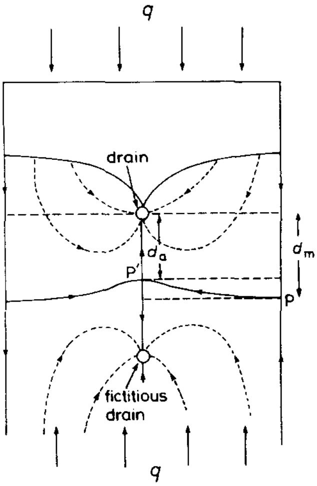

Fig. 7. Shape of the impermeable barrier that emerges in List's analysis.

in shape, highest under the drains and lowest midway between drains, but, unlike the one in Childs' treatment, does not pass through the drain.

The analysis gives the maximum water-table height midway between drains as:

$$\frac{H_m}{D} = \frac{2}{\pi}\left\{\mu\eta_0 - \nu \ln\left[\frac{\sinh 2\eta_0 \sinh(\eta_0 + \eta_1)}{\cosh \eta_0 \cosh \eta_1}\right]\right\} \tag{20}$$

where $\mu = (K + q)/(K-q)$, and $\nu = q/(K-q)$, and η_0 and η_1 are related by

$$\coth \eta_0 + \coth \eta_1 = (K + q)/q \tag{21}$$

for optimum conditions when the water table is drawn down to the top of the drain that is assumed to be running full without back pressure. Equation (20) is for a depth of impermeable barrier d_a immediately below the drain given by:

$$\frac{d_a}{D} = -\frac{H_m}{D} + \frac{2}{\pi}\left\{\mu\alpha - \nu \ln\left[\frac{\sinh(\alpha + \eta_0) \sinh(\alpha + \eta_1)}{\cosh \eta_0 \cosh \eta_1}\right]\right\} \tag{22}$$

where $\eta = -\alpha$ is the root of the equation

$$\mu[\coth(\eta-\eta_0) + \coth(\eta-\eta_1) + 2] + \coth(\eta+\eta_0) + \coth(\eta+\eta_1) = 0 \tag{23}$$

and for a depth of impermeable barrier d_m midway between drains given by:

$$\frac{d_m}{D} = -\frac{H_m}{D} + \frac{2}{\pi}\left\{\mu\beta - \nu \ln\left[\frac{\cosh(\beta + \eta_0) \cosh(\beta + \eta_1)}{\cosh \eta_0 \cosh \eta_1}\right]\right\} \tag{24}$$

where $\mu = -\beta$ is the root of the equation

$$\mu[\tanh(\eta-\eta_0) + \tanh(\eta-\eta_1) + 2] + \tanh(\eta+\eta_0) + \tanh(\eta+\eta_1) = 0 \tag{25}$$

Consideration of the extent of the flow region that emerges in the analysis, using the same arguments as for the trend of the error in Childs' hodograph analysis, show that the water-table height given by Equation (20) is smaller than that for a horizontal barrier at a depth d_a below the drains but larger than that for one at a depth d_m. List's analysis thus gives upper and lower bounds for the relationship between water-table height and depth of impermeable barrier at a given q/K value. No exact relationship is possible, even though the boundary conditions at the water table are exactly satisfied, because of the undulation of the pseudo-impermeable barrier. This undulation becomes significant for a shallow flow region; that is, for small d/D and q/K values.

Youngs' inequality

From arguments based on the potential and streamline patterns for drainage to ditches, Youngs (1975) concluded that the water-table height in drained lands with drains laid above an impermeable barrier was given by:

$$\left\{\left[\frac{q}{K} + 2\left(\frac{d}{D}\right)^2\right]\Big/\left(1 - \frac{q}{K}\right)\right\}^{1/2} - \frac{d}{D} > \frac{H_m}{D} > \left[\frac{q}{K} + \left(\frac{d}{D}\right)^2\right]^{1/2} - \frac{d}{D} \qquad (26)$$

In view of Van Deemter's (1950) observations concerning the little difference between water-table heights in lands drained by ditches and by drains of optimum radius, Inequality (26) may be considered to give bounds for drains of optimum size as well as for ditches. Youngs' (1975) calculations showed that Inequality (26) was useful for small values of d/D, giving little difference in bounds when $(d/D)\sqrt{(K/q)} < 0.5$, especially for small values (<0.01) of q/K. This is in contrast to List's treatment where the bounds are closer for large values of d/D.

Hooghoudt's potential analysis

Hooghoudt (1940) extended the potential analysis that he used for an infinite depth of soil to the situation where there is a finite depth of soil below the drain by considering a series of image drains at a distance d below the impermeable barrier. As before, the flow from the water table to the horizontal plane through the drain is considered to be vertical without resistance so that the flow region is bounded on top by a horizontal plane through which the flux is the uniform rainfall rate that should properly be considered to pass through the water table. Hooghoudt also simplified the analysis by considering the superimposition of one horizontal row of images instead of the infinite array that would give the correct result for his assumed flow region. The drainage equation that results is:

$$\frac{H_m}{D} = \frac{2q}{\pi K}\left\{\ln\frac{D}{r} - 0.454 + \frac{1}{32}\sum_{n=1}^{\infty}\ln\frac{[(2n-1)^2D^2 + 4d^2]^2}{(n^2D^2+d^2)[(n-1)^2D^2+d^2]}\right\} + \frac{r}{D} \qquad (27)$$

Because of the approximate nature of the analysis, as well as the restricted flow region assumed, Equation (27) cannot be expected to give accurate results.

Kirkham's potential solution

Kirkham (1958) assumed the same restricted flow region as did Hooghoudt (1940) in his potential analysis, but correctly considered the infinite array of images required to give the correct potential distribution in the assumed flow region. In subsequent work (Kirkham, 1960) he accounted for the head loss within the water-table arch by applying a correction factor $1/(1 - q/K)$, giving the revised drainage equation:

$$\frac{H_{\mathrm{m}}}{D} = \frac{2q}{\pi K}\left\{\ln\frac{2D}{\pi r} + \sum_{n=1}^{\infty}\left[\frac{1}{n}\left(\cos\frac{n\pi r}{2D} - \cos n\pi\right)\left(\coth\frac{n\pi d}{D} - 1\right)\right]\right\}\Bigg/\left(1 - \frac{q}{K}\right) \tag{28}$$

This is an upper bound for the water-table height at a given q/K value, since the water is assumed to follow longer path lengths than in the actual situation, requiring a greater head from a higher water table.

Dagan's solution

Dagan (1954) attempted to overcome the inaccuracy of applying the Dupuit-Forchheimer theory to the flow of water to drains where the flow is not approximately horizontal because of the convergence of flow towards the drain channels. He accepted that the Dupuit-Forchheimer analysis was sufficiently accurate to describe the flow in the region midway between drain lines. For the flow in the vicinity of the drains, he supposed that the flow from the incident rainfall through the soil surface was also that through the horizontal plane at the level of the lowest point of the water-table arch. The width of this region was assumed to be $2d$. With these boundary conditions, he used potential theory on this region. The drainage equation that he obtained combining both regions, is:

$$\frac{H_{\mathrm{m}}}{D} = \frac{q}{2K}\left[\frac{D}{d} - \frac{2}{\pi}\ln\left(2\cosh\frac{\pi r}{d} - 2\right)\right] \tag{29}$$

The boundary conditions of the two flow regions assumed in the derivation of this equation do not match at their common boundary. This fact, together with the approximations used, makes the use of this equation of uncertain value.

Ernst's equation

Ernst (1962) obtained a solution to the drainage problem by supposing the total resistance to flow within the groundwater region to be the sum of a vertical resistance impeding the flow from the water table to a horizontal plane through the lowest point on it, a horizontal resistance impeding lateral flow, and a radial resistance associated with the convergence of flow towards the drain channels. The drainage equation that he proposed, becomes for uniform soil:

$$\frac{H_m}{D} = \left[\frac{qD}{2(d+r)K} + \frac{2q}{\pi K} \ln\left(\frac{d+r}{2\pi r}\right) + \frac{2gr}{KD} \right] \Big/ \left(1 - \frac{q}{K}\right) \tag{30}$$

Ernst's equation was derived originally for layered soils. It is to be regarded as an empirical equation based on physical arguments.

Hammad's equation

Hammad (1962) obtained a solution to the potential problem posed by land drains fed by uniform rainfall, that satisfied exactly all boundary conditions except at the water table. At the latter boundary the potential is assumed to be the same over the whole surface; that is, it is assumed to be horizontal. The potential condition of a free surface is thus satisfied, but only for very high water tables can the uniform flux condition be approximately satisfied. Hammad's analysis resulted in the drainage equation:

$$\frac{H_m}{D} = \frac{q}{\pi K} \ln\left(\frac{H_m}{r} + \frac{2D^2}{\pi^2 dr}\right) \tag{31}$$

The condition assumed by Hammad of a flat water table is not observed even approximately for the optimum drainage situation when the water table is drawn down to the top of the drain. The larger flow region assumed in deriving Equation (31) leads to lower water-table heights being predicted than do actually occur.

COMPARISON OF DRAINAGE EQUATIONS FOR FINITE DEPTH TO IMPERMEABLE BARRIER

Maximum water-table heights at mid-drain position are given in Table I for a range of depths of soil between the drains and a horizontal impermeable barrier below them for $q/K = 0.001$, 0.01 and 0.1, calculated from all the drainage equations reviewed in the last section. All calculations have been made using the optimum drain radius obtained by the hodograph analysis for infinite depth of soil for the particular q/K value. The results from List's (Equations (20) to (25) analysis and Youngs' Inequality (26) give lower and upper bounds for the relationship between H_m/D and d/D; those from Childs' hodograph (Equations (18) and (19)) and Kirkham's potential (Equation (28)) treatment give over-estimates of H_m/D, whereas those from Hammad's (Equation (31)) analysis give under-estimates of

TABLE I

Calculated mid-drain water-table heights

Drainage equation	H_m/D for d/D =						
	0	0.01	0.02	0.04	0.08	0.16	0.32
$q/K = 0.001$, $r_o/D = 0.00020$							
Childs (empirical)	0.0316	0.0269	0.0241	0.0196	0.0134	0.0076	0.0049
Hooghoudt (equiv. depth)	—	0.0234	0.0181	0.0126	0.0085	0.0062	0.0053
Childs (hodograph)	0.0434	0.0393	0.0355	0.0296	0.0217	0.0141	0.0087
List (upper bound)	—	—	—	—	—	—	—
(lower bound)	—	0.0068	0.0064	0.0060	0.0056	0.0052	0.0049
Youngs (upper bound)	0.0316	0.0247	0.0224	—	—	—	—
(lower bound)	0.0316	0.0232	0.0174	0.0110	0.0060	0.0031	0.0016
Hooghoudt (potential)	—	0.0055	0.0055	0.0055	0.0055	0.0055	0.0055
Kirkham	—	0.0447	0.0242	0.0141	0.0091	0.0066	0.0056
Dagan	—	0.0518	0.0272	0.0152	0.0093	0.0067	0.0055
Ernst	—	0.0504	0.0266	0.0147	0.0089	0.0062	0.0051
Hammad	—	0.0037	0.0035	0.0032	0.0030	0.0028	0.0026
$q/K = 0.01$, $r_o/D = 0.0020$							
Childs (empirical)	0.1002	0.0924	0.0848	0.0723	0.0555	0.0398	0.0326
Hooghoudt (equiv. depth)	—	0.0905	0.0823	0.0696	0.0543	0.0416	0.0347
Childs (hodograph)	0.1206	0.1160	0.1120	0.1046	0.0931	0.0721	0.0543
List (upper bound)	—	—	—	—	—	—	0.0588
(lower bound)	—	0.0490	0.0472	0.0441	0.0407	0.0372	0.0343
Youngs (upper bound)	0.1005	0.0915	0.0844	0.0755	0.0718	—	—
(lower bound)	0.1000	0.0905	0.0820	0.0677	0.0481	0.0287	0.0153
Hooghoudt (potential)	—	0.0402	0.0402	0.0402	0.0402	0.0401	0.0401
Kirkham	—	0.4338	0.2292	0.1271	0.0765	0.0519	0.0406
Dagan	—	0.5029	0.2573	0.1368	0.0787	0.0519	0.0406
Ernst	—	0.4206	0.2332	0.1281	0.0737	0.0477	0.0366
Hammad	—	0.0294	0.0272	0.0250	0.0228	0.0206	0.0184
$q/K = 0.1$; $r_o/D = 0.021$							
Childs (empirical)	0.323	0.306	0.290	0.266	0.232	0.201	0.186
Hooghoudt (equiv. depth)	—	0.306	0.296	0.277	0.245	0.205	0.172
Childs (hodograph)	0.341	0.335	0.332	0.324	0.309	0.284	0.246
List (upper bound)	—	—	—	—	—	0.274	0.206
(lower bound)	—	0.274	0.267	0.256	0.239	0.218	0.198
Youngs (upper bound)	0.333	0.324	0.315	0.299	0.274	0.250	—
(lower bound)	0.316	0.306	0.297	0.279	0.246	0.194	0.130
Hooghoudt (potential)	—	0.252	0.252	0.252	0.252	0.252	0.251
Kirkham	—	4.599	2.351	1.229	0.673	0.403	0.284
Dagan	—	4.786	2.395	1.210	0.634	0.367	0.255
Ernst	—	1.672	1.264	0.854	0.533	0.332	0.233
Hammad	—	0.218	0.197	0.175	0.154	0.133	0.113

H_m/D. The other equations are founded on approximations and assumptions that make any judgement on the calculated values difficult. Equation (15) of Childs is wholly empirical, being a fit of the experimental results of Collis-George and Youngs (1958).

From Table I it is seen that Childs' empirical equation lies outside the known bounds for $q/K = 0.001$. For this value of q/K, the relationship between H_m/D and q/K obtained experimentally by Collis-George and Youngs had to be extrapolated. Hooghoudt's equivalent-depth equation (Equation (16)), on the other hand, is contained mainly between the known bounds for all q/K values. However, the other equations that have been derived with some attempt at satisfying physical requirements, do not give such good predictions. Thus Hooghoudt's equation (Equation (27)) which is an extension of the potential analysis that he applied in the case of infinite soil depth, gives water-table heights that are little affected by the depth of the impermeable barrier, and hence the equation does not represent the known physical behaviour. Dagan's (Equation (29)) and Ernst's (Equation (30)) results both give values of H_m/D greater than the known upper bound.

In Figs. 8, 9 and 10 the shaded areas give the known bounds for the relationship between H_m/D and d/D, obtained by combining the results from the several analyses of known merit, for $q/K = 0.001$, 0.01 and 0.1, respectively. Results from Childs' (Equation (15)) and Hooghoudt's (Equation (16)) relationships are also shown.

Computations show that Kirkham's equation gives generally high values of H_m/D, particularly for large values of q/K. Childs' hodograph analysis gives,

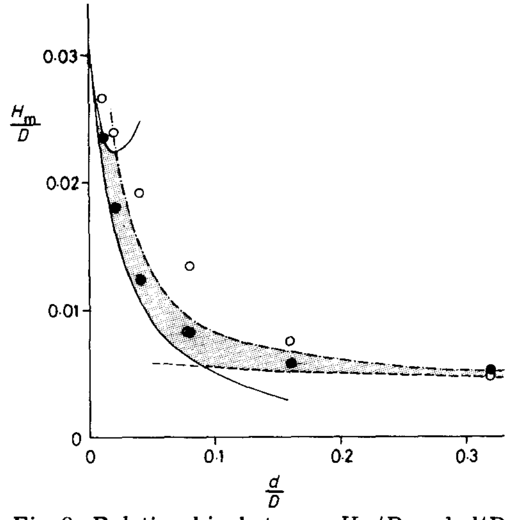

Fig. 8. Relationship between H_m/D and d/D for q/K = 0.001. The shaded area defines the uncertainty in the relationship obtained from Youngs' bounds (Inequality (26)) (full line), List's bounds (equations (20) to (25)) (dashed line) and Kirkham's Equation (28) (dash-dot line). Calculated values from Childs' empirical Equation (15) are shown by open circles, and those from Hooghoudt's Equation (16) by closed circles.

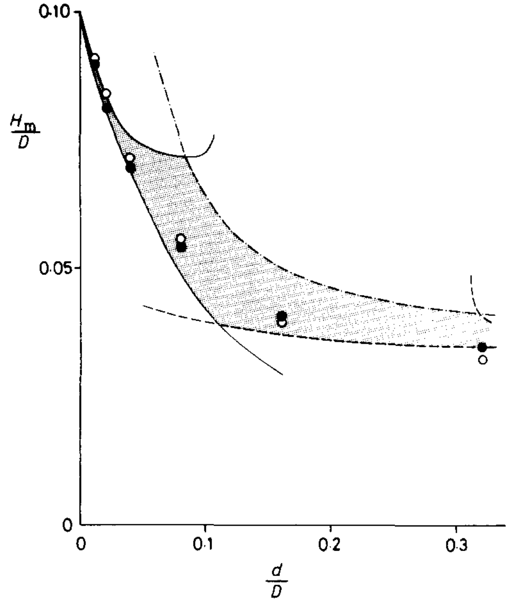

Fig. 9. Same as Fig. 8 but for $q/K = 0.01$.

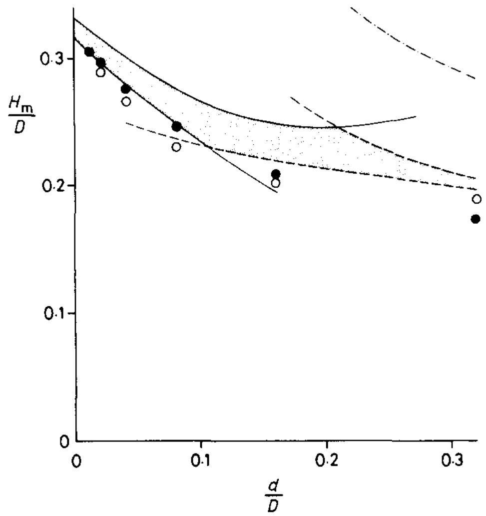

Fig. 10. Same as Fig. 8 but for $q/K = 0.1$.

as expected, also too high values. List's analysis gives an upper bound only for large values of d/D because of the shape of the undulation in the boundary that emerges in the solution. Youngs' prediction (Inequalty (26)) gives close bounds for small values of d/D, especially for small values of q/K. Hammad's Equation (31) gives values lower than the lower bound predicted by Youngs or List.

The results of the computations with List's analysis show that the water-table height when $q/K = 0.1$ is 8.0% higher when the impermeable barrier is located with $d_a = 0.30$ below the drain than at an infinite depth. For $q/K = 0.01$ it is 7.8% higher, and for $q/K = 0.001$ it is 6.2%. Because of the greater flow region assumed in List's analysis than occurs in the situation of a horizontal impermeable barrier at a depth $d = d_a$, the water-table height in the latter situation must be even higher than in the former. Thus, the finding from experimental work that drain performance is the same for depths of impermeable barrier with $d > 0.3D$ as for infinite soil depth is seen not be to be strictly correct. For the water-table height to be not more than 1% greater than for infinite soil depth, List's analysis gives $d_a > 0.65D$ for $q/K = 0.1$, $d_a > 0.66D$ for $q/K = 0.01$, and $d_a > 0.63D$ for $q/K = 0.001$.

CONCLUDING DISCUSSION

Figs. 8, 9 and 10 that show the relationships between H_m/D and d/D for q/K values of 0.001, 0.01 and 0.1, respectively, for optimum drain size conditions, summarise the present theoretical state concerning the solution to the drainage problem most commonly analysed and shown in Fig. 2. Some available solutions appear to be adequate where an impermeable barrier lies close to the drain and where it lies very much deeper. None is very satisfactory at intermediate depths when the physical assumptions and mathematical approximations in the analyses lead to a greater uncertainty in the values of H_m/D at a given value of d/D and for a given value of q/K. Thus an accurate general analytical solution to the drainage problem is still awaited. Kirkham and Powers (1964) claimed to have derived a procedure to obtain a series solution to the general potential problem of groundwater flow to drain lines. However, because of the complexity in obtaining the coefficients of the series and the slow convergence of the series that pose difficulties in computing, the procedure does not appear to have produced a solution to the drainage problem discussed in this paper. Indeed, it would appear that a numerical solution would give a solution with only a fraction of the effort that is required for the evaluation of this analytical solution. Our calculations show that Hooghoudt's (1940) equivalent depth solution, although founded on approximate physical theory, gives generally values within the known limits of uncertainty when used with the optimum drain radius obtained by hodograph theory. Since it has been argued that the optimum drain radius is the more realistic drain radius to use in drainage theory, Hooghoudt's equation with this radius of drain can be used with reasonable confidence for drain design purposes.

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