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Modelling of effective thermal conductivity for a nonhomogeneous anisotropic porous medium

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Abstract—Based on the microscale temperature, the effective thermal conductivity in the principal direction of an anisotropic porous medium is evaluated. A correlation of the effective thermal conductivity then is employed to predict the volume-averaged temperature of a two-dimensional nonhomogeneous anisotropic porous medium. Through the example, the present model is found to possess an excellent performance even when the principal direction of the anisotropic medium is not parallel to the physical coordinates. The results also reveal that using the classic mixing rule could lead to a significant error when the thermal conductivity ratio of the solid and fluid is large. The present numerical procedure is believed to be able to determine the effective thermal conductivity for most porous media as long as their microstructure is regular and known. © 1997 Elsevier Science Ltd.

INTRODUCTION

Numerous systems in industry and in nature can be modelled as anisotropic porous media. Fibrous materials, nuclear reactor cores, stratified rocks, tube bundles, mushy zone of a solidifying alloy are some of the examples. In the previous investigations dealing with anisotropic porous media [1–3], however, the anisotropy of the thermal conductivity is commonly neglected due to the use of mixing rule. This could give rise to a significant error for the temperature and heat transfer as remarked by Ni and Beckermann [4].

In their study, McKibbin and Tyvand [5] modelled a stratified system as a homogeneous anisotropic porous medium. The system of coordinates (x, y) was defined such that $k_{xy} = 0$, while the effective thermal conductivities parallel (k_{xx}) and normal (k_{yy}) to the layers were simply estimated by the theory of parallel and series thermal resistance. Numerical results of thermal convection and critical Rayleigh number were obtained. However, their correctness and accuracy were not verified.

The purpose of the present study is to develop a modelling for the effective thermal conductivity of a nonhomogeneous anisotropic porous medium. The modelling will be employed to predict the volume-averaged temperature of a two-dimensional nonhomogeneous anisotropic porous medium. The performances of the present model are then examined by comparing the predicted temperature with the available 'exact solution'.

A MODEL FOR EFFECTIVE THERMAL CONDUCTIVITY

Consider a periodically wedge-shaped solid in a two-dimensional enclosure filled with a fluid as shown in Fig. 1(a). Dimension of the wedges is illustrated in Fig. 1(b). The width of each wedge p is assumed very small as compared to that of the enclosure (i.e. $p \ll w$). Let the uniform temperatures T_0 and T_∞ be maintained on the top and the bottom surfaces of the enclosure, while both vertical surfaces are insulated. The thermal conductivities of the solid and fluid are, respectively, k_s and k_ℓ.

Due to symmetricity, only a half wedge $(0 \leqslant x \leqslant p/2)$ is needed to solve for the temperature distribution. The heat conduction equation thus is expressible as

$$(2\alpha)^2 \frac{\partial}{\partial \xi}\left(\kappa \frac{\partial \theta}{\partial \xi}\right) + \frac{\partial}{\partial \eta}\left(\kappa \frac{\partial \theta}{\partial \eta}\right) = 0$$

$$\partial\theta(0,\eta)/\partial\xi = 0, \quad \partial\theta(1,\eta)/\partial\xi = 0$$

$$\theta(\xi, 0) = 0, \quad \theta(\xi, \lambda) = 1 \qquad (1)$$

in a domain as shown in Fig. 1(c), where the dimensionless variables are defined by

$$\theta = (T - T_\infty)/(T_0 - T_\infty), \quad \xi = 2x/p, \quad \eta = y/a$$

$$\alpha = a/p, \quad \beta = b/p, \quad \gamma = c/a, \quad \lambda = L/a$$

$$\kappa = 1 \text{ for fluid}, \quad \kappa = k_s/k_\ell = \sigma \text{ for solid}. \qquad (2)$$

It appears that the system of equations (1) deals with a thermal conductivity jump from σ to unity across a solid–fluid interface of irregular shape. Such a problem can be easily solved by using the weighting function scheme [6, 7] along with the SIS solver [8]. The numerical procedure should be iterated until the temperature converges within a prescribed tolerance. After the solution converges, the average heat flux is evaluated from

NOMENCLATURE

a	height of the wedges [m], Fig. 1(b)
b	width of the wedge base [m], Fig. 1(b)
c	thickness of solid region [m], Fig. 1(b)
f	a function defined in equation (9)
k	thermal conductivity [W m^{-1} K^{-1}]
k_{eff}	effective thermal conductivity [W m^{-1} K^{-1}]
L	height of the sample [m], Fig. 1(b)
m, n	parameters for correlation (9)
p	period of the wedges [m]
Q	total heat flux in a half wedge $0 \leqslant x \leqslant p/2$ [W]
q	dimensionless heat flux defined in equation (3)
T	temperature distribution [K]
T_∞,T_0	constant temperatures [K], Fig. 1(a)
w	width of the sample [m], Fig. 1(a)
(x, y)	physical coordinates [m].

Greek symbols

α	aspect ratio, a/p
β	dimensionless quantity, b/p
γ	dimensionless quantity, c/a
Δx, Δy	grid size
ε	fraction of fluid
θ	dimensionless temperature, $(T-T_\infty)/(T_0-T_\infty)$
$\bar{\theta}$	average temperature, equation (6)
κ	dimensionless thermal conductivity, k/k_ℓ
λ	dimensionless quantity, L/a
(ξ, η)	dimensionless coordinates
σ	thermal conductivity ratio, k_s/k_ℓ
ϕ	inclination angle of the wedges, Fig. 5.

Subscript

1,2	principal axes of the wedges, Fig. 5
i,j	quantity at the grid point (x_i, y_j)
ℓ	fluid
s	solid
xx, xy, yx, yy	index of tensor for anisotropic thermal conductivity.

$$q = \frac{2Qa}{pk_\ell(T_0-T_\infty)} = -\int_0^1 \kappa\frac{\partial\theta}{\partial\eta}\mathrm{d}\xi \qquad (3)$$

where Q is the total heat flux in the half wedge $(0 \leqslant x \leqslant p/2)$. In the present computation, the numerical result of q was found independent of η. This is evidence for the correctness of the numerical procedure.

Figure 2 reveals the numerical result of isotherms for the case $(\alpha, \beta, \gamma, \lambda, \sigma) = (3, 0.8, 0.25, 1.5, 100)$. For convenience, the fraction of fluid $\varepsilon(\eta)$ is provided on the right-hand ordinate. Note that the isotherm shown in Fig. 2 is the temperature result on a microscale for the heat transfer problem described in Fig. 1(a). In many practical applications, however, there is no need to examine the temperature distribution on microscale. Under such a situation, the two-phase region in Fig. 1(a) can be simply treated as a porous medium with an effective thermal conductivity k_{eff}.

Conventionally, the classic mixing rule [9]:

$$k_{eff} = \varepsilon k_\ell + (1-\varepsilon)k_s \qquad (4)$$

or

$$\kappa_{eff} = k_{eff}/k_\ell = \varepsilon + \sigma(1-\varepsilon) = \sigma + (1-\sigma)\varepsilon \qquad (5)$$

is adopted to estimate the effective thermal conductivity of a porous medium. Unfortunately, the mixing rule could give rise to a considerable error especially when $\sigma \gg 1$. To properly define the value of κ_{eff} for the anisotropic porous medium depicted in Fig. 1(a), one evaluates the average temperature $\bar{\theta}$ from

$$\bar{\theta}(\eta) = \int_0^1 \theta(\xi, \eta)\,\mathrm{d}\xi \qquad (6)$$

such that the effective thermal conductivity can be determined from the volume-averaged Fourier law

$$q = -\kappa_{eff}\frac{\mathrm{d}\bar{\theta}}{\mathrm{d}\eta}. \qquad (7)$$

Results of the average temperature $\bar{\theta}(\eta)$ and the effective thermal conductivity κ_{eff} are illustrated in Fig. 3. The effective thermal conductivity from the mixing rule (5) is also plotted in Fig. 3 for comparison.

It is undoubted that based on the κ_{eff} value from equation (7) the same result of $\bar{\theta}$ as defined in equation (6) can be regenerated by solving the heat conduction equation

$$\frac{\mathrm{d}}{\mathrm{d}\eta}\left(\kappa_{eff}\frac{\mathrm{d}\bar{\theta}}{\mathrm{d}\eta}\right) = 0, \quad \bar{\theta}(0) = 0, \quad \bar{\theta}(\lambda) = 1 \qquad (8)$$

without recourse to the system of equations (1). Such a particular κ_{eff} was found always less than the conventional linear function (mixing rule) as observable from Fig. 3. Nevertheless, it equals to k_s in the pure solid region ($\varepsilon = 0$), while becomes k_ℓ in the pure fluid region ($\varepsilon = 1$) as does the mixing rule.

Figure 4(a) shows the effect of the aspect ratio α

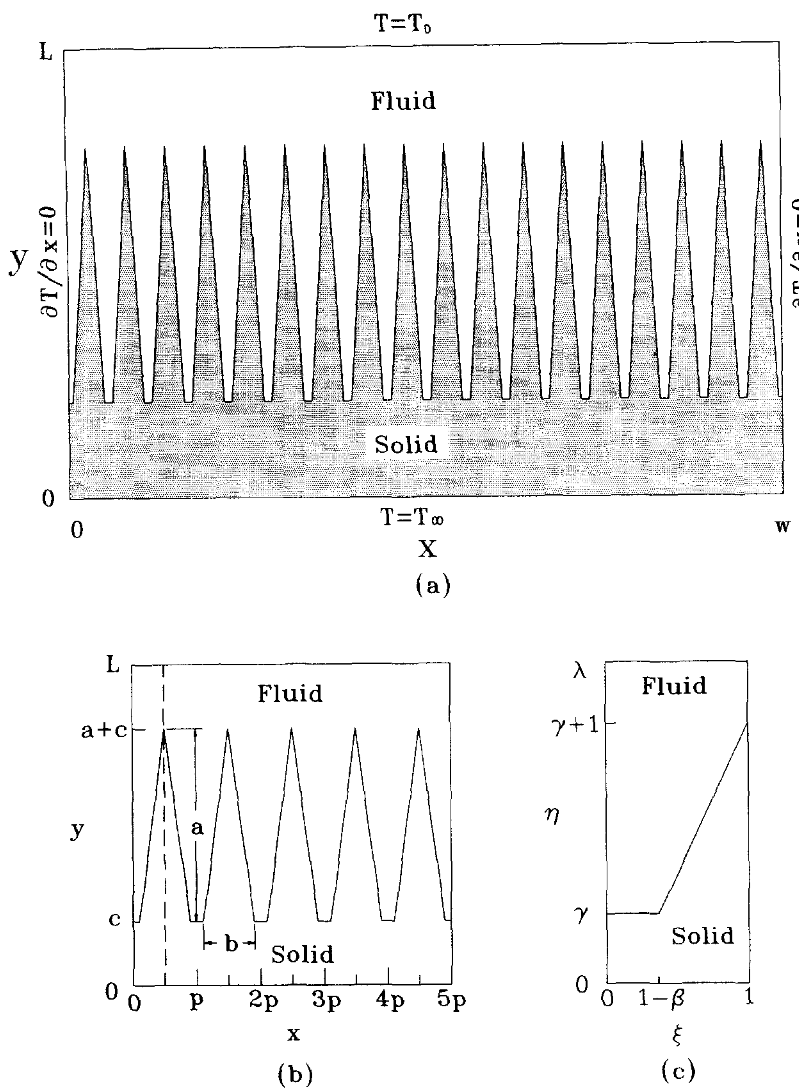

Fig. 1. (a) Microstructure; (b) dimension; and (c) computational domain of a nonhomogeneous porous medium.

on the function $\kappa_{eff}(\varepsilon)$ while the parameter β and the conductivity ratio σ are maintained at constant. Similarly, Figs. 4(b) and (c) depict, respectively, the effects of σ and β on the function $\kappa_{eff}(\varepsilon)$, while other parameters are constant. It is observable from Fig. 4 that the mixing rule provides a good approximation for κ_{eff} when the aspect ratio of the wedges α is sufficiently large ($\alpha > 6$) and/or there is no significant conductivity jump ($\sigma \approx 1$). However, the mixing rule could give rise to a considerable error at large thermal conductivity jump ($\sigma \gg 1$) specially at small α and large β.

It is interesting to note from Figs. 3 and 4 that the function $\kappa_{eff}(\varepsilon)$ seems independent of both λ and γ, and receives little influence also when a parabolic profile is used instead of the wedge-shaped solid–fluid interface. For convenience, the effective thermal conductivity $\kappa_{eff}(\varepsilon, \sigma, \alpha, \beta)$ is correlated by

$$\kappa_{eff} = 1 + (\sigma - 1)(1 - \varepsilon) f(\varepsilon, \sigma, \alpha, \beta)$$

$$f(\varepsilon, \sigma, \alpha, \beta) = (1 + m\varepsilon^n)^{-1}$$

$$m = (2/\alpha + 3/\alpha^2)\beta(1 + 6\beta)(\ln \sigma)^5 \times (2.522 \times 10^{-3} - 8.957 \times 10^{-4}(\ln \sigma) + 9.714 \times 10^{-5}(\ln \sigma)^2)$$

$$n = (20 - 29/\alpha + 15/\alpha^2)(40 + 3\beta - 13\beta^2)(0.0095 + 0.001666(\ln \sigma)) \quad (9)$$

for the range of $0 \leqslant \varepsilon \leqslant 1$, $1 \leqslant \sigma \leqslant 1000$, $\alpha \geqslant 1$,

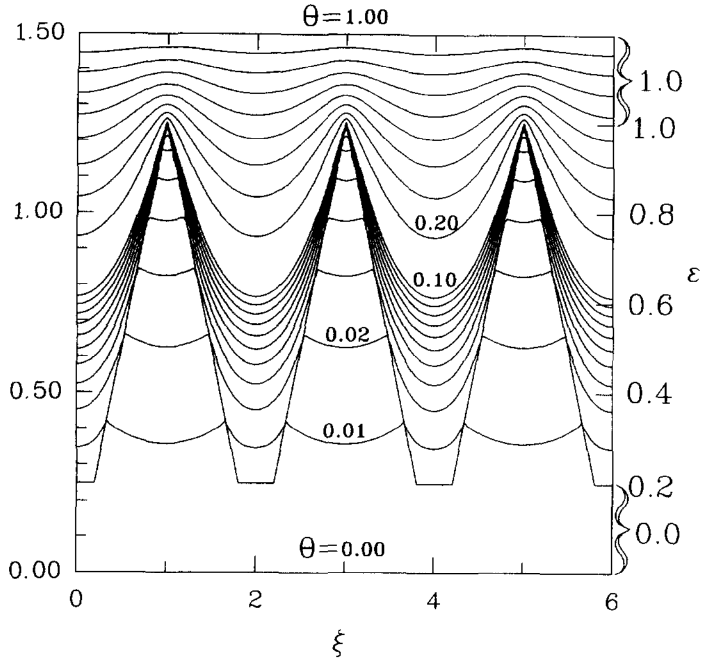

Fig. 2. Isotherms with increment $\Delta\theta = 0.01$ (for $0 \leqslant \theta \leqslant 0.10$) and $\Delta\theta = 0.10$ (for $0.10 \leqslant \theta \leqslant 1.00$) for the problem defined in Fig. 1.

$0.2 \leqslant \beta \leqslant 1$. The maximum error of the correlation (9) is less than 5%. Note that the special case $f(\varepsilon, \sigma, \alpha, \beta) = 1$ corresponds to the classic mixing rule.

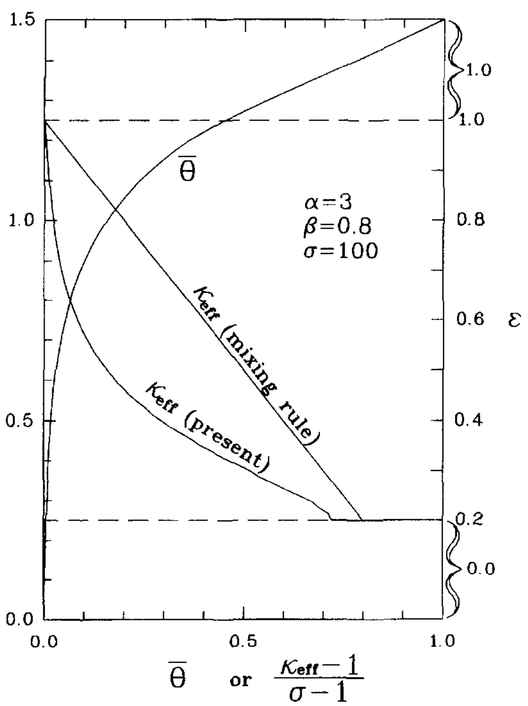

Fig. 3. The averaged temperature $\bar{\theta}(\eta)$ and effective thermal conductivity $\kappa_{eff}(\eta)$ for Fig. 1.

NONHOMOGENEOUS ANISOTROPY OF A POROUS MEDIUM

Implementation of the effective thermal conductivity formulated previously will be demonstrated in this section through a two-dimensional heat conduction problem dealing with a nonhomogeneous anisotropic porous medium. Figure 5 illustrates a wedged-shaped solid in a square enclosure ($0 \leqslant x \leqslant 1$, $0 \leqslant y \leqslant 1$) filled with a fluid. The conductivity ratio is $\sigma = 100$. The geometric parameters of the wedges are $1 \leqslant \alpha \leqslant 9$ and $\beta = 1$ with an inclination of $\phi = 45°$. Width of the wedges is assumed very small ($p \ll 1$). The same boundary conditions as in Fig. 1(a) are assumed to impose on the surfaces of the enclosure.

To determine the 'microscale' temperature, a two-dimensional heat conduction equation was solved on a 45°-inclined Cartesian grid system covering the entire enclosure. The grid size employed in the numerical procedure ($\Delta x = \Delta y = 0.0141421$) was found adequate for the present problem. Based on the result of the 'microscale' temperature θ, a locally averaged temperature was computed from

$$\bar{\theta}_{i,j} = \frac{1}{25} \sum_{n=j-2}^{j+2} \sum_{m=i-2}^{i+2} \theta_{m,n} \tag{10}$$

where the notation (i, j) denotes the location of a grid point in the two-dimensional Cartesian grid system. For convenience, this locally averaged temperature $\bar{\theta}_{i,j}$ will be referred to as the 'exact solution' in this section.

As in the previous section, the problem shown in

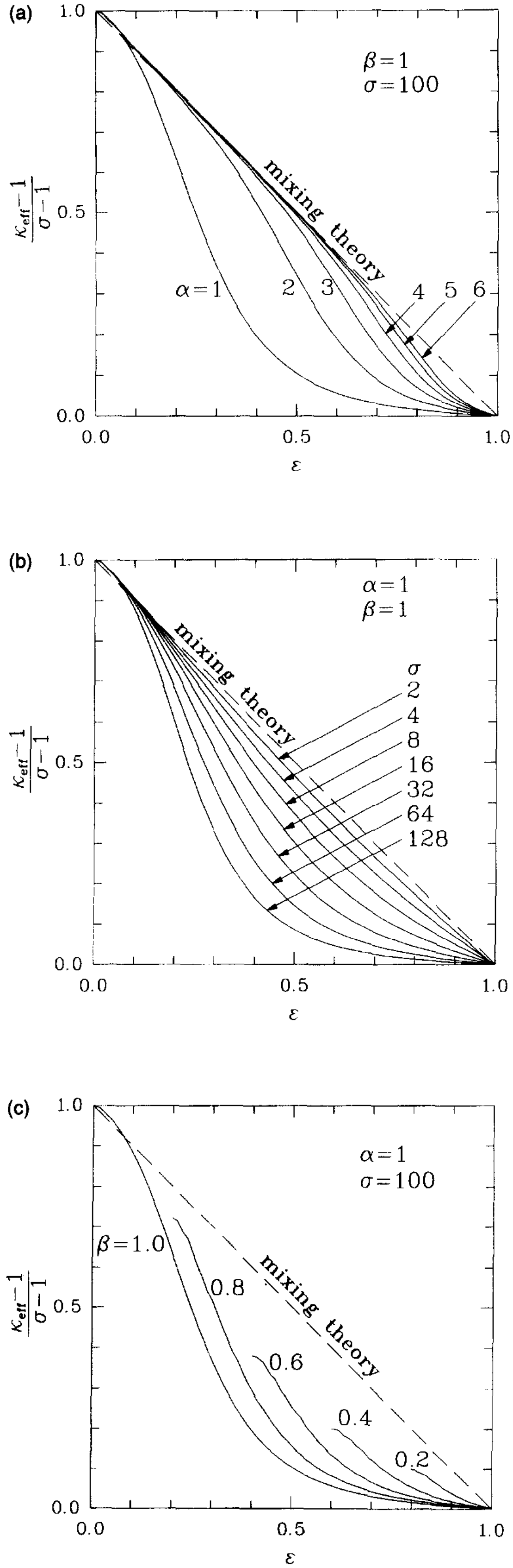

Fig. 4. Influences of: (a) the aspect ratio α; (b) the thermal conductivity ratio σ; and (c) the parameter β on the effective thermal conductivity $\kappa_{eff}(\varepsilon)$.

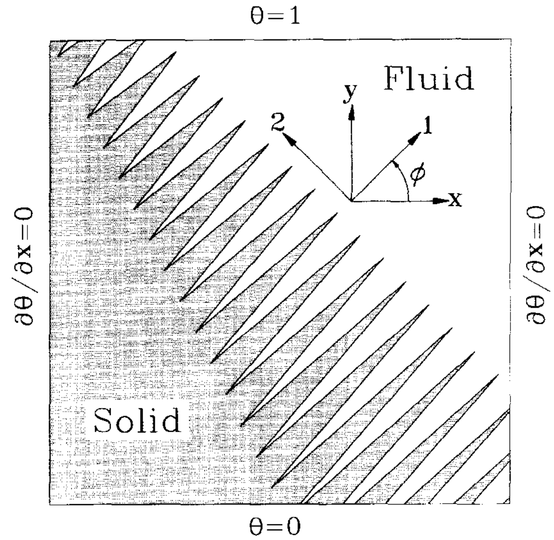

Fig. 5. Test problem dealing with a nonhomogeneous anisotropic porous medium.

Fig. 5 involves a pure solid, a nonhomogeneous anisotropic porous medium and a pure fluid. For an anisotropic porous medium, the heat conduction equation can be written as [10]

$$\frac{\partial}{\partial x}\left(\kappa_{xx}\frac{\partial\theta}{\partial x}\right)+\frac{\partial}{\partial y}\left(\kappa_{yy}\frac{\partial\theta}{\partial y}\right)+\frac{\partial}{\partial x}\left(\kappa_{xy}\frac{\partial\theta}{\partial y}\right)+\frac{\partial}{\partial y}\left(\kappa_{yx}\frac{\partial\theta}{\partial x}\right)=0 \quad (11a)$$

and

$$\kappa_{xx} = (\kappa_1+\kappa_2)/2+(\kappa_1-\kappa_2)\cos(2\phi)/2$$
$$\kappa_{yy} = (\kappa_1+\kappa_2)/2-(\kappa_1-\kappa_2)\cos(2\phi)/2$$
$$\kappa_{xy} = \kappa_{yx} = (\kappa_1-\kappa_2)\sin(2\phi)/2 \quad (11b)$$

where κ_1 and κ_2 are the effective thermal conductivity on the principal axes of the wedges (see Fig. 5). Obviously, equations (11) would reduce to

$$\frac{\partial}{\partial x}\left(\kappa\frac{\partial\theta}{\partial x}\right)+\frac{\partial}{\partial y}\left(\kappa\frac{\partial\theta}{\partial y}\right)=0 \quad (12)$$

for isotropic media ($\kappa_1 = \kappa_2 = \kappa$), and thus apply also to both pure solid and pure fluid regions.

It should be noted here that the conventional mixing rule (5) deals with only the porosity ε of the porous medium. The anisotropy of the porous matrix, however, is ignored ($\kappa_{xy} = \kappa_{yx} = 0$). This could lead to a significant error. To clarify this point, let

$$\kappa_1 = \kappa_2 = \sigma+(1-\sigma)\varepsilon \quad (13)$$

and solve equations (11) along with the associated boundary conditions

$$\partial\theta(0,y)/\partial x = 0, \quad \partial\theta(1,y)/\partial x = 0$$
$$\theta(x,0) = 0, \quad \theta(x,1) = 1 \quad (14)$$

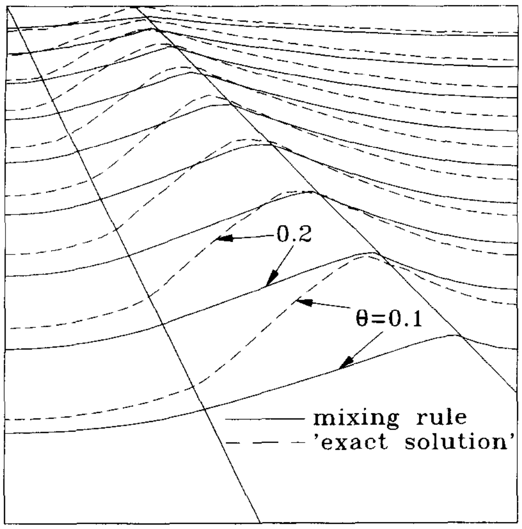

Fig. 6. Significant error due to the use of mixing rule.

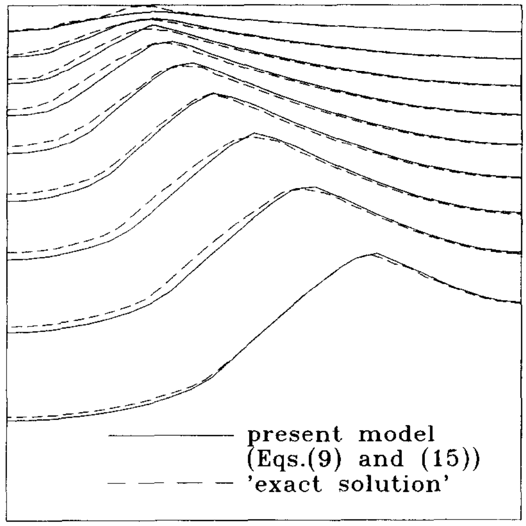

Fig. 7. Performance of the present model based on equations (9) and (15).

on a convectional Cartesian grid system of 26×26 grid points. The result of isotherms as well as the 'exact solution' is plotted in Fig. 6 for comparison. As expected, the isotherms of the 'exact solution' are essentially parallel to the principal axis of the wedges in the two-phase region. The predicted temperature from the conventional mixing rule, however, turns clockwise by about 30°. This evidences that ignoring the anisotropy of an anisotropic porous matrix could lead to a significant error.

In the present study, the correlation (9) proposed in the previous section is employed to estimate the effective thermal conductivity κ_1 in the direction parallel to the axes of the wedges, while the theory of series thermal resistance for composite wall is used to model the transverse conductivity κ_2 (see Fig. 5), i.e.

$$\kappa_2 = \left(\frac{\varepsilon}{1} + \frac{1-\varepsilon}{\sigma}\right)^{-1} = \frac{\sigma}{1+(\sigma-1)\varepsilon}. \tag{15}$$

The result of isotherms is presented in Fig. 7 that shows a satisfactory agreement between the present result and the 'exact solution.' Finally, let κ_1 be simply modelled with the theory of parallel thermal resistance

$$\kappa_1 = \varepsilon + \sigma(1-\varepsilon) = \sigma + (1-\sigma)\varepsilon \tag{16}$$

instead of the correlation (9), while κ_2 is estimated from equation (15). The resulting isotherms are shown and compared with the 'exact solution' in Fig. 8.

It should be noted here that the parallel resistance theory (16) is identical to the mixing rule (5). Hence, equation (16) is not expected to give good approximation for κ_1 unless $\alpha > 6$ and $\sigma \approx 1$ as does the mixing rule (see Fig. 4). This might account for the considerable error when equations (16) and (15) are employed to approximate the principal thermal conductivities (κ_1, κ_2) as observable from Fig. 8, especially in the pure fluid region.

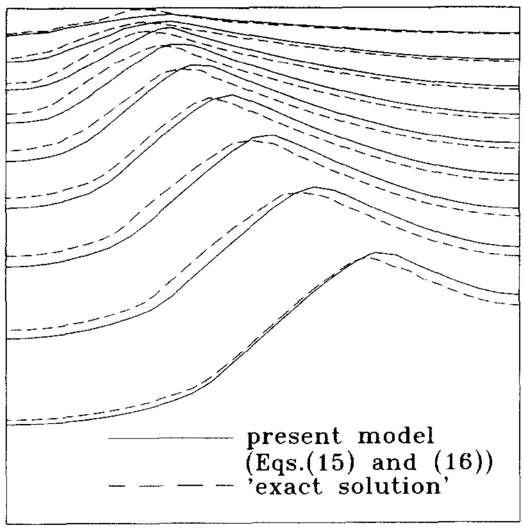

Fig. 8. Performance of the present model based on equations (15) and (16).

As a final note, it is mentioned that instead of directly evaluating the effective thermal conductivity, considerable efforts have been undertaken to find the upper and lower bounds of the effective thermal conductivity for random heterogeneous media [11, 12]. However, the microstructure of a porous medium could have a strong influence on its effective thermal conductivity, especially when the thermal conductivity ratio is large ($\sigma \gg 1$). As a result, the bound width for the effective thermal conductivity of a random heterogeneous medium could be very large [12]. Fortunately, the present numerical procedure provides a simple and efficient methodology to determine the effective thermal conductivity for most porous media as long as their microstructure is regular and known.

CONCLUSION

A microscale temperature dealing with a non-homogeneous anisotropic porous medium is computed in the present investigation. Based on such a microscale temperature, the effective thermal conductivity in the principal direction of the anisotropic medium is evaluated. A correlation with good accuracy is then obtained for the effective thermal conductivity. Finally, the present model as well as the conventional mixing rule is applied on a two-dimensional heat conduction problem dealing with a non-homogeneous anisotropic medium. Through the example, the conventional mixing rule is found to produce a local average temperature at a completely unacceptable level, while the present model shows an excellent performance even when the principal direction is not parallel to the physical coordinates. For most porous media, the present numerical procedure is able to compute the effective thermal conductivity as long as the microstructure is regular and known.

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