Modeling Dynamic Systems

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Dynamic Modeling

Second Edition

With a Foreword by Donella H. Meadows

With 273 Illustrations

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Foreword

I have been lucky enough to run across four Great Learnings in my life, the third of which was dynamic modeling. Maybe I should go through these Learnings chronologically, putting modeling into what I think of as its place.

The first Great Learning for me was about nature. I absorbed it as a child simply from being outdoors. I flopped down on my back in my suburban back yard and watched the trees, the clouds, the constellations, the Northern Lights. Soon I was riding my bike to nearby forest preserves and learning about wildflowers and birds and bugs. I was stunned by the beauty, the diversity, the complexity. Why such a variable world? I wondered. Why so many kinds of life? How do they work? How do they fit together?

Those questions led to the second Great Learning, the world of science. In high school, college, and graduate school I soaked up all the science I could get—chemistry, physics, biology. I loved the careful logic of it all, the precise definitions, the critical thinking. Science answered few of my questions about the world—in many ways it just deepened the mysteries. But it taught me rules for thinking, for assembling evidence, for separating assumptions from conclusions, and for questioning assumptions. These rules will always be a part of me, though I haven't practiced research science for 25 years now.

The third Great Learning came when I started worrying about not only nature, but about how nature interacts with human society, human population growth, human economic aspirations. My questions were getting even more complicated! I was in real trouble! Fortunately, my new set of concerns led me to the systems folks, the dynamic modelers who were using computers to try to encompass greater complexities than could ever be held all together at one time by the unaided human mind.

Here was another careful, critical way of thinking, but instead of taking things apart, as science did, it put things back together and looked at them as wholes. It forced me to make new and important distinctions. A stock is different from a flow, and that difference is especially important over *time*. Flows can be turned on and off quickly, but stocks *take awhile* to fill up or empty out.

An information flow is different from the flow of a real, physical substance, such as the flow of water through a watershed, or the flow of traffic along a highway, or the flow of mail through the postal system. Again the difference has to do with *time.* Information can move almost instantly, it can be in several places at once, it can disappear into nowhere. A real physical flow must be conserved, it comes from somewhere and goes to somewhere else, and it *takes a characteristic amount of time* to do it.

You might think basic systems rules like these are obvious, but it still comes as a shock to people to discover that the stuff they buy actually comes from somewhere (such as a forest or a mine) and that when they throw it away, it goes somewhere. It may come as even more of a shock to learn how long it spends in that place to which it is thrown. And maybe the biggest shock is the way what we throw out, and what we do in many other ways, can *feed back over time* (another primary systems principle) and re-enter our lives.

Like my direct experience of nature, like my training in science, my work with dynamic modeling has stuck in my head. It affects all my thinking, especially my sense of *timing.* I have a healthy respect now for the way things that grow exponentially (from populations to epidemics to rumors) can *get very big very suddenly.* I have a feel for the way oscillating things (from pendulums to predator-prey systems to economies) can *turn around and go the other direction.* I even have some dawning insight into my original questions about why nature is so variable and diverse, how natural systems work, and what might happen as the human economy grows exponentially into them.

These, and more, are the kinds of insights you can get from this book. You will be amazed at the doors of understanding they will open. I wish for you the same excitement and joy that learning about dynamic systems instilled in me.

I haven't yet mentioned my fourth Great Learning, which, like the others, followed naturally from the learnings before. It had to do with ethics, morality, goodness. Most people wouldn't think that a progression of questions that began with nature and then led to science and then to complex systems would end up with questions about virtue. But for me it did. As I saw how systems work, and how they break down, I began to feel a sense of responsibility for them. I got more and more committed to making them work. For everyone. For human society and for nature. For the present and the future. Again, fortunately, there were people who had thought deeply about these things and who could help me learn.

I don't know whether your own path will follow mine. Maybe one can learn about the ever-changing, equilibrating, oscillating, evolving, orderly-and-yet-chaotic complexity of the world without taking a stand for that world. Or maybe not. I just thought I had better warn you.

Donella H. Meadows

Series Preface

The world consists of many complex systems, ranging from our own bodies to ecosystems to economic systems. Despite their diversity, complex systems have many structural and functional features in common that can be effectively modeled using powerful, user-friendly software. As a result, virtually anyone can explore the nature of complex systems and their dynamical behavior under a range of assumptions and conditions. This ability to model dynamic systems is already having a powerful influence on teaching and studying complexity.

The books in this series will promote this revolution in "systems thinking" by integrating skills of numeracy and techniques of dynamic modeling into a variety of disciplines. The unifying theme across the series will be the power and simplicity of the model-building process, and all books are designed to engage readers in developing their own models for exploration of the dynamics of systems that are of interest to them.

Modeling Dynamic Systems does not endorse any particular modeling paradigm or software. Rather, the volumes in the series will emphasize simplicity of learning, expressive power, and the speed of execution as priorities that will facilitate deeper system understanding.

Matthias Ruth and Bruce Hannon

Preface to the Second Edition

Since the publication of its first edition, *Dynamic Modeling* has nationally and internationally found widespread use across academic disciplines, adoption as a coursebook and a guide for practitioners. A modeling community is rapidly developing around *Dynamic Modeling* and the *Modeling Dynamic Systems* book series that it spawned.

The second edition answers to the calls from the modeling community for more examples from theory and applications in the many areas that have been covered by the first edition. For example, we have expanded our treatment of population cohort models and included a detailed application to the dynamics of the US population. We enriched our discussion of positive feedbacks with a model of various fixed points. We enlarged our section on genetics and provided more applications to the modeling of diseases. The economics part of the book for the first time presents models from game theory and of market dynamics that result from the combined effects of inventory changes and producer expectations. The set of engineering models of gravity and acceleration include examples of mechanical amplifiers that are illustrated by the workings of a playground swing.

In addition to these expansions, many chapters have been revised and updated to make use of the new features that STELLA® provides for model development. All models and a run-time version of the software are included with this book on a CD ROM that is compatible across Macintosh and Windows platforms.

More than before, we are stressing the need to learn and apply concepts and tools developed in one area of inquiry to other areas with the purpose of generating new insights, streamlining the problem-solving process and fostering creative thinking and modeling. But as before, we see dynamic modeling as both a skill and art that can be readily acquired to solve the many problems laid out inside the book and outside.

Bruce Hannon and Matthias Ruth

Preface to the First Edition

Modeling of dynamic systems on the computer has become a real possibility for the typical college student. Until now, students needed significant preparation in mathematics and computer programming to prepare such models. It is possible that because of this hurdle, many thought little of the utility and wisdom of preparing and manipulating computer modes of dynamic events in the world around them. Without such models we were often left to manipulate real systems in order to understand the relationships of cause and effect. One could, if necessary, just change things a little (e.g., introduce a little pesticide, CO_2, etc. into the environment) and observe the effects. If no significant adverse effects are noted (in a "reasonable" time), one is free to increase the level of the system change. This is an exceedingly common paradigm. It is an elaboration of the way an auto mechanic repairs an engine, a kind of trial and error method. But social and ecological systems are not auto engines. Errors in tampering with these systems can have substantial costs, both in the short and long terms. In spite of the growing evidence, the trial and error approach remains the meter of the day. We trust that, just like the auto mechanic, we will be clever (or "rich") enough to clear up the problems created by the introduced change. Hand-eye tinkering is the American way. We let our tendency toward optimism mask the incurred problems.

However, the level of intervention in social and ecological systems has become so great that the adverse effects are beginning to seem unstemmable. As our optimism about repair begins to crumble, we take on the attitude of patience toward the inevitable—unassignable cancer risk, global warming, fossil fuel depletion—the list is long. At the same time as we become increasingly pessimistic about our ability to successfully influence cause and effect relationships we are increasingly aware of the complexity that underlies the multitude of possible interrelationships among the components of the system we intend to influence. Science, often perceived as the bearer of truth, led us to identify ever more specific issues surrounding ever smaller parts of the system. Yet, we need to understand the interactions of many or all of these parts simultaneously in order to guide our

actions. As a result, we are often overwhelmed by details, frequently failing to see the forest for the trees.

There is something useful that we can do to turn from this path. We can experiment using computer models. We can use these to give us predictions of the short-and long-term future outcomes of proposed actions. The consequences of the discovery of adverse systemic effects on the computer are no more than a ruffled pride. The potential for change of such discoveries should rival that of our famous technological inventions.

Computer modeling has been with us for nearly 40 years. Were it not for the possibility of wrestling the power of modeling from the domain of the academic, corporate or government expert, we would not be very enthusiastic about modeling. Fortunately, the personal computer and programs such as the one on which this book is based are increasingly available. These two innovations, software and hardware available to a large number of individuals, mean that almost anyone can now begin to model real world phenomena on their own, in terms that are easily explainable to others. They can use these skills to explain the inscrutable and to challenge the ex cathedra.

This democratization of modeling the real world indicates that we are on the verge of a revolution in education and thought. For centuries we have taught people how to read and write in ways that are understandable to large groups of people. Via this standardization and the subsequent development of these skills we were able to develop verbal descriptions for others of dynamic phenomena in the way that we saw them in our minds— the literate description of a mental dynamic simulation. People possess different mental models of the same phenomenon and there understandably ensues a clash of views. Just why people have differing mental models in such cases is not as important as our attempts to rationalize the broad use of one mental model instead of another.

Long ago, the use of numbers began to augment the arguments over various mental models. For several centuries we have developed numerate skills in our students to let them set priorities on the assumptions in the competing mental models. This seems like an attempt to rationalize the selection process and the widespread importance attached to skills in numeracy and it is, at least in part, an attempt to bring many minds to the challenge of selecting the most appropriate mental model of the way the world works.

The ecologist Garrett Hardin and the physicist Heinz Pagels have noted that an understanding of system function, as a specific skill, needs to be and can become a third integral part of general education, alongside literacy and numeracy. It requires the recognition (easily demonstrable with exceedingly simple computer models) and admission that the human mind is not capable of solving very complex dynamic models by itself. Just as we need help in seeing bacteria and distant stars, we need help in solving complex dynamic models. We do solve the crucial dynamic modeling prob-

lem of ducking stones thrown at us or of safely crossing busy streets. We learned to solve these problems by being shown the logical outcome of mistakes or through survivable accidents of judgement. We experiment with the real world as children and get hit by hurled stones and/or we let adults play out their mental model of the consequences for us and we believe them. These actions are the result of experimental and predictive models and they begin to occur at an early age. In the complex social and ecological world, however, we cannot rely on the completely mental model for individual or especially for group action, and often we cannot afford to experiment with the system in which we live. We have got to learn to simulate, to experiment and to predict with complex models. Computers are the only tool for such a purpose.

Now comes a new generation of personal computers with their easy-to-learn format and the software STELLA® with its iconographic programming style. STELLA and the personal computer are slowly changing the way in which we think. They do enable each of us to wrest out the mental model we have of a particular phenomenon, to augment it, elaborate it and then to do something we cannot otherwise do, to run it, to let it yield the inevitable dynamic consequences hidden in the structure of the model. STELLA and the Macintosh, as well as the new, easy-to-use, Windows-based personal computers, are not the ultimate mechanisms in this process of mind extension. They are just the first and their presence and relative ease of use makes clear the path to freer and more powerful intellectual inquiry and challenge by every student.

These are the arguments for this book on dynamic modeling. We consider such modeling as the most important task before us. To help students learn to extend the reach of their minds in this unfamiliar yet very powerful way is the most important thing we can do.

Bruce Hannon and Matthias Ruth

Contents

Part 1

Introduction

1

Modeling Dynamic Systems

> Indeed, from Pythagoras through pyramidology, extreme irrationalities
> have often been presented in numerical form. Astrology for centuries
> used the most sophisticated mathematical treatments available—and is
> now worked out on computers: though there is, or used to be, an En-
> glish law which provided that "every person pretending or professing to
> tell Fortunes, or using any subtle Craft, Means or Device . . . shall be
> deemed a Rogue and Vagabond."[1]

1.1 Model Components

Model building is central to our understanding of real-world phenomena.
We all create mental models of the world around us, dissecting our obser-
vations into cause and effect. Such mental models enable us, for example,
to successfully cross a busy street. Engineers, biologists, and social scien-
tists simply mimic their observations in a formal way. With the advent of
personal computers and graphical programming, we can all create more
complex models of the phenomena in the world around us. As Heinz Pa-
gels has noted,[2] the computer modeling process is to the mind what the
telescope and the microscope are to the eye. We can model the macro-
scopic results of microphenoma, and vice versa. We can simulate the vari-
ous possible futures of a dynamic process. We can begin to explain and
perhaps even to predict.

   Frequently, the phenomena occurring in the real world are multifaceted,
interrelated and difficult to understand. In order to deal with these phe-
nomena, we abstract from details and attempt to concentrate on the larger
picture—a particular set of features of the real world or the structure that
underlies the processes that lead to the observed outcomes. Models are
such abstractions of reality. Models force us to face the results of the struc-
tural and dynamic assumptions we have made in our abstractions.

   The process of model construction can be rather involved. However, it is
possible to identify a set of general procedures that are followed frequently.
These general procedures are shown in simplified form Figure 1.1. Real
events stimulate our curiosity about a particular phenomenon. This curios-
ity can be translated into a question or set of questions about observed

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[1]Conquest, R. 1993. *History, Humanity and Truth*, 22nd Jefferson Lecturer in the
Humanities, Washington, DC, May 5, 1993.
[2]Pagels, H. 1988. *Dreams of Reason*, Simon and Schuster, New York.
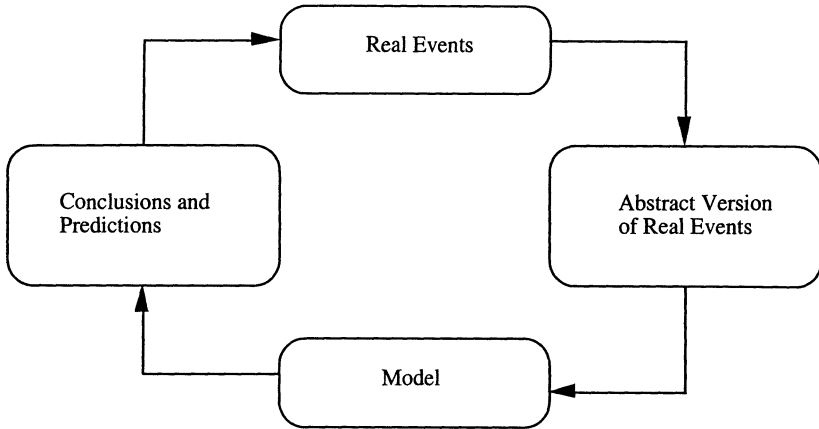
FIGURE 1.1

events and the processes that brought these events about. Key elements of processes and observations can be identified to form an abstract version of real events. Particularly, we may want to identify variables that describe these events, and outline the relationship among variables, thereby establishing the structure of the model. Based on the performance and the results of operating or "running" the model, we can draw conclusions and provide predictions about events yet to be experienced or observed. These conclusions and predictions, in turn, can be compared with real events and may lead to the falsification of a model, its acceptance, or more likely, its revision. When crossing the busy street, we make an estimate of the width of the street, the number and speed of the approaching cars, and our own speed. We may abstract away from extraneous details such as the color of the cars or the species of birds in the trees on the other side of the street. Once we made our observations or estimates, and our abstractions, we relate the various pieces of information to each other—we develop a model. Before we cross the street we "execute" our model in our mind, consider the outcome, and then decide whether we have a fair chance of arriving unharmed on the other side. If we do, we likely use this model again. If we don't, but are lucky enough, we revise the model and use that revised version for our next decision. Perhaps the birds in the trees on the other side of the street are vultures and we should have used that information as a sign that this is a particularly dangerous spot to cross. We should have been more precise in the estimate of the speed of the cars or the width of the street.

Modeling is a never-ending process—we build, revise, compare, and change models. With each cycle, our understanding of the reality improves.

Two general types of models can be distinguished. The first type is models that represent a particular phenomenon at a point of time. For example,

a map of the United States depicts the location and size of a city or the rate of infection with a particular disease, each in a given year. Other models describe and analyze the very processes by which a particular phenomenon is created. We may develop a mathematical model describing the change in the rate of migration to or from a city, or the change in the rate of the spreading of a disease. Similarly, we may develop a model that captures the change of these rates *over time*. The latter type of models are dynamic models that attempt to capture and represent the change in real or simulated time.

An understanding of the dynamics and changing interrelationships of systems, such as social, biological, and physical systems, is of particular importance in a world in which we face increasing complexity. In a variety of disciplines, scientists ask questions that involve complex and changing interrelationships among systems. What are the impacts of a vaccination program on the rate of infection in a population? How does the profit-maximizing rate of oil exploration in the lower 48 United States change with a change in the interest rate? What are the time paths of toxins carried in a river and how do these toxins affect the local wildlife? All good modeling processes begin (and end) with a good set of questions. These questions keep the modeler focused and away from the miasma of random exploration.

Models help us understand the dynamics of real-world processes by mimicking with the computer the actual but simplified forces that are assumed to result in a system's behavior. For example, it may be assumed that the number of people migrating from one country to another is directly proportional to the population living in each country and decreases the farther these countries are apart. In a simple version of this migration model, we may abstract away from a variety of factors that impede or stimulate migration, besides those directly related to the different population sizes and distance. Such an abstraction may leave us with a sufficiently good predictor of the known migration rates, or it may not. If it does not, we reexamine the abstractions, reduce the assumptions, and retest the model for its new predictions.

It is an elementary preprinciple in modeling that one should keep the simulation simple, even simpler than one knows the cause and effect relationship to be, and only grudgingly complexify the model when it does not produce the real effects. After all, it is not the goal to develop models that capture all facets of real-life systems. Such models are useless because they are as complicated as the systems we wanted to understand in the first place.

Computer models are causal in the sense that they are built by using general rules that describe how each element in a system will respond to the changes of other elements. In the migration example above, the number of migrants was assumed to be proportional to the size of the population in each country. With migration, these population sizes change over time, thereby leading to new levels of migration over time.

When a model is simulated with a computer, each element of the model is specified by initial conditions and the computer works out the system's responses according to the specified relations among elements. These initial conditions may be based on actual measurement, such as the size of a population in a city or the number of people affected with a disease, or estimates, such as estimates for the contact rate of infected people with uninfected ones. The estimates, in turn, may be based on empirical information or are just reasonable guesses by the modeler and are used to illustrate the particular processes, rather than provide exact empirical information.

Models help us in the organization of our thoughts, data gathering, and evaluation of our knowledge about the mechanisms that lead to the system's change. For example:

One can create a computer model of a forest ecosystem, consisting of a group of assumptions and information in the form of computer language commands and numbers. By operating the model the computer faithfully and faultlessly demonstrates the implications of our assumptions and information. It forces us to see the implications, true or false, wise or foolish, of the assumptions we have made. It is not so much that we want to believe everything that the computer tells us, but that we want a tool to confront us with the implications of what we think we know.[3]

Some of the elements that make up the system for which a model is being developed are referred to as state variables. State variables may or may not be conserved. Each conserved state variable represents an accumulation or stock of materials or information. Typical conserved state variables are population, resource endowments, inventories, and heat energy. Nonconserved state variables are pure indicators of some aspects of the system's condition. Typical nonconserved state variables are price and temperature. System elements that represent the action or change in a state variable are called flows or control variables. As a model is run over time, control variables update the state variables at the end of each time step. Examples for control variables are the number of births per period, a variable that changes the state variable "population," or the number of barrels of crude oil extracted changing the state variable "reserves."

Typically, components of the system that is being modeled interact with each other. Such interactions of system components are present in the form of feedback processes. Feedback processes are said to occur if changes in a system component initiate changes in other components that, in turn, affect the component that originally stimulated the change. Negative feedback exists if the change in a component leads to a response in other components that counteracts the original change. For example, the increase in the density of a prey species leads to an increase in predator density that, in turn, reduces prey density. Analogously, positive feedback is present if

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[3]Botkin, D. 1977. Life and Death in a Forest: The Computer as an Aid to Understanding, in: C. Hall and J. Day (eds.) *Ecosystem Modeling in Theory and Practice: An Introduction with Case Studies*, John Wiley and Sons, New York, p. 217.

the change in a system component leads to changes in other components that then strengthen the original process. For example, if the valve of a boiler is defective it may not open properly when the pressure of steam inside the boiler increases. If with increasing pressure the valve gets stuck more tightly, pressure will increase even further, thereby restricting further the opening of the valve. The result may be that the boiler ultimately explodes. Positive feedback, when uncontrolled, results in "explosive" dynamics. The defunct boiler is an apt example, and so is the case of population "explosion."

Negative feedback processes tend to counteract any disturbance and lead systems toward steady state. One possible steady state for interacting predator and prey populations would be that the size of each population stabilizes in the long run. Such stabilizing dynamics are in contrast to the positive feedback processes that tend to amplify any disturbance, leading systems away from equilibria.

Typically, systems exhibit both positive and negative feedback processes that have different and varying strengths. Variation in feedback processes can be brought about by nonlinear relationships. Such nonlinear relationships are present if a control variable does not depend on other variables in a linear fashion but changes, for example, with the square root of some other variable. As a result of nonlinear feedback processes, systems may exhibit complex dynamic behavior.

Once a stimulus on a system occurs, the response of the system may not be instantaneous. Rather, there may be a time lag between the stimulus and the response. In some cases, the length of the time lag is rather well known. For example, a power outage during winter in the American Northeast is typically followed by an increase in the number of births 9 months later. How the power outage translates into demand for classrooms or school buildings 6 years later when the children are of school age, is less obvious and depends on a large number of other factors, such as migration behavior of families, availability of teachers, and availability of public funds.

People often lack an understanding of time-lagged system behavior and they have a chronic inability to control such behavior both with regard to the systems that humans create and with regard to natural systems. The less well entrenched these systems are and the shorter they have been operating, the easier and less expensive it is to change them. Changing the power supply for a new residential development can be relatively straightforward, but changing a country's dependency on petroleum resources is extremely difficult—it involves changes to the entire infrastructure that supports our current lifestyles, ranging from petroleum refineries and power generation to automobile manufacturing and public transportation. Phasing out chlorofluorocarbons required an understanding of their effect on stratospheric ozone depletion as well as the time lags associated with their release and their damage on the environment. Understanding and managing carbon

flows through the fossil-fuel-based economy and the global ecosystem requires an understanding of multiple, interdependent, time-lagged systems. Yet, by the time ignorance of environmental impacts has been reduced, it is often too expensive and too difficult to influence system behavior.

Systems modelers pay special attention to nonlinearities and time lags in their models. Throughout their lifetime they try to sharpen their perception of nonlinearities and other systems features, and they improve their skills in modeling them. The eloquence of their models inspires other modelers and opens their eyes to see the world in a new way.

1.2 Dynamic Modeling as a Skill and Art

The intricacies of many real-world systems overwhelm the ability of humans to adequately understand these systems. Our mental models are often inadequate to provide a comprehensive perspective on the many interrelated aspects of systems and to anticipate their behavior. This is why we need to develop formal models, and why we need to develop skills to play out the assumptions of our formal models—we need help in handling uncertainty, feedbacks, and lags.

Many of the decisions that society faces also require that its members are effective in sharing their information and knowledge with each other—that they communicate their assumptions about system behavior and that they identify the likely system responses under alternative assumptions. One approach to societal decision making would be to identify a group of experts and ask them for advise. This is typically done in management decision making where consultants are brought in to find solutions for problems, and in policy decision making where studies are commissioned to chart the likely behavior of a social, economic, technological, or environmental system. In either case, it is the experts who define the problem such that it can be addressed with their problem-solving expertise. Once they have nailed the problem down, they provide advice on how to address it.

If you ask different groups of experts, they may look at the problem differently and they may come up with different solutions. After all, disagreement among narrowly chosen perspectives on a complex system is a likely result of complexity itself. But once the experts disagree, the question *What should I do?* changes into the question *Which advisor should I believe?* and this new question is often as difficult to answer as the first one.

Of course, one could always add layers to this process—such as have an advisory staff help with decision making or with the selection of advisors. Obviously that would not resolve your problem, but rather move it to a different level in the decision-making hierarchy.

The advice on which experts base their judgment is typically derived from models of the respective system. Consultants develop databases and simulation tools to help managers make decisions. In some cases, these

models have not been developed from scratch by the people who use them, but simply modified or combined to provide an answer to a specific question. Noticing their reliance on models, you may therefore be tempted not to ask experts for the answers that they generate with their models, but instead to ask them to give you their models so you can form your own opinions.

Expert systems, simulation games, and learning laboratories are three examples of model environments produced by consultants and scientists to provide decision makers with an ability to play out the consequences of alternative actions in what-if scenarios. Although these decision support tools are a step forward in empowering decision makers, they still are based on the understanding that an outside expert brings to the problem, rather than on the knowledge of the people directly involved. The question *What should I do?* now changes to *What does the model do?* The problem is then not whether to believe the experts' answers, but whether to believe the assumptions they put into their models. And obviously, we all can find fault with some assumption, and therefore disregard a model's validity.

Another strategy is to go all the way and develop computer models yourself to address the specific problems that you face. The usual response by decision makers is that problem-specific model development in-house would be too costly and time consuming, and that there is no guarantee that at the end the model would be a better decision support tool than one developed by outside experts. But that does not need to be so. Today there are powerful methods and tools of computer modeling available that enable virtually anyone to develop dynamic models of complex systems, to effectively communicate different assumptions among the various stakeholders—such as the decision makers, the scientists and other experts, and the public. You will learn these methods and tools as you work through this book. And you should use them to develop models *with* those who have a problem to resolve. Work with them, help them identify the questions to be answered by the modeling process, help them arrive at an agreeable solution and finally, help them formulate new questions about their system. In this way you will learn a lot from others, and you will help people become modelers, rather than skeptical users of models developed for them—models whose construction is a mystery to them and models they do not fully understand or believe in. The very nature of this book and the books of the Dynamic Modeling Series is to help you in learning how to translate your mental models into rigorously based computer ones and how to engage yourself and others in a continuous learning process.

Besides helping people to handle uncertainty, feedbacks, lags, and group decision making, the development of formal and computer models provides authentic tasks that are intellectually challenging and rewarding. Through exchange of models among modelers, the learning process turns into a cognitive apprenticeship in which all members of the modeling group can learn from each other.

Computer modeling becomes "dynamic" not only when feedback pro-
cesses among system components are captured through time, but when
model development is based on the dynamic exchange of data and infor-
mation among a group of model developers and users. It is the pluralism in
perspectives that helps identify key features and behaviors of complex sys-
tems. That pluralism is also an important ingredient for the usefulness of
models in generating new knowledge and in providing decision support.
Pluralism in perspectives is typically not promoted or maintained to derive
the solutions that outside experts bring to a problem.

The process and product of dynamic modeling can help highlight gaps in
an organization's or a society's understanding of its processes, and it helps
identify the most important parameters in a system. As models are devel-
oped, they provide a record of the existing understanding. When the mod-
els are run, they reveal "normal" system behavior if no interference into the
system takes place, and they may reveal emergent properties of the system.
We may see smooth dynamics, or perhaps erratic transitions from one type
of dynamics to another. Such knowledge is useful in helping us make deci-
sions. If, for example, the system exhibits dynamics such as the solid line in
Figure 1.2, an interference into that system with the intent to smooth out
the rapid transitions may actually exacerbate the dynamics leading to the
more pronounced up-and-down turns of the dotted line. Knowing what is
"normal" for a system may help you maintain your calm and may even
mean that you leave the system alone—after all, you know it will soon
come back from its extreme behavior. If, however, the erratic changes in a
system's dynamics are deemed unacceptable, we can use the model to play
out alternative what-if scenarios in order to find those controls that smooth
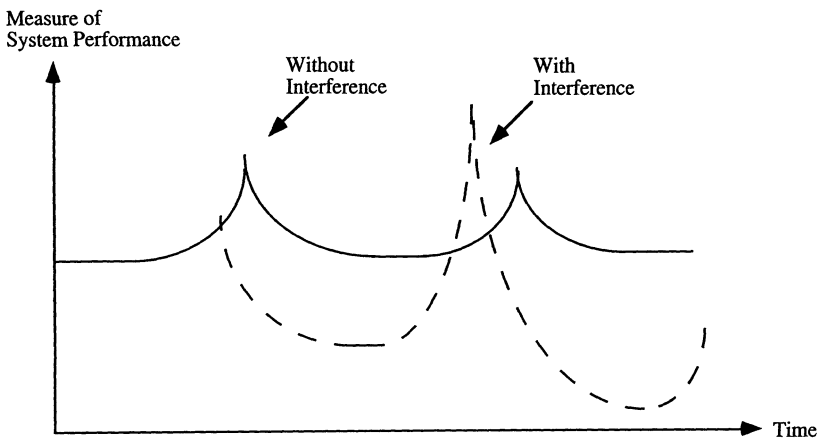out the peaks.

FIGURE 1.2

Perhaps there is one set of controls that makes the model behave more erratically and another set that makes it behave more smoothly. Playing with the controls in the model is easier, and the consequences of it are typically less costly, than playing with the controls in a real systems. This is why we train pilots on flight simulators. But we have not done this yet for people who make decisions about the course of ecological, social, and economic systems.

The fact that the model may be sensitive to one set of assumptions rather than another also can be exploited for data collection purposes. If there is some set of assumptions to which the dynamics are very sensitive, we may want to collect more information on that part of the model that uses the respective assumptions. If the model does not respond much if one part of it uses different assumptions, then we may not want to spend our time and effort refining that part further. Unfortunately, a lot of data get regularly collected before we know whether we actually need them. Some data are very costly to collect and ultimately get used in models, but a good guess could have sometimes done equally well.

The model facilitates not only insight into but also communication of likely system behavior by generating quantitative results. Learning and communication are also facilitated through the structure of the model itself: while the art of Dynamic Modeling requires that one is skillfully identifying system components and their interactions, the technique of Dynamic Modeling requires a master plan for the development of the model's structure—not the details, but the layout of model components. With increasing experience in building models for a wide variety of problems, similarities among systems structure may become apparent to the modelers. The more interdisciplinary the modeling approach, the more likely it is that knowledge from different disciplines is brought to bear on the development of the master plan according to which a model is designed.

For example, very successful models of the spread of a disease have been developed by using analogies from chemistry. In a chemical reaction, two reactants may react with each other to form a product. Similarly, the individuals in a population carrying a disease "react" with individuals who are susceptible to the disease and generate a "product"—sick individuals. The principles that can be used to model and understand chemical reactions can, by analogy, be used to understand the spread of a disease. Effectively using analogies can significantly reduce the effort that is necessary to develop models.

Throughout this book we encounter a variety of nonlinear, time-lagged feedback processes that give rise to complex system behavior. Such processes can be found in a large range of systems. The variety of models of this book naturally span only a small range—but the insights on which these models are based can (and should!) be used to inform the development of models for systems that we do not cover here. The models of this book provide a basis for the formation of analogies.

Formation of analogies is one way of dealing with complexity. A great many new insights are generated by learning something from the structure or behavior of one system, which is well understood, about another system, of which we have less knowledge. The formation of analogies forces us to choose different systems perspectives. We identify the structure of one problem and compare it with the structure of another problem. We note their similarities and their differences. We lay open the assumptions that make the analogy work, and we contrast the insights we generate about one system with the findings about the processes that govern another system. The similarities between the systems generate one set of insights, while the dissimilarities prompt the adoption of a different, but complementary perspective, and help place bounds on complexity.

Analogy is different from identity in that there may be identical features but different substructures. The art of analogy is to realize what abstractions are important—which substructures may safely be disregarded—to answer particular questions. The creation of knowledge through analogy, however, is not solely based on abstraction and the subsequent recognition of similarities. Rather, true knowledge comes from the recognition of the dissimilarities, alongside the similarities.

It is the intention of this book to show you how to model, not just how to use models. In the following section we introduce to you the computer language that we'll use throughout the book, and that will be immensely helpful to foster our understanding of dynamic systems and develop our skills of analogy formation. We close this chapter with a discussion of the structure of the model-building process and an identification of a set of easy-to-follow modeling steps. These steps are not sacred; they are intended as a guide to get you started in the process.

1.3 Modeling in STELLA

STELLA® was chosen for this book as the computer language with which we model dynamic systems because it is a very powerful, yet easy-to-learn tool. The software provides also an "Authoring" feature that enables you to develop models for use by others who are uninterested in, or ignorant of, the underlying details of the model. However, since model development and understanding are the purpose of this book, we do not discuss the Authoring feature here.

The basics of the STELLA programming language are outlined briefly in the next section. In order to easily follow that introduction, install STELLA on your disk. The installation procedure is explained in the Appendix. Then open the STELLA program by double-clicking on the STELLA program icon on your disk.

STELLA is a commercially available graphical simulation program developed by High Performance Systems. Run-time versions of STELLA for Macintosh and Windows are enclosed with this book including the models

FIGURE 1.3

developed here. Note that the run-time version alows you to open existing models, to modify them, to build your own models, but not to save changes or new models. To order a full version of the software, contact High Performance Systems, Inc.

As an introduction to the basic elements and the process of the modeling with STELLA, we provide here the stepwise development of a basic model on the growth of a population. This construction employs all of the four graphical "tools" needed for programming in STELLA. A Quick Help Guide to the workings of the software is provided in the Appendix. When you open the software, you will see the symbols (or building block objects) for stocks, flows, converters, and connectors (or information arrows) that make up a STELLA model (Fig. 1.3).

If a globe appears on the lefthand side of the screen, click on the globe to switch to modeling mode (Fig. 1.4). In this mode you can specify the initial conditions and functional relationships of your model.

Let's begin with the first tool, a stock, representing, for example, the number of people living on an island of 1 square kilometer. Just click on the rectangle, move it to the center of the diagram page and click again with the mouse. Then type in the name "POPULATION." What you should get is shown in Figure 1.5.

In STELLA, this type of stock is called a "Reservoir," which is true if we are dealing with conserved quantities. But the appropriate name is "state variable," indicating and recording at least one of the states or conditions of our system. A state variable is used in the program to make all the other calculations in the model. More about that later. Just note for now that this stock represents the number of people in the population on the island. But also note that, since the area of the island is 1 (square kilometer), the value of the population, to be changed and stored in the computer every little DT of time throughout the running of the model, is also the *density* of the population. Population is a stock, something containable and conserved in
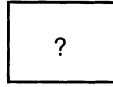

FIGURE 1.4

POPULATION

FIGURE 1.5

the "reservoir" sense, and density is not a stock, density is not conserved. Yet, both variables are *state* variables. Thus, due to our choice of area in this model, population and density are representable by the same rectangle.

Note the question mark in the rectangle. STELLA is saying that you must provide an initial or starting value for all state variables. If we double-click on the rectangle, a dialog box appears, a typical event in the use of STELLA. STELLA is requesting the initial value. You can specify an initial value of your choice, e.g. 10, using the keyboard or the mouse and the dialogue keypad. Click OK, close the dialogue box, and you should have lost the question mark.

The next question is: What controls the addition (or subtraction) of people from the island? Let's assume, for simplicity only, that the people on this island are born adult and that they never die. Later we will develop more realistic models. Here, we have only "BIRTHS" and that is the name we give to the "Control Variable." We use the second tool—called the "Flow"—to represent such control of the states. Just click on the second symbol, then click (about 2 inches left of the stock) and drag the arrow to the "POPULATION" until the state variable box becomes spotted, and release. You should get the picture shown in Figure 1.6.

In this simple model, the arrow only points into the stock, indicating an inflow. If and when necessary, the arrow can be selected to point both ways. Just double-click on the circle attached to the flow symbol and choose "Biflow." In our case, however, BIRTHS should only be going *into* the POPULATION. The control variable BIRTHS is a uniflow, say "new people per year."

Next we need to know how it is that these people reproduce. Not the real details of course—we only need to know how to represent with a sat-
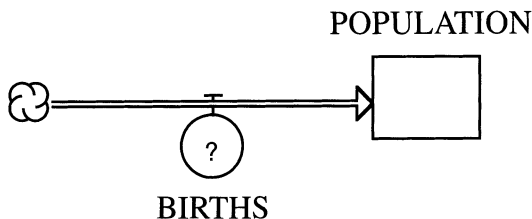
POPULATION

BIRTHS

FIGURE 1.6

isfactory accuracy the flow of new people per year. One way to do that is to look up birth rates in the Census statistics. There we might find that number given as, let's say, 3 new people per 100 existing people in the population per year. This number can be represented as a "Translation Variable" or "Convertor" (a lone circle) from the STELLA tool bar. Our BIRTH RATE is not yet a variable—that comes later. The same process of clicking and dragging with which you got the rectangle will get you the circle; just open it and enter the number 0.03 (3/100) in the indicated space. Note the impressive list of "built-in" functions to choose from for more complex models, down the side of the dialogue box.

The last tool is the "Information Arrow" (or "Connector"). This arrow conveys the information from a circle or rectangle, information about the state, control, or transformation variable, to a circle, to a control or transforming variable. In this case, we want to convey the information about the BIRTH RATE to BIRTHS and we want to convey information about the level of POPULATION to the variable BIRTHS. Two arrows must be added and now the diagram looks like Figure 1.7.

The last modeling step is to specify BIRTHS. Open it and note the list of required inputs. Naturally—because of the information arrows—these required inputs are BIRTH RATE and POPULATION. Recall the meaning and the units of these two variables: BIRTHS must equal BIRTH RATE*POPULATION. By clicking on these variables and the * sign, we have the needed relationship and the variable can be closed, completing the model for a moment.

The next item on our agenda is to set the size of the time step, DT, over which stocks should be updated and the length of the simulation run. Let's arbitrarily choose DT = 1 year and the length of time as 100 years.

In order to display the results of your model, click on the icon that represents a graph and move it to the diagram. Similarly, you can plot the model results in a table by choosing the icon that represents a table. The STELLA icons for graphs and tables are shown, respectively, in Figure 1.8.
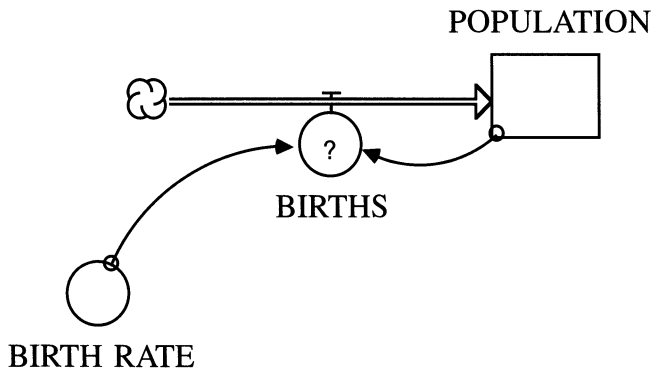
FIGURE 1.7

FIGURE 1.8

Upon selection, the graph pad opens automatically (or can be opened by double-clicking on it), and shows you the list of stocks, flows, and parameters of your model. Each of these can be plotted. Select POPULATION to be plotted, add it with the >> arrow to the list of selected items, set the scale from 0 to 200 and check OK. Stop for a moment and ask yourself: What is the likely result of your model? Now RUN the model by choosing RUN in the pull-down menu. The result is shown in Figure 1.9.

The graph shows exponential growth—just as we should have expected. And you should always say what you think the model should produce before you run it. This sort of speculation builds your intuition about system behavior and it helps you correct programming errors. When things go differently from your anticipation, something is wrong and it is up to you to explain it.

What is going on here? How does STELLA do this? It is really very simple. At the beginning of each DT, say at the initial period (@time=0), STELLA looks at all the circles for the necessary calculations. Ultimately, the calculations will probably be based on the value of the state variables. Only the variable BIRTHS depends on the state variable POPULATION. So, to get the current value of BIRTHS, STELLA multiplies 0.03 times the value of POPULATION(@time=0) or 10 (which are known thanks to the infor-

FIGURE 1.9

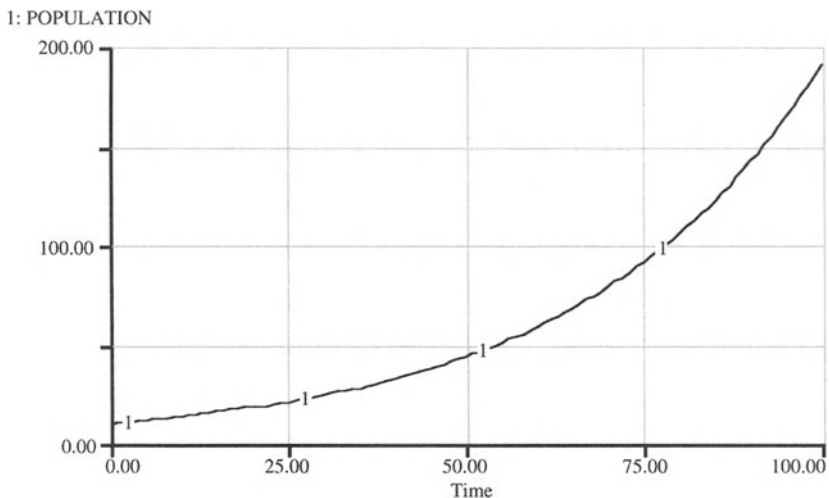mation arrows) to get 0.3. Then STELLA adds the current BIRTHS calcula-
tion to the POPULATION(@time=0) to get an updated value for the POPU-
LATION(@time=1) of 10.3. Forget the fractional people problem—there
are easy ways to interpret or handle this. For the next DT, from time = 1
to time = 2, the entire process is repeated, and so on, to the end of the
specified time.

You should see the special role of the state variable here: it is the only in-
formation that the computer uses to carry from the end of one DT to the
beginning of the next. This is why we like to call it the variable that rep-
resents the *condition* of the system.

This model is so simple that we could have solved it by analytical or sym-
bolic techniques with pencil and paper. It is a linear model and it is unreal-
istic. To add one more dimension of realism and to show some of the flex-
ibility of STELLA, let's incorporate a feedback linkage from the population
to the birth rate. First, an information arrow is needed to connect POPULA-
TION with BIRTH RATE. Once this connection is made (Figure 1.10), a
question mark appears in the symbol for BIRTH RATE because the original
specification is no longer correct but requires POPULATION as an input.

Now open the BIRTH RATE and click on the required input POPULA-
TION. We can specify the relationship between BIRTH RATE and POPULA-
TION in mathematical form or just make an educated guess about their re-
lationship. Such an educated guess between variables may be represented
by drawing a graph that reflects the anticipated value of one variable (here
BIRTH RATE) that corresponds to the value assumed by another variable
(here POPULATION). This feature is called a "Graphical Function."

Click on "Become Graph" to specify the relationship between BIRTH
RATE and POPULATION in a graphical form, set the limits on the popula-
tion at 2 and 200 and set the corresponding limits on the BIRTH RATE at 0
and 0.06 to represent a change of the birth rate when POPULATION is be-
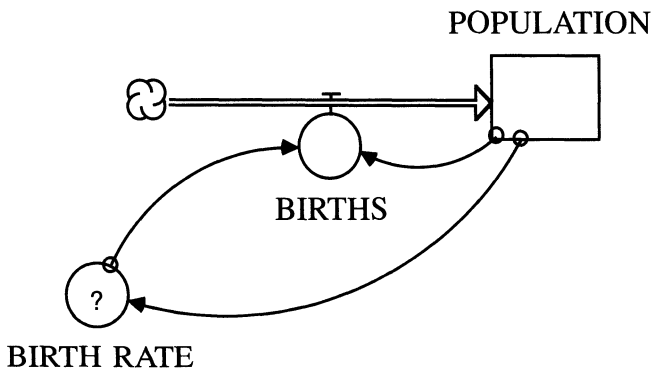tween 0 and 200. We are just making these numbers up, of course. Finally,

FIGURE 1.10

use the mouse arrow to "draw" a curve from the maximum birth rate and population of 2, to the point of zero birth rate and a population of 200.

Imagine that we had three observed points from various sources: these two and one somewhere in the middle. So we have sketched in a curve that goes through all three points and assumed that a gradual transition existed at all the other points (Fig. 1.11). Such a sketch is good enough given the state of our knowledge. OK this graph.

You are ready to run the model again but first guess the general shape of the graph for POPULATION through time. First of all it should in general, rise. Secondly, it should rise fast at first since the initial population is only 10 and the initial birth rate is therefore very high, and rise slowly later. Finally, the population should level off at 200 since that is the level when the density would become so great that all birthing stops. RUN and you will see that we are right (Fig. 1.12).

There is clearly no analytic solution to this problem. It has been solved by the only available technique, namely a numerical solution method. We should continue to investigate how sensitive the answer is to changes in the graph and to variations in the size of DT. The time step doesn't have to be 1 year. In general, the smaller the DT, the more accurate the numerical methods of updating the state variables, and thus, the more accurate the answer. Keep making DT smaller until you see that the change in the critical variable is within the measuring tolerances.

In general, keep your models simple, especially at first. Compare your results with measured values where at all possible. Only complicate your

FIGURE 1.11

1: POPULATION
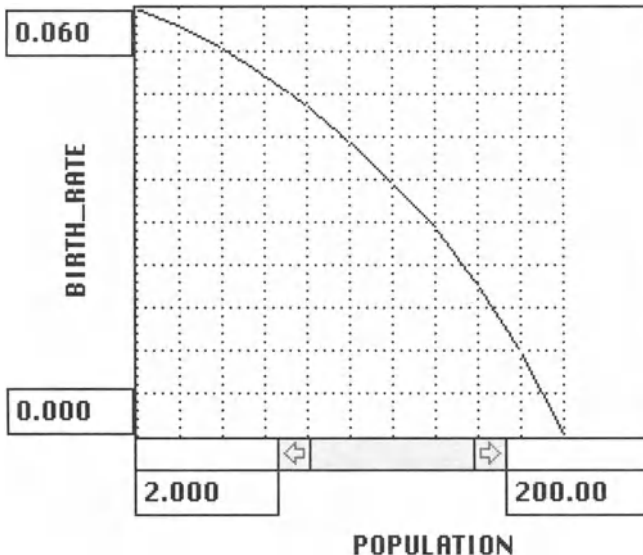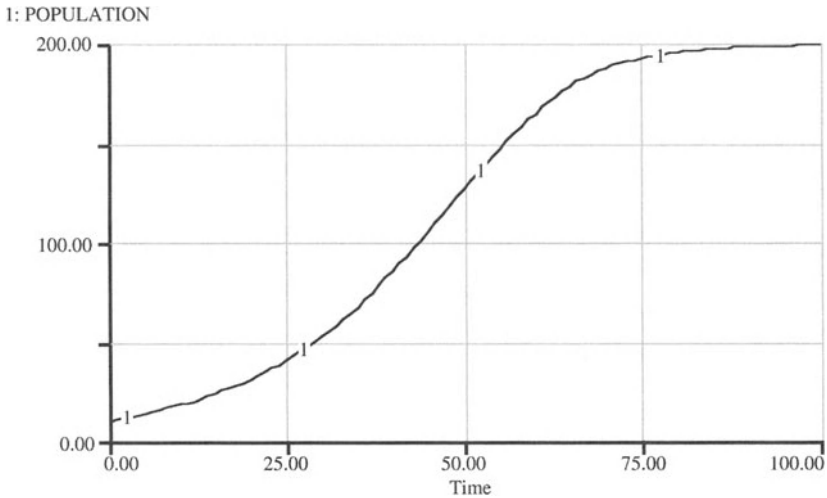
FIGURE 1.12

model when it does not produce results that predict the available experimental data within a sufficient level of accuracy.

As your model becomes increasingly complicated, you should attempt to maintain a structure in the STELLA diagram that clearly conveys interdependencies of the model parts. After all, its visually oriented approach to modeling makes STELLA such a powerful tool. Move model parts with the "Hand" symbol around in the diagram or change the color of icons with the "Paint Brush" (Fig. 1.13). Just click on these symbols to activate them. Use the "Dynamite" to bomb away unnecessary parts of the model (see Fig. 1.13).

Once the model becomes larger and contains an increasing number of modules, or submodels, you may want to prevent some of these parts from changing. Click on the "Sector" symbol—the symbol on the left in Figure 1.14—and drag it over that part of the model you want to isolate. If you then go to the Run pull-down menu you will see that you can now run sectors individually or all of them together. Execution of the individual sectors does not alter the values of the variables in other sectors.

Annotate the model to remind yourself and others of the assumptions for your model and its submodels. To do so, click on the "Text" symbol, identified by the letter "A," drag it into the diagram, and type in your text.

FIGURE 1.13

FIGURE 1.14

These tools prove very helpful when you develop increasingly compli-
cated models and when you want to share your models and model results
with others. We encourage you to make extensive use of these tools pro-
vided by STELLA. You may also want to explore features in the STELLA
software that we have not mentioned in the text, but are still useful when
building STELLA models. These features include Drill Down (visual hierar-
chy), Space Compression, High-Level Mapping Layer, as well as a number
of Authoring features. A brief overview of STELLA features is found in the
Appendix.

STELLA has excellent sensitivity analysis procedures. You should always
exercise your model thoroughly, checking your intuition against the results.
Run the model to extremes by changing the initial conditions, for example.
Soon you will want to try a formal sensitivity analysis.

Before you move on to "experiment" with the systems described in the fol-
lowing chapters you should sufficiently familiarize yourself with the basics of
the STELLA software. Make sure you can select, initialize, and connect the
various icons of that graphical programming language. In the remainder of
this book we will explore with you the use of these icons, or tools, for the
modeling of the dynamics of a variety of systems. Additional information on
the use of STELLA is provided in the Appendix. The models are also enclosed
on the CD ROM that accompanies this book. To open these models, start
STELLA, then close the blank model that STELLA provides for you—simply
go to the "File" pull-down menu and select "Close Model"—then open a
model of your choice through the "File" menu "Open Model."

POPULATION DYNAMICS

```
POPULATION(t) = POPULATION(t - dt) + (BIRTHS) * dt
INIT POPULATION = 10

INFLOWS:
BIRTHS = POPULATION*BIRTH_RATE
BIRTH_RATE = GRAPH({.03} POPULATION)
(2.00, 0.06), (21.8, 0.0576), (41.6, 0.0546), (61.4,
0.0507), (81.2, 0.0465), (101, 0.0414), (121, 0.0354),
(141, 0.0291), (160, 0.021), (180, 0.0117), (200, 0.00)
```

1.4 Principles of Modeling

Though our title of this section, Principles of Modeling, may seem some-what ostentatious, we think we have learned something general about the modeling process after many years of trying. So here is our set of 10 steps for the modeling process. We expect you to come back to this list once in a while as you proceed in your modeling efforts, and to challenge and re-fine these principles of modeling.

1. Define the problem and the goals of the model. Frame the questions you want to answer with the model. If the problem is a very large one, define subsystems of it and goals for the modeling of these subsystems. Think now: Is my model to be descriptive or predictive?
2. Designate the state variables. (These variables will indicate the status of the system.) Keep it simple. Purposely avoid complexity in the begin-ning. Note the units of the state variables.
3. Select the control variables, the flow controls into and out of the state variables. (The control variables are calculated from the state variables in order to update them at the end of each time step.) Note to yourself which state variables are donors and which are recipients with regard to each of the control variables. Also, note the units of the control vari-ables. Keep it simple at the start. Try to capture only the essential fea-tures. Put in one type of control as a representative of a class of similar controls. Add the others in step 10.
4. Select the parameters for the control variables. Note the units of these parameters and control variables. Ask yourself: Of what are these con-trols and their parameters a function?
5. Examine the resulting model for possible violations of physical, eco-nomic, etc. laws, for example, the conservation of mass, energy, and momentum; any continuity requirements. Also, check for consistency of units. Look for the possibilities of division by zero, negative volumes or prices, etc. Use conditional statements if necessary to avoid these violations.
6. To see how the model is going to work, choose some time horizon over which you intend to examine the dynamic behavior of the model, the length of each time interval for which state variables are being up-dated, and the numerical computation procedure by which flows are calculated. (For example, choose in the STELLA program Time Step = 1, time length = 24.) Set up a graph and guess the variation of the state variable curves before running the model.
7. Run the model. See if the graph of these variables passes a "sanity test." Choose alternative lengths of each time interval for which state vari-ables are updated. Choose alternative integration techniques. (For ex-ample, reduce in STELLA the time interval DT by half and simulate the mode again to see if the results are the same.)

8. Vary the parameters to their reasonable extremes and see if the results in the graph still make sense. Revise the model to repair errors and anomalies.
9. Compare the results with experimental data. This may mean shutting off parts of your model to mimic a lab experiment, for example.
10. Revise the parameters, perhaps even the model, to reflect greater complexity and to meet exceptions to the experimental results, repeating steps 1 to 10. Frame a new set of interesting questions.

Don't worry about applying all of these steps in this order as you develop your models and improve your modeling skills. However, check back to this list now and then to see how useful, inclusive, and reasonable these steps are.

Remember that modeling has three possible general uses: first, you can experiment with models. A good model of a system enables you to change its components and see how these changes affect the rest of the system. Second, a good model enables prediction of the future course of a dynamic system. Third, a good model stimulates further questions about the system behavior and the applicability of the principles that are discovered in the modeling process to other systems.

The models developed in this book are all built with the graphical programming language STELLA. In contrast to the majority of computer languages available today, STELLA enables you to spend the majority of your time and effort on understanding and investigating the features of a dynamic system, rather than writing a program that must follow some complicated, unintuitive syntax. Eventually, any modeling enterprise may become so large that the program STELLA is too cumbersome. For example, in spatial ecological modeling we use STELLA to capture the expertise of a variety of life science professionals. We then electronically translate that generic model into C+ or FORTRAN and apply it to a series of connected cells, for example as many as 120,000 in a model of the sage grouse.[4] The next step is to electronically initialize these now-cellularized models with a specific Geographic Information System and run the results on a large parallel processing computer or a large network of paralleled workstations. In this way, the knowledge-capturing features of STELLA can be seamlessly connected to the world's most powerful computers.

The cellular approach to building dynamic spatial models with STELLA and running those models on ever more powerful computers is receiving increasing attention in landscape ecology and environmental management. An alternative, but closely related approach has been chosen by Ruth and Pieper[5] in their model of the spatial dynamics of sea level rise. Their model

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[4]Westervelt, J. and B. Hannon. 1993. A Large-Scale, Dynamic Spatial Model of the Sage Grouse in a Desert Steppe Ecosystem, *Mimeo*, Department of Geography, University of Illinois, Urbana, Illinois.
[5]Ruth, M. and F. Pieper. 1994. Modeling Spatial Dynamics of Sea Level Rise in a Coastal Area, *System Dynamics Review*, Vol. 10, p. 389.

consists of a relatively small set of interconnected cells, describing the physical processes of erosion and sediment transport. Each cell of the model is initialized with site-specific data. These cells are then moved across the landscape to create a mosaic of the entire area to be covered by the model. In its use of an iconographic programming language, its visual elements for data representation, and its representation of system dynamics, the model is closely related to pictorial simulation models[6] and cellular automata models.[7] The approach is flexible, computationally efficient, and typically does not require parallel-processing capabilities. Though slightly awkward, it is also possible to use STELLA to carry out object-oriented modeling.

In the next chapter we begin with identifying features of dynamic models that are repeated throughout this book and fundamental to many dynamic processes. Then, we turn our attention to the particulars of models for chemical, genetic, ecological, economic, and engineering processes. As we develop these models, we introduce the general principles that underlie the respective disciplines, enabling you to pursue the investigation of these systems further, and to ask and answer questions of your own about the dynamics of increasingly complex real-world phenomena.

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[6]Câmara, A.S., F.C. Ferreira, J.E. Fialho, and E. Nobre. 1991. Pictorial Simulation Applied to Water Quality Modeling, *Water Science and Technology*, Vol. 24, pp. 275–281.
[7]Toffoli, T. and N. Margolus. 1987. *Cellular Automata: A New Environment for Modeling*, MIT Press, Cambridge, MA.

Part 2

Some General Methods for Modeling

2
Four-Model Set

> So self starts nothing, but what tends apace, / Home to the goal, where
> it began the race.
>
> —William Cowper, "Charity"

2.1 Stimulus-Response Model

In this chapter we discuss basic model forms that underlie a variety of models developed in this book. In order to easily follow our discussion, you should open the respective model from the CD ROM that came with this book, or even better, create the models from scratch as we proceed.

There are four basic model forms that continually reappear in the modeling process. They are presented here as both an introduction to STELLA and as a general introduction to modeling. For these basic forms of models we use populations as an example of (conserved) state variables.

The first of these basic model forms is the STIMULUS-RESPONSE form. Here the contribution to the population is independent of the stock or state variable. The likely analogy is the immigration of people to a host population: these people arrive without a clue as to the condition of the host, presuming that nothing could be worse than the conditions where they are. In this case, the arriving population (a flow) provides a simple stimulus for the change in the host population (the stock). This stimulus leads to the growth of the population that absorbs the immigrants. The STELLA model for the STIMULUS-RESPONSE model is shown in Figure 2.1.

The state variable POPULATION has as its inflow the NET IMMIGRATION that does not depend on the state variable itself:

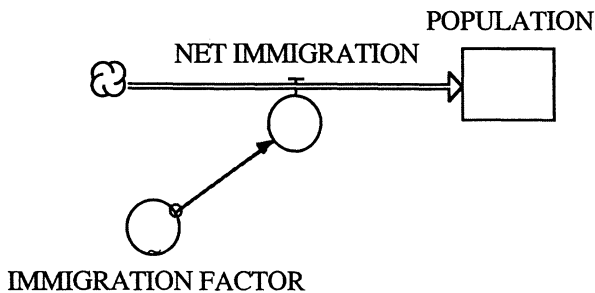$$\text{NET IMMIGRATION} = \text{IMMIGRATION FACTOR} \tag{1}$$

FIGURE 2.1

POPULATION is measured in numbers of individuals. NET IMMIGRATION is measured in numbers of people per time period. The units of IMMIGRATION FACTOR are here equal to those of NET IMMIGRATION. At this point we do not concern ourselves with the origin of the immigrants. Rather, the immigrants just appear, and we model this appearance as a flow that originates in a "cloud" and feeds into the stock of POPULATION.

The flow of NET IMMIGRATION is modeled as a function of the variable IMMIGRATION FACTOR, which is, in turn, a function of TIME. TIME is a built-in function of STELLA and is just the current time period. Open IMMIGRATION FACTOR and either type in TIME as the "right-hand side of the equation" or select TIME from the list of built-in functions. We used the graphical relationship in Figure 2.2 to specify the change in NET IMMIGRATION with TIME.

The way in which the immigration factor depends on the time period is specified in a graphical relationship. Alternatively, we may have chosen an explicit mathematical relationship. In this model, the immigration factor assumes a value of zero in time period zero and reaches the value of 1.6 in time period 100. Consequently, net immigration increases steadily, and population increases at an increasing rate.

Running the STIMULUS-RESPONSE model over 100 periods with an initial population of 10 yields the levels of population over time that are shown in Figure 2.3.
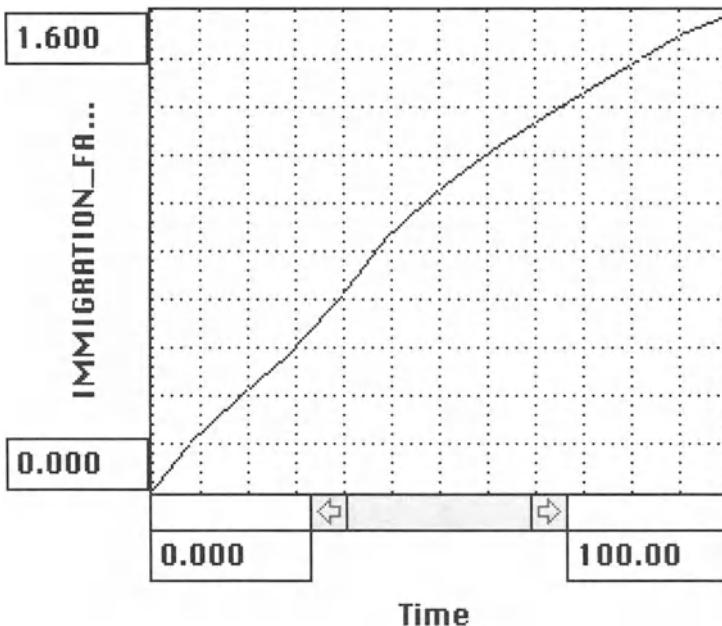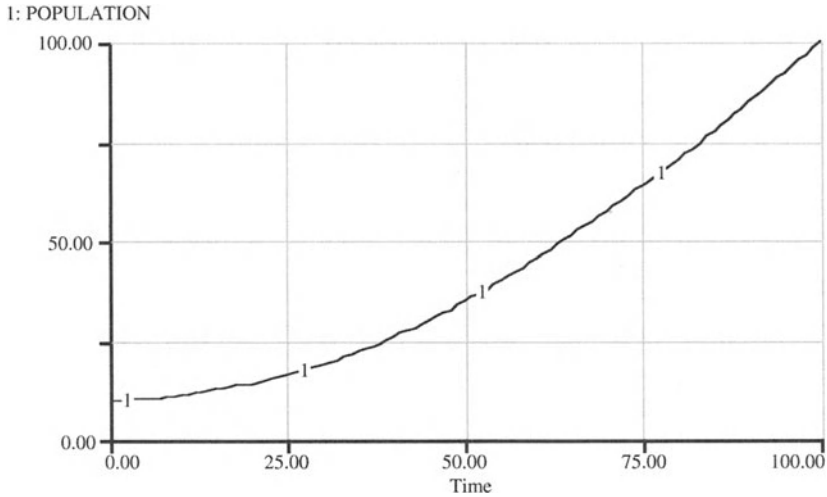

FIGURE 2.2

1: POPULATION

FIGURE 2.3

Before we move on to introduce other model forms, let us verify that the units in which the parameters and variables are measured are compatible with each other. Checking the consistency of the units is a very important step in model building and application. For a summary of the variables and constants in our model see Table 2.1.

Given these units, we can check their consistency by "calculating" the units of the population stock derived from the reproductive rule

POPULATION(t) = POPULATION(t – dt) + (NET IMMIGRATION) * dt   (2)

with

NET IMMIGRATION = IMMIGRATION FACTOR   (3)

If we disregard for the moment the time indices and substitute in this rule the units in which the variables and parameters are measured we get

NUMBER OF INDIVIDUALS = NUMBER OF INDIVIDUALS
        + NUMBER OF INDIVIDUALS PER TIME PERIOD * Time Period
                = NUMBER OF INDIVIDUALS   (4)

Since "Time Period" is equal to DT, "Time Period" cancels out and the units of our model "sum up" correctly.

TABLE 2.1

Variable	Units
POPULATION	Number of Individuals
NET IMMIGRATION	Number of Individuals per Time Period
IMMIGRATION FACTOR	Number of Individuals per Time Period

```
                    STIMULUS-RESPONSE MODEL

POPULATION(t) = POPULATION(t - dt) + (NET_IMMIGRATION)
* dt
INIT POPULATION = 10 {Individuals}
INFLOWS:
NET_IMMIGRATION = IMMIGRATION_FACTOR {Individuals per
Time Period}

IMMIGRATION_FACTOR = GRAPH(TIME)
(0.00, 0.00), (8.33, 0.2), (16.7, 0.344), (25.0, 0.48),
(33.3, 0.656), (41.7, 0.864), (50.0, 1.01), (58.3,
1.12), (66.7, 1.23), (75.0, 1.33), (83.3, 1.42), (91.7,
1.52), (100, 1.58)
```

2.2 Self-Referencing Model

The second form is the SELF-REFERENCING model, where, for example, the population level is influential in forming its own growth rate. In this case, the population levels off at a maximum. This maximum is predetermined, yet the path along which it is approached or reached is determined by the processes occurring in the system. The self-referencing process may or may not have an implicit limit. Here the specific limit is the net birth rate of zero when the population has reached a certain level. We might think of this limit as a "goal" of the population but nothing in the model explicitly requires that it will be reached. In this particular form of the model, the net birth rate is set at zero when the population reaches 100. The model will show that the population approaches 100 but never quite gets there. Suppose that the NET BIRTH RATE had a small positive value at the point where the population is at a level of 100. The population would continue to rise at this rate indefinitely. The model is still of the SELF-REFERENCING form.

The SELF-REFERENCING model is shown in Figure 2.4. The inflow into the POPULATION stock, NET BIRTHS, depends now explicitly on the level of the population a time step earlier: this year's population is equal to last year's population plus the births that occur this year minus the deaths of this year. NET BIRTHS is measured in numbers of individuals per time period. The net effect of births and deaths for the population is taken together by multiplying a NET BIRTH RATE by the population.
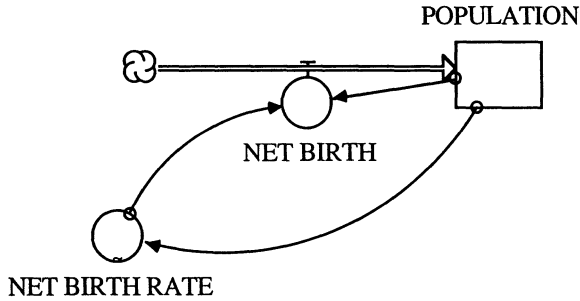
POPULATION

NET BIRTH

NET BIRTH RATE

FIGURE 2.4

NET BIRTHS = NET BIRTH RATE * POPULATION   (5)

The units of NET BIRTH RATE are 1/number of individuals in the population/time period.

As in the previous model, we do not concern ourselves with the origin or destination of the people who are borne or die, and we model their appearance and disappearance jointly, for the case of a positive net birth rate, as a flow that originates from a "cloud."

The NET BIRTH RATE in our model is not a fixed number but depends, in turn, on the population. As in the previous chapter, this dependence is specified, for simplicity, graphically. Alternatively, anthropologists may have told us the statistical relationship between NET BIRTH RATE and POPULATION in functional form. We could have then expressed the influence of the population size on the magnitude of NET BIRTH RATE by this function instead of the graph. The graphically expressed relationship between NET BIRTH RATE and POPULATION is shown in Figure 2.5.

The resulting population dynamics are shown in Figure 2.6. The population levels off over time since NET BIRTH RATE becomes zero as POPULATION reaches 100.

Before you move on with this model, make sure you know the units of each parameter and variable of the model. Verify that the units are compatible. Use the method applied in the previous section to check for the compatibility of the units.

Look again at the model we actually have here. Note well that if the NET BIRTH RATE graph were a straight line between the present end points, the resulting population grows according to the formal term "logistic." The differential equation formed is nonlinear and it has an analytic solution. Most nonlinear differential equations are not solvable analytically, so our model is a rare case. If, however, the net birth rate is dependent on the population in a nonlinear way, i.e., the NET BIRTH RATE line is curved in some way, the resulting differential equation has no analytic solution.
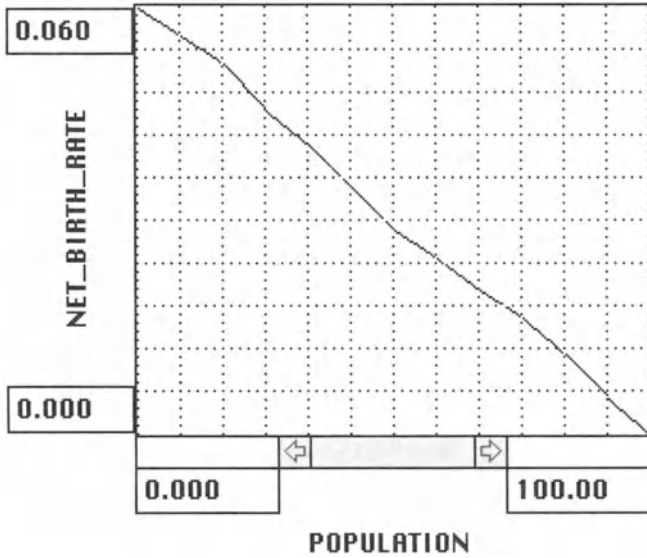
FIGURE 2.5

The very existence of the logistic definition reflects our past struggle be-tween models with analytic solutions and reality-in-modeling. Frequently, a logistic relationship among system components is postulated in order to make the analytical problem solvable. With the advent of computers and numerical solution techniques and such programs as STELLA, we need to worry less about analytic tractability and more about modeling reality.
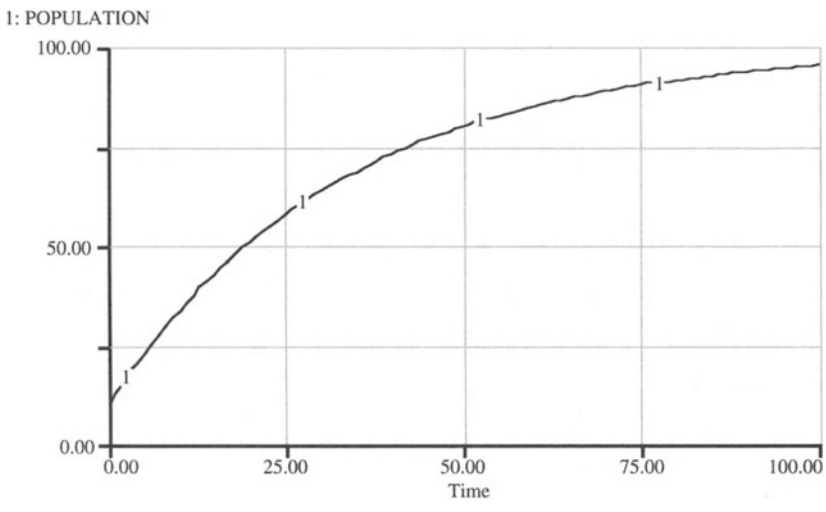

FIGURE 2.6

```
                    SELF-REFERENCING MODEL

POPULATION(t) = POPULATION(t - dt) + (NET_BIRTH) * dt
INIT POPULATION = 10 {Number of Individuals}
INFLOWS:
NET_BIRTH = NET_BIRTH_RATE*POPULATION {Number of
Individuals per Time Period}

NET_BIRTH_RATE = GRAPH(POPULATION)
(0.00, 0.0597), (8.33, 0.0558), (16.7, 0.0519), (25.0,
0.0453), (33.3, 0.0405), (41.7, 0.0345), (50.0,
0.0288), (58.3, 0.0249), (66.7, 0.0204), (75.0,
0.0165), (83.3, 0.0114), (91.7, 0.0051), (100, 0.00)
```

2.3 Goal-Seeking Model

The third basic model form is called GOAL SEEKING. A target population is the goal and the difference between the current population and the target drives the population toward the target. Unlike in the previous models, the process that underlies the GOAL SEEKING model is one of explicitly seeking the target. There are many examples of this kind of process. The decay of a radioactive source (the target is zero radiation), the cooling of a hot brick (the target is the ambient temperature), and the diffusion of a concentrated gas (the target is the room concentration, which is increasing as the gas escapes from the original container).

The GOAL SEEKING model is shown in Figure 2.7. Here, the flow of NET BIRTHS depends not only on the stock of the population but also on the exogenously defined TARGET POPULATION:

$$\text{NET BIRTHS} = \text{NET BIRTH RATE}$$
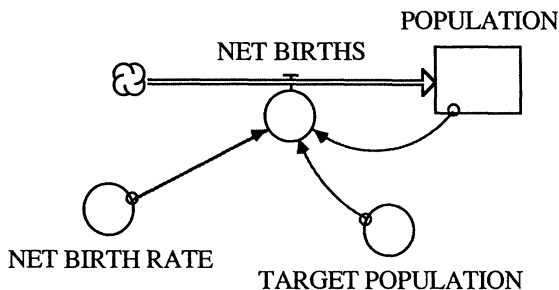$$* (\text{TARGET POPULATION} - \text{POPULATION}) \qquad (6)$$

FIGURE 2.7

Here we arbitrarily set the exogenous population target to 100 and our starting population equal to 10. Consequently, the flow NET BIRTHS is positive at the outset of the model run, and remains positive as long as the population size lies below the target. As the population grows, the difference to the target vanishes and approaches zero. Subsequent increases in population size become ever smaller, leading to a population that approaches 100 individuals.

   Again, note the units in which the variables and parameters of the model may be recorded and make sure that these units are compatible with each other. Once you checked for the compatibility of the units, make an educated guess of the model behavior. The population dynamics for our parameters and initial conditions set in the goal seeking model are plotted in Figure 2.8.

FIGURE 2.8

```
                        GOAL-SEEKING MODEL


   POPULATION(t) = POPULATION(t - dt) + (NET_BIRTHS) * dt
   INIT POPULATION = 10 {Number of Individuals}
   LINFLOWS:
   NET_BIRTHS = NET_BIRTH_RATE*(TARGET_POPULATION -
   POPULATION) {Number of Individuals per Time Period}


   NET_BIRTH_RATE = .03 {Net Addition of Individuals per
   100 Individuals in Population per Time Period}
   TARGET_POPULATION = 100 {Number of Individuals}
   .
```
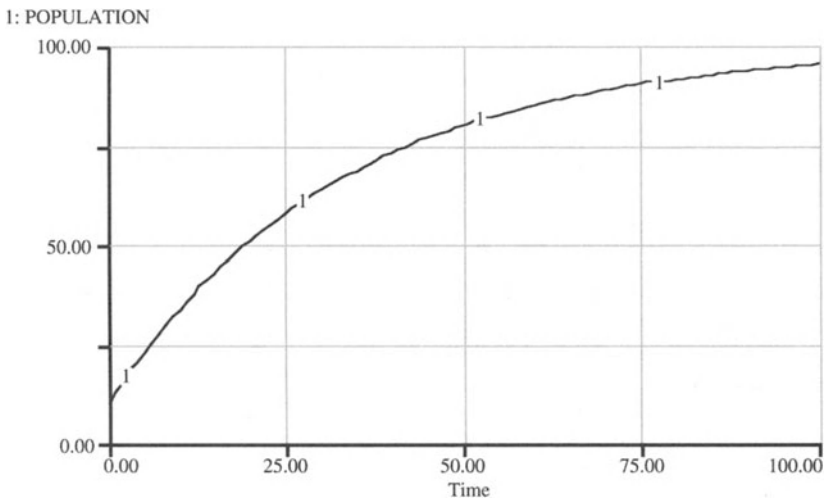
2.4 Goal-Setting Model

The most sophisticated form of the basic models is called the GOAL SET-TING model. Here the POPULATION state variable is involved in setting the population density, along with external forces. In this model, these external forces are affecting the area that is inhabitable by the population. Thus, together with the state variable POPULATION, these external forces determine the population density. That density is, in turn, controlling the target population: an increase in the population leads to an increase in the population density, and consequently, to a lower net birth rate such that a steady population size might be achieved in the long-run.

The population density is calculated simply as the ratio of the number of individuals per area

$$\text{POPULATION DENSITY} = \text{POPULATION/VARIABLE AREA} \qquad (7)$$

and the relationship between density and the VARIABLE TARGET POPULATION is graphically specified as in Figure 2.9.

We now use the VARIABLE POPULATION TARGET to calculate the NET BIRTHS as we have done similarly in the previous section of this chapter:

$$\text{NET BIRTHS} = \text{NET BIRTH RATE}$$
$$* (\text{VARIABLE TARGET POPULATION} - \text{POPULATION}) \quad (8)$$

FIGURE 2.9

FIGURE 2.10
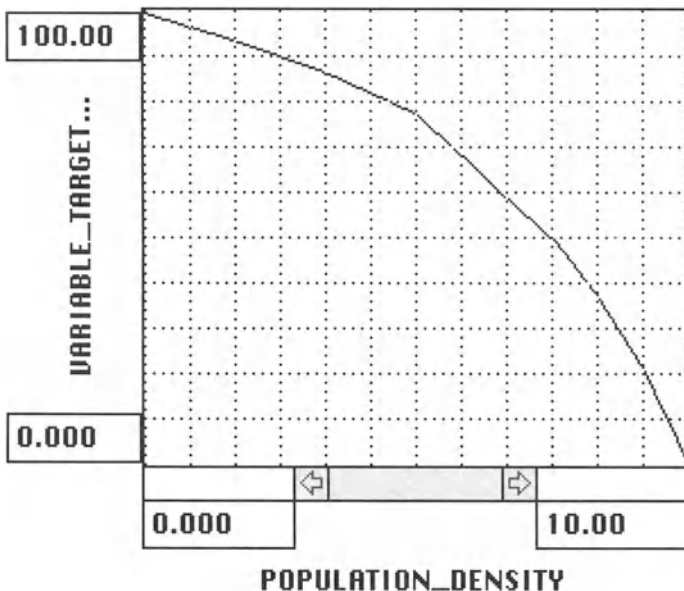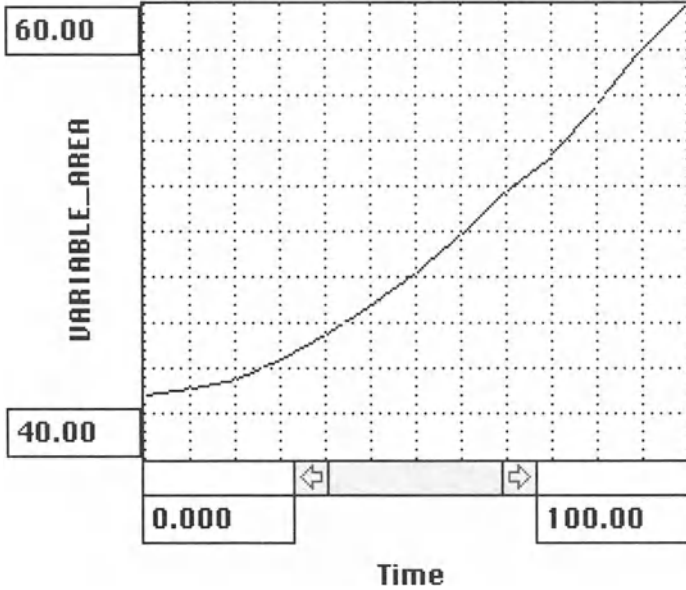
We further assume that external forces affect the size of the area that can be inhabited, thereby influencing the population density and, thus, the level for a steady population size (the target) that may be achieved. The change in area over time is graphically defined as in Figure 2.10. The resulting model is shown in Figure 2.11, and its dynamics are depicted in Figure 2.12.
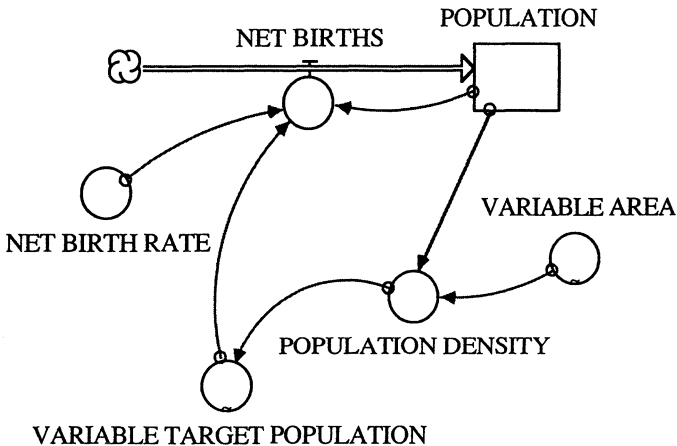

FIGURE 2.11

1: POPULATION
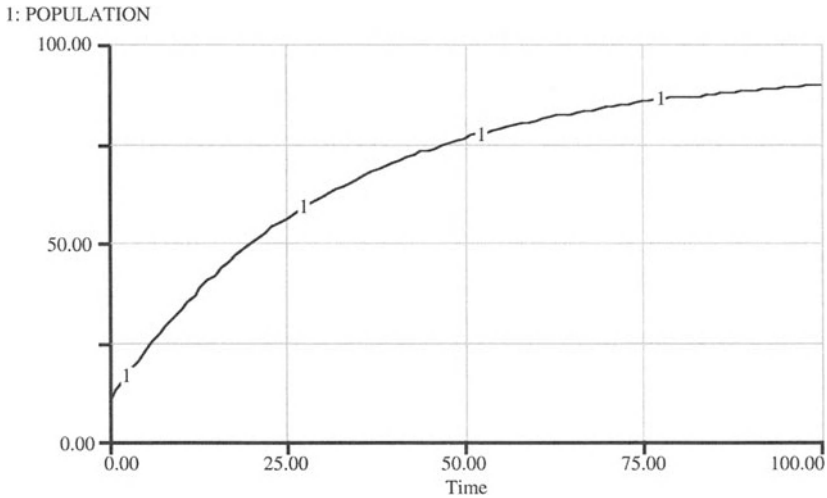
FIGURE 2.12

You should be able to find these basic forms of STIMULUS-RESPONSE, SELF-REFERENCING, GOAL SEEKING, and GOAL SETTING in many different types of systems. For example, the plight of the manufacturer of a widget may depend on a goal that is set endogenously rather than exogenously. To maximize profits, this person uses a given price for widgets taken from the marketplace. This price and the profit-maximizing procedure combine to yield an optimal size for the producer's company. But suppose foreign competition is affecting the price in unpredictable ways. Then our producer must continually adjust the price in the model to account for the price variation. In this case, the goal is profit (the most possible) but the price enters directly into the profit calculation. Other examples are provided in the following section.

```
                        GOAL-SETTING MODEL

POPULATION(t)  =  POPULATION(t - dt) + (NET_BIRTHS) * dt
INIT POPULATION = 10 {Number of Individuals}
INFLOWS:
NET_BIRTHS =
NET_BIRTH_RATE*(VARIABLE_TARGET_POPULATION -
POPULATION) {Number of Individuals per Time Period}
```

```
NET_BIRTH_RATE = .03 {Net Addition to, or Subtraction
of, Individuals per 100 Individuals in the Population
per Time Period}
POPULATION_DENSITY = POPULATION/VARIABLE_AREA
{Individuals per Square Kilometer}
VARIABLE_AREA = GRAPH(TIME)
(0.00, 42.8), (8.33, 43.1), (16.7, 43.5), (25.0, 44.4),
(33.3, 45.5), (41.7, 46.7), (50.0, 48.1), (58.3, 49.9),
(66.7, 51.7), (75.0, 53.3), (83.3, 55.5), (91.7, 58.0),
(100, 60.0)
VARIABLE_TARGET_POPULATION = GRAPH(POPULATION_DENSITY)
(0.00, 99.5), (0.833, 96.5), (1.67, 93.5), (2.50,
90.0), (3.33, 86.5), (4.17, 82.0), (5.00, 77.5), (5.83,
68.5), (6.67, 59.0), (7.50, 50.0), (8.33, 37.0), (9.17,
21.0), (10.0, 0.00)
```

2.5 Examples

2.5.1 Exponential Decay of a Stock

Assume that there is a stock of some substance that decays over time. The rate of decay is proportional to the remaining concentration. The proportionality factor is a physical constant that is characteristic for that particular substance. How would we model the process of decay in order to find the time period at which the concentration falls below a certain level?

This is one of the simplest models that we can build in STELLA. It consists of a stock that is named CONCENTRATION. With time, the concentration is depleted at a DECAY RATE. The decay rate is proportional to the level of the concentration. The constant of proportionality is called the DECAY CONSTANT. The DECAY RATE is calculated by multiplying the CONCENTRATION and the DECAY CONSTANT:

$$\text{DECAY RATE} = \text{DECAY CONSTANT} * \text{CONCENTRATION} \qquad (9)$$

We arbitrarily set the initial concentration to 100 and the decay rate to .1733. The corresponding model is shown in Figure 2.13.

Before you run the model, note the units of the decay rate, decay constant, and concentration. Make sure that these units are compatible with each other. Also, guess the dynamic path of CONCENTRATION.

The result is shown in the graph of Figure 2.14—an exponential decay toward zero from the original concentration. We can now find on our graph the time period at which the concentration falls below a certain level. For example, after 4 hours, the concentration is less than 50%.

This model is of the simple self-referencing type discussed above. It is as though the concentration is the driving force. Does the concentration ever
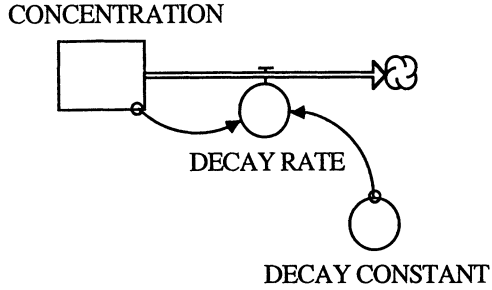
CONCENTRATION

FIGURE 2.13

reach zero? It is like Zeno's paradox, or the frog that jumps half the remaining distance out of a well with each jump, or the movement of a chemical across an osmotic membrane. With each time period we get slightly closer but the amount by which we approach zero concentration decreases also.

Redraw the flow of this model so that it adds, rather than subtracts, from the stock, and then run the model again after rescaling. In the case of an inflow, the model yields some form of attractor, whose force of attraction is proportional to its size, very similar to compounded interest on an initial investment. We will discuss this interpretation of the reverse of diffusion in a later chapter.

Change the time step and run the model again. Do so by running the model first at a DT = 1, then at DT = 0.5, DT = 0.25, DT = 0.125. You should always explore the sensitivity of the model to changes in the time step, since we are using numerical approximation techniques to solve for the dynamics of the system. When changing the DT for a successive run, follow the rule
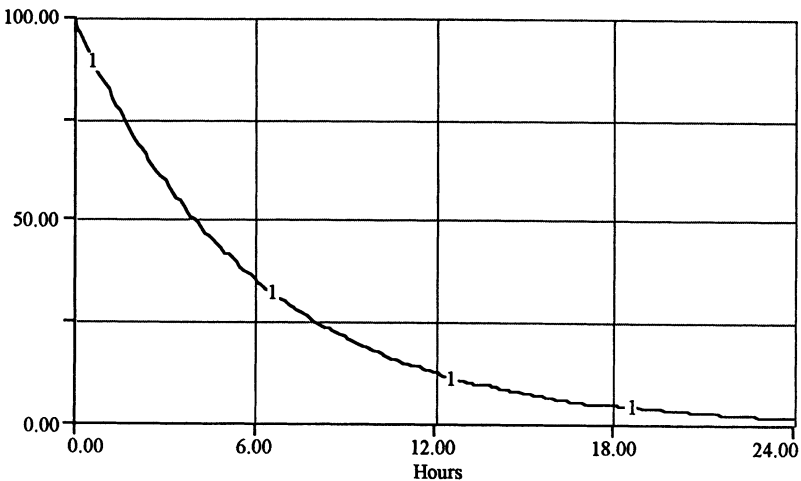

FIGURE 2.14

$$DT_{new} \; 5 \; \frac{DT_{old}}{2} \tag{10}$$

Keep on reducing the DT until the results of your simulation do not change significantly. You recognize that the speed of calculation is reduced with

---

EXPONENTIAL DECAY

```
CONCENTRATION(t) = CONCENTRATION(t - dt) + (-
DECAY_RATE) * dt
INIT CONCENTRATION = 100 {micro curies}
OUTFLOWS:


DECAY_RATE = DECAY_CONSTANT*CONCENTRATION {Rule: the
decay rate of the initial concentration is linearly
proportional to the concentration level.}
DECAY_CONSTANT = .1733{1/days}
```

---

smaller DT. Thus, you may not want to start out with a very small DT but try to find the one that is just small enough for the purposes of your model in the sense that further decreases in DT do not alter the results significantly.

2.5.2 Newtonian Cooling

This model presents a slight complexification of the problem outlined in the previous example. The system for which the model is designed consists of some object of a particular temperature surrounded by a cooler environment of ambient temperature. Let us assume that this environment is large in comparison to the object and that the ambient temperature is not increased to a measurable extent as the object cools. The object's temperature is determined by the difference between the temperature of the object and that of the surrounding environment. The ambient temperature becomes the target final temperature of the object.

In order to model the cooling process we must answer the following two questions: What is the state variable (stock) of this system? How is the state variable affected by other components of the system? The state variable is the temperature of the object that is above the environmental or ambient temperature. We represent this state variable by a stock. The scale for the temperature does not matter. Scale differences are compensated for in the size of the decay constant, which we call COOLING CONSTANT in this model.

TEMPERATURE

COOLING RATE

COOLING CONSTANT   AMBIENT TEMPERATURE
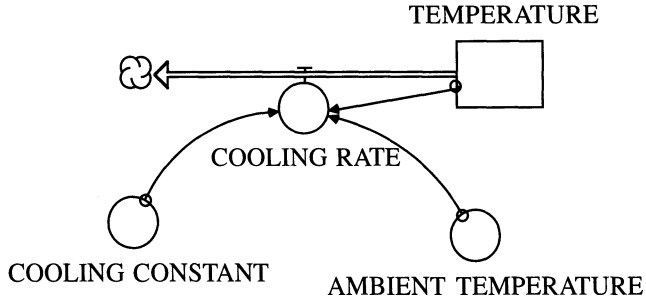
FIGURE 2.15

For simplicity, the ambient temperature is set constant—it does not change as the object cools. The cooling rate of the object then is

COOLING RATE = COOLING CONSTANT
                    * (TEMPERATURE − AMBIENT TEMPERATURE)   (11)

The STELLA diagram is shown in Figure 2.15. The model is of the simple goal seeking type. Its behavior is shown in Figure 2.16.

The resulting graph is similar to the exponential decay of the first example, except that the target is not zero here. Try to build a model in which a cool object is heated. You can easily develop such a model by reversing the direction of the flow pipe.

Also, try to change the time step of the model simulation. How fast can you run the model and not sacrifice the level of your desired accuracy? As we discussed above, the rule is that you keep cutting the time step in half

1: TEMPERATURE

FIGURE 2.16

as long as the level of desired accuracy is maintained. Another way to increase the accuracy of the solution is to change the solution technique to Runge-Kutta 2 or 4 (second or fourth order). Orde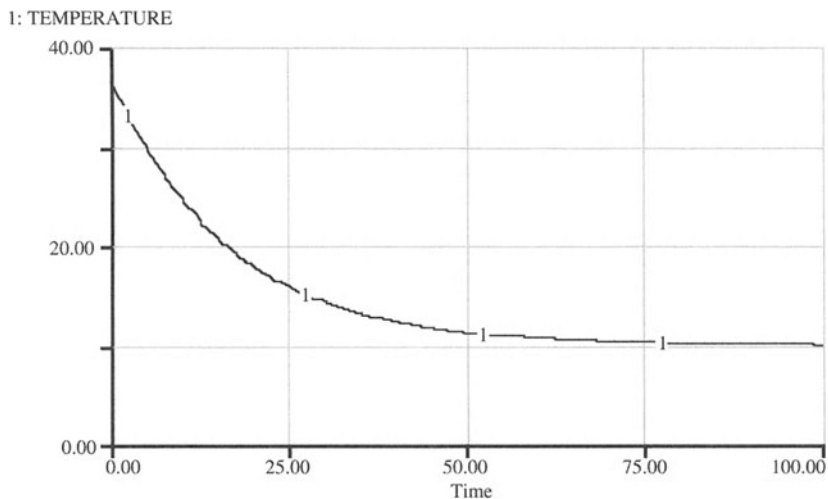r refers to the number of subdivisions on the time step over which approximations of the variable are made. These last two techniques have certain limits as we shall see later.

Can you construct a two-stage reduction in the temperature where the first body rejects its heat to a second body, which in turn rejects it to the environment? Assume the two bodies are the same size and material. What adjustments of the model are necessary to reflect temperature changes in the environment? How do these changes in the environment's temperature affect the final temperature of the object? What will happen, when the environmental temperature fluctuates over time due to some natural processes (e.g. the diurnal or seasonal warming and cooling of the surroundings)?

```
                    NEWTONIAN COOLING

TEMPERATURE(t) = TEMPERATURE(t - dt) +
(- COOLING_RATE) * dt
INIT TEMPERATURE = 37 {Initial temperature of a body
in degrees centigrade cooling to the surrounding
temperature of the environment: 10 Degrees C}

OUTFLOWS:
COOLING_RATE = COOLING_CONSTANT*(TEMPERATURE-
AMBIENT_TEMPERATURE)
{Newton's law of cooling: the rate of change of a body
is linearly proportional to the temperature difference}
AMBIENT_TEMPERATURE = 10 {Degrees C}
COOLING_CONSTANT = .06 {1/Time Period}
```

3
Gradual Development
of a Dynamic Model

> Imitation cannot go above its model.
> —Ralph Waldo Emerson, *Collected Works*

3.1 Modeling Industrialization
for a Simple Agrarian Society

In the previous chapter we modeled simple dynamic systems and identified a set of basic model forms. While the previous chapter outlined how to model a simple system under alternative assumptions, this chapter shows how to expand and refine a model. Additionally, we want to stress the need for treating consistently the units in which parameters and variables are measured.

We can best demonstrate the process of dynamic modeling by describing first the process that we wish to model and the parts of the model that determine its dynamic behavior. Then we may want to abstract away from a large number of details that characterize the real system, thereby concentrating on the main driving forces. Once these forces are modeled appropriately, we may gradually account for the details that we left out originally.

For example, let us consider the status of an agrarian population of 10 people on an island of unit area, who are unaccountably blessed with a cache of food. This cache of food is their only source of nutrition and, therefore, controls the level of their population. The wiser members of the village realize that the cache will soon be empty and propose that the community take up the planting of seeds, which will provide them with a continuous food supply. After some years at such labor, they discover the fact that the use of tools in their primitive agricultural process will ease the labor on the field and make these same fields more productive. They eventually build factories that produce the implements for their agriculture, changing forever some of their Auburn stalwarts into factory lads, beginning the trend of the great cities of the world.

How can we possibly even begin to model the economic civilization process of the industrial revolution? We recommend starting small and simple. Once the basic features of the agrarian village's decision process are captured in a simple model, we may proceed to refine that model and extend it to include the process of growing food and crafting tools.

From some of the previous very simple models, we can guess that we need some measure of the level of the population. That level will determine

much of what happens in the entire model. As before, we introduce a state variable called POPULATION that is controlled by a flow into POPULATION. We call this flow controller BIRTHS. The stuff from which people are actually made, all that carbon, hydrogen, and nitrogen, are in such great supply that we need not worry about depleting them. So they come from the external environment, from the "cloud," that marks the system boundary. At this time we do not worry about the fact that people are not born fully adult; we leave the concept of age cohorts to a later model. The specification for the flow of BIRTHS is given as

$$\text{BIRTHS} = \text{BIRTH RATE} * \text{POPULATION} \qquad (1)$$

The flow control takes its signals from several places. One of them is the BIRTH RATE variable. The units of this variable are important, and the lesson of checking units in these models cannot be overstressed. The units are "new persons born per person already in the population, per year." Here we have decided to make this birth rate a function of the population density. Our little island will someday be quite crowded and it surely occurs to even the most virile that a suspension of the birth rate is needed. Let's assume that our anthropologists have found that under such conditions, the maximum population will be 200; the people will stop reproducing when their island population reaches 200. The anthropologists further tell us that when space is absolutely no problem, the birth rate is 0.1. The relationship between birth rate and population size is shown in Figure 3.1.
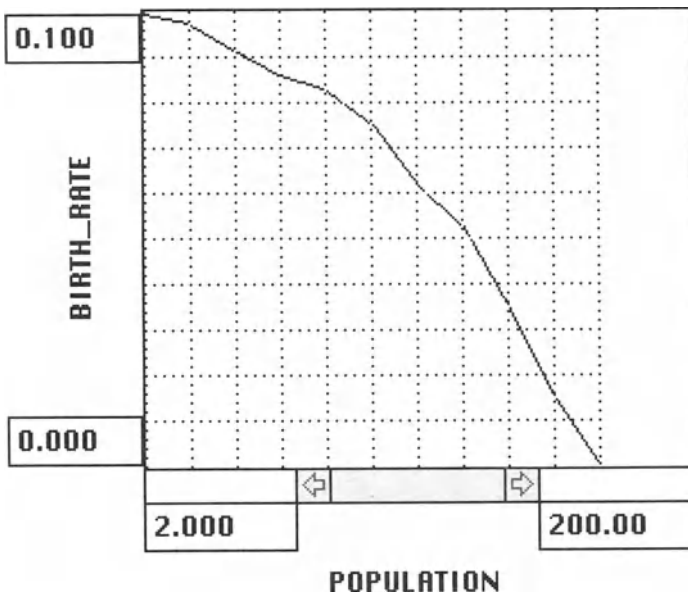

FIGURE 3.1

For simplicity, we may define or normalize the size of the island as 1.0. Since the island area is one, the population and the density are the same—just the units are different. Such an assumption keeps the model simpler in form than it should be in reality.

We specified birth rates at various population densities as a graph. We did not know more about the reproductive habits of these people so we have sketched in the data between these two given points. This is not an unusual case. Sometimes we have some data and the rest of it must be reasoned. STELLA contains a sensitivity analysis procedure and we can see how sensitive our results of an ultimate population level will be to different interpretations of the intermediate forms of this graph. We will do that later.

The flow control needs the BIRTH RATE information but as it is a rate "per person," the controller also needs to know the current level of the POPULATION. Arrows, the tools that carry the information in the direction of the arrowheads, must convey the information on the birth rate and the population to the controller. Once we make the connections between population, births, and birth rate, our model looks like Figure 3.2.

We are now ready to test our model. But first we must guess the form of our results. What will the population dynamics reveal? First we can guess that the population will reach a steady level of 200. If the population is to become steady, it will not do so suddenly. Rather, the population growth will slow gradually toward zero as it approaches the 200-person level. Yet, with an initial population of only 10, the initial growth rate will be very high. We expect a sigmoidal curve, approaching 200 after a long passage of time.

Set the DT for 1 (year) and the upper bound on the time at 160 years. Select a graph and then select the POPULATION variable. Run this model and see what you get. The results of the assumptions above lead to the dynamics in Figure 3.3. We do get what we reasoned the answer to be.

Redraw the BIRTH RATE curve into a convex form and note the different results. The intermediate results are sensitive to the shape of this curve. In order to develop a reliable model of the population dynamics of our island population, we therefore need to better know the values of the birth rate at different population levels. We send our anthropologists back to the library for more research.
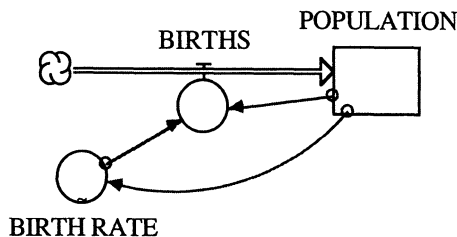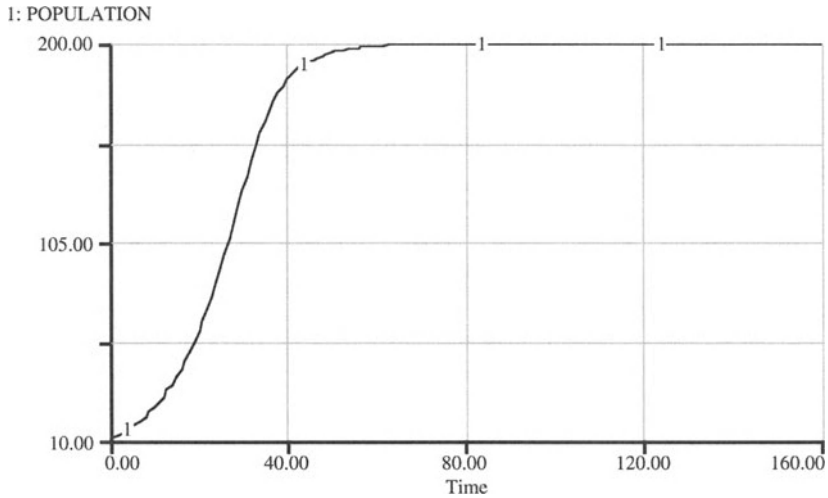

FIGURE 3.2

1: POPULATION

FIGURE 3.3

If you are using the Macintosh version of STELLA you can have your an-thropologists provide you with a data set relating birth rates to various population sizes. This data set can be made available to us in a separate file, created, for example, with some spreadsheet software that enables data sets to be "published," i.e., put in a format that can be recognized by other software. Let us assume that the data set consists of a column with birth rates for 21 different population sizes. With the Macintosh computer we can directly read this data set into the STELLA program for the simple agrarian society. Double-click on the translation variable BIRTH RATE. Then click on "Output" at the upper right-hand corner in the dialog box of the graph. Next choose "Subscribe To" in the "Edit" pull-down menu. A new dialog box appears that requires you to select a data set (or "edition") to which STELLA should subscribe. Choose "Birth Rate Publisher" as the data set of your choice—this is the one provided by our anthropologists and it comes with the disk for this book. The data set we are using here is, of course, made up. Scroll along the new graph that has been created using the data set from "Birth Rate Publisher" to see how birth rates change with increas-ing population sizes. Now click on OK and rerun the model. Figure 3.4 shows what you should get.

In STELLA we can subscribe to data sets in order to specify graphs, as we have done in this chapter, or to specify flows, translation variables, or in-itialize stocks. In each case, we can subscribe to editions that contain numbers only—graphics or equations are not permissible. For the specifi-cation of graphs and flows the editions to which we subscribe must be in the form of a single column. Translation variables and reservoirs, in con-trast, require only a single initial value. Conveyors or queues—we will dis-
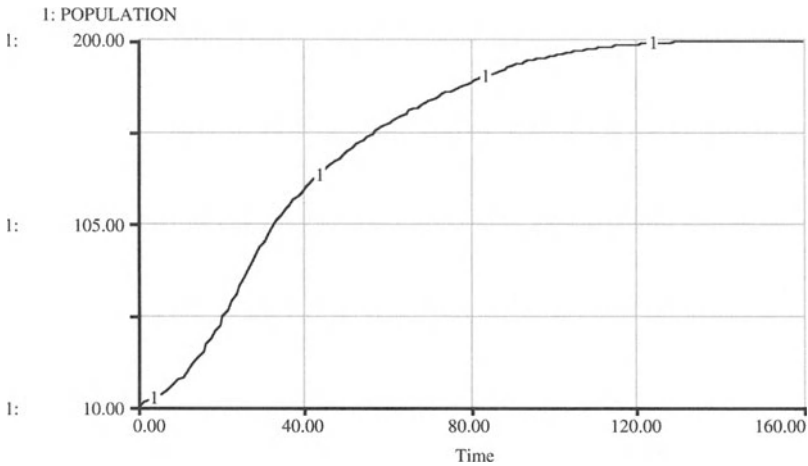
FIGURE 3.4

cuss them in Chapters 17 and 35—require initial values that are separated by commas.

If the edition you are subscribing to in your model does not conform with the requirements for the specification of your STELLA model, STELLA will let you know in a dialog box once you attempted to subscribe to an edition and clicked on OK. To fix the problem, choose "Subscription Options . . ." from the "Edit" menu, click on "Cancel Subscriber," and then click OK.

With STELLA we can not only subscribe to editions published by other software, we can also publish our own results in editions that are then saved on disk. These editions may then be used as inputs into other programs. Each table created in STELLA can be a publisher of only one edition. If you wish to publish separate editions of your data, you need to plot this data in separate tables. To learn how to publish an edition, create, for example, a table for the model above and plot in this table POPULATION. Run the model, then click on the heading POPULATION in the table so that the whole column is highlighted (or highlight several columns if you have plotted them in the table and wish to publish them). Next go to the "Edit" menu and select "Create Publisher," give your edition a name and click on "Publish." This edition will be saved on your disk if and when you save your model.[1]

While the model of the agrarian society developed above is overly simplified, we have solved a nonlinear differential equation. This nonlinear

---

[1]Of course, saving your model is not possible with the run-time version of STELLA. Nevertheless, you can access your edition within STELLA as long as the model with which you created the edition remains open.

differential equation is analytically intractable and can be solved only numerically.

What is the very first step that you would take to increase the reality of this population model? Why not consider how the people leave the island? Since the climate here is idyllic, no one wants to leave. But people do leave through death of course. The death process is nearly a mirror of the birth process and is uninstructive to add alone. In the next problem, we add the food cache and tie food availability to the death rate.

SIMPLE AGRARIAN SOCIETY

```
POPULATION(t) = POPULATION(t - dt) + (BIRTHS) * dt
INIT POPULATION = 10 {Individuals}
INFLOWS:
BIRTHS = BIRTH_RATE*POPULATION {Addition of
Individuals per Time Period}

BIRTH_RATE = GRAPH(POPULATION)
(2.00, 0.0995), (21.8, 0.0975), (41.6, 0.091), (61.4,
0.0855), (81.2, 0.083), (101, 0.075), (121, 0.062),
(141, 0.053), (160, 0.035), (180, 0.015), (200, 0.00)
```

3.2. Impacts of Per Capita Food Consumption on Population Growth

Here we have added the flow control DEATHS, which leads to a cloud (another system boundary) since we don't care in this model to remember the number of deads. Deaths are controlled by the DEATH RATE, a graphical function as before except that now the death rate is controlled by the amount of food consumed per head, EAT PER HEAD (Fig. 3.5). This variable, in turn, is a function of the available FOOD PER CAPITA, the ratio of the remaining FOOD (initially 100 units) to the current population (Fig. 3.6). The idea here is that the leaders of our little community will begin to ration the food according to an agreed-upon rule when the food supply gets low.

Finally, we realize that the state variable FOOD must decline as food is eaten. The amount eaten per person times the number of people gives the rate EAT, which is the flow control for FOOD consumed:

$$EAT = EAT\ PER\ HEAD * POPULATION \tag{2}$$

The resulting STELLA model is shown in Figure 3.7.

Build and run the model and form a graph, as before, for the state variables POPULATION and FOOD. But first, guess the graphical form for this
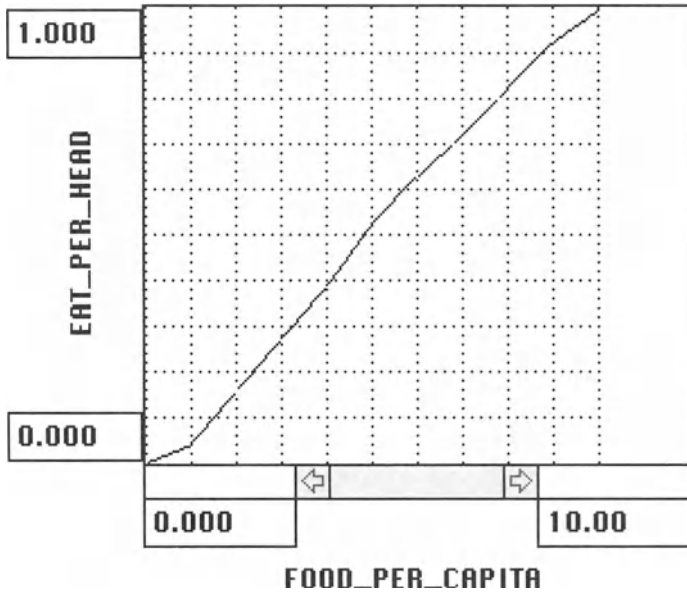
FIGURE 3.5
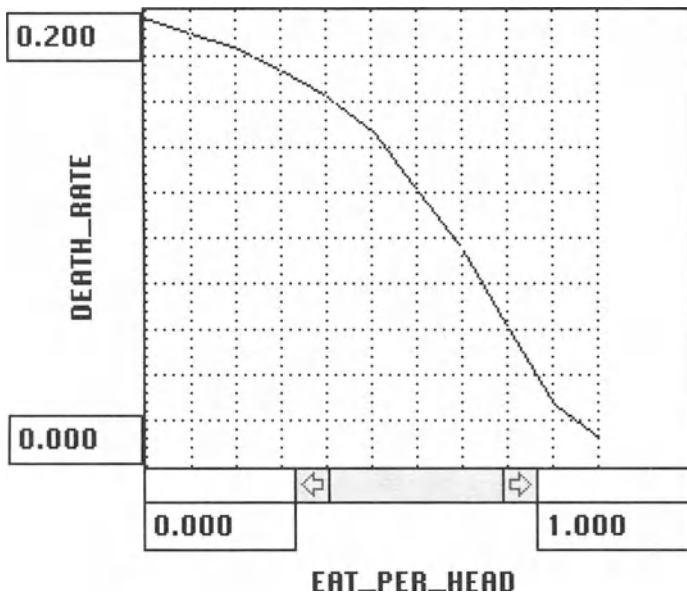
FIGURE 3.6
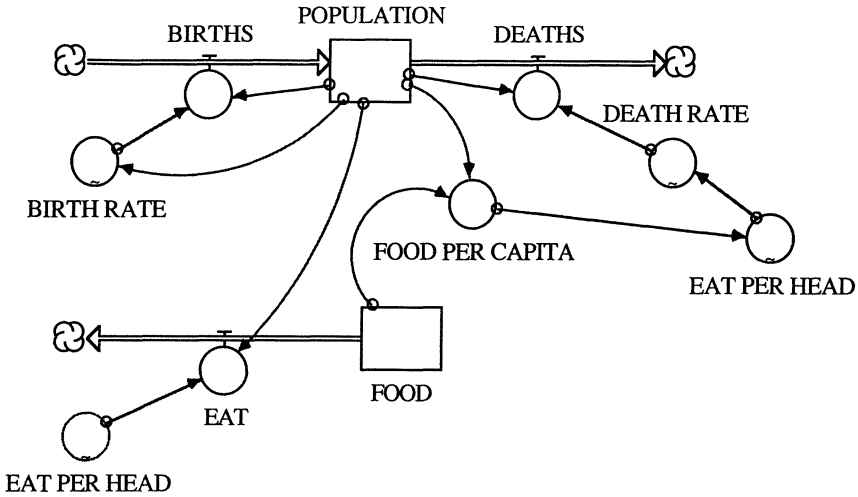
FIGURE 3.7

population as our more complex model utters its revelations of the future. Will the ultimate population be 200 or 0 or something in between? Remember, we have only a limited supply of food. The food variable should monotonically decline since there is no replenishment process. If we are running the model long enough we will see the end of our people! But not before they rise to a great peak (less than 200 of course). Compare your model output with ours, which is shown in Figure 3.8.

Try changing the shape of the EAT PER HEAD graph to represent different scarcity control strategies. Suppose a very democratic regime was in

FIGURE 3.8

control and whatever was left is shared equally. Try to model that case. Suppose, on the other hand, that the community leaders are weak and let a few strong members hoard food while others starve. How would a model for this situation appear? Should the birth rate be linked somehow to the remaining food in an attempt to avoid the sharp collapse? Would the per capi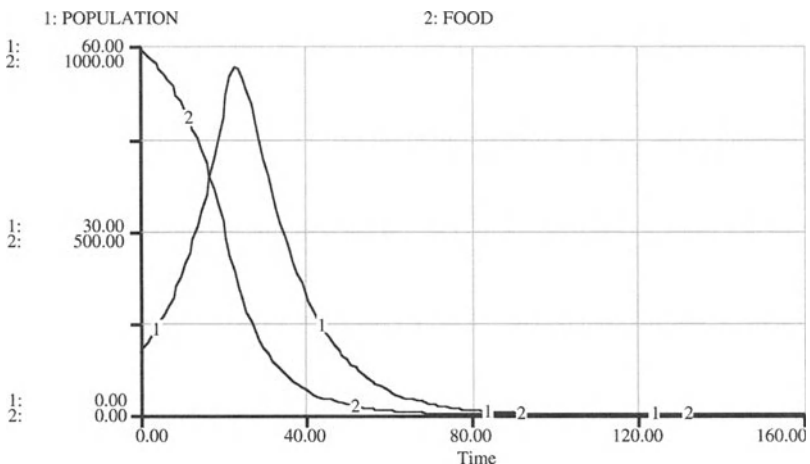ta consumption be better represented as a delayed function of the available food per capita? Try these mode extensions and try introducing a 2-year lag in the consumption response.

   In reality, these people are hunter-gathers and as such they could live from the natural larder with more or less a steady population. They would be in tune with nature, and they would have occasional times of drought or flood. Their population may fluctuate somewhat during these times and we will suppose that during one such time, this particular community opted for agriculture. This is the subject of the next part of our model for the developing society.

FOOD CONSUMPTION AND POPULATION GROWTH

```
FOOD(t) = FOOD(t - dt) + (- EAT) * dt
INIT FOOD = 1000 {Kilograms; reduce to eliminate the
first peak of the population.}
OUTFLOWS:
EAT = EAT_PER_HEAD*POPULATION {Kilograms per Time
Period}

POPULATION(t) = POPULATION(t - dt) + (BIRTHS - DEATHS)
* dt
INIT POPULATION = 10 {Individuals}
INFLOWS:
BIRTHS = BIRTH_RATE*POPULATION {Individuals per Time
Period}
OUTFLOWS:
DEATHS = DEATH_RATE*POPULATION {Individuals per Time
Period}

FOOD_PER_CAPITA = FOOD/POPULATION {Kilograms per
Capita}
BIRTH_RATE = GRAPH(POPULATION)
(0.00, 0.0995), (20.0, 0.0975), (40.0, 0.091), (60.0,
0.0855), (80.0, 0.083), (100, 0.075), (120, 0.062),
(140, 0.053), (160, 0.035), (180, 0.015), (200, 0.00)
DEATH_RATE = GRAPH(EAT_PER_HEAD)
(0.00, 0.197), (0.1, 0.19), (0.2, 0.184), (0.3, 0.173),
(0.4, 0.163), (0.5, 0.147), (0.6, 0.121), (0.7, 0.095),
(0.8, 0.061), (0.9, 0.028), (1.00, 0.013)
```

```
EAT_PER_HEAD = GRAPH(FOOD_PER_CAPITA)
(0.00, 0.00), (1.00, 0.04), (2.00, 0.155), (3.00,
0.27), (4.00, 0.385), (5.00, 0.52), (6.00, 0.63),
(7.00, 0.72), (8.00, 0.82), (9.00, 0.92), (10.0, 0.995)
```

3.3 Adding Agriculture

To add agriculture we must put people to work in the fields. FOOD PRO-
DUCTION draws upon the available labor and land in the population. It is
another flow control variable for the state variable FOOD. Let's assume that
half of the people are available for food production. Our problem becomes
one of converting person-years of labor into a rate of food production. We
could just revert to a graph that converts person-hours into a food produc-
tion rate. But here we call in economists and their idea of production func-
tions. These production functions are mathematical relationships among
the inputs that are used in a production process and outputs—they are the
recipes for the production of a particular good. The total amount of land for
this population is assumed to be fixed and of variable quality. Our group
will no doubt farm the best land first and yet even as they bring more land
into production through the use of more labor, they suffer the diminishing
returns of the increasingly poor land.

   In order to arrive at a mathematical description of the relationships
among inputs and outputs, economists look over the historical data of pro-
duction sectors in an economy. The economists of our simple society, for
example, may have observed over time different levels of labor input into
the agricultural production process and outputs of food. Based on these
historical records they fit the data to the following kind of function:

$$\text{FOOD PRODUCTION} = A*(\text{LABOR})^{\wedge}\text{ALPHA} \qquad (3)$$

In our case, the economists report values for the parameters A and ALPHA.
The estimates used in our model are A = 5 and ALPHA = 0.3. Additionally,
we assume that half of the population is in the labor force. Thus, LABOR is
taken as POPULATION divided by 2.

   It is useful to pause here and consider how ecologists model this type of
phenomenon. In the terms of this model, they would call A the maximum
possible food production rate, the rate under ideal conditions. They would
substitute for LABOR^ALPHA a concave graph whose maximum value was
1 and whose minimum was 0, a graph representing the "effectiveness" of
labor. Where there are other inputs to producing food, there would be ad-
ditional such graphs, each with a maximum effectiveness of 1, each multi-
plying A. This is essentially the same approach that we are using here and
that economists commonly use.

FIGURE 3.9

With each additional feature, the diagram of the model increases in size. In larger models that have highly interdependent components we need to make a large number of connections. With an increasing number of connections, or information arrows, the readability of the model can be seriously reduced. You may end up with a model that looks more like a pile of spaghetti than a computer program. Make use of "Ghosts" of icons to avoid crossing arrows and increase the transparency of the model structure. You can create a ghost, for example, of a state variable by first clicking on the ghost icon (Fig. 3.9) in STELLA's tool bar.

Once you clicked on the ghost icon, move it to the variable that you want to duplicate. Click on the symbol you want to duplicate. The ghost icon then changes its appearance into that of the symbol you clicked on. You can now place this duplicate of the original anywhere in the diagram and connect it with information arrows to the relevant parts of the model.

In our model, we created a duplicate, or "Ghost," of POPULATION for the society with agriculture (Fig. 3.10). The ghost always assumes the value of the original. Be aware that all changes to the variable must be made to the original, not the ghost. For example, if you wish to model emigration
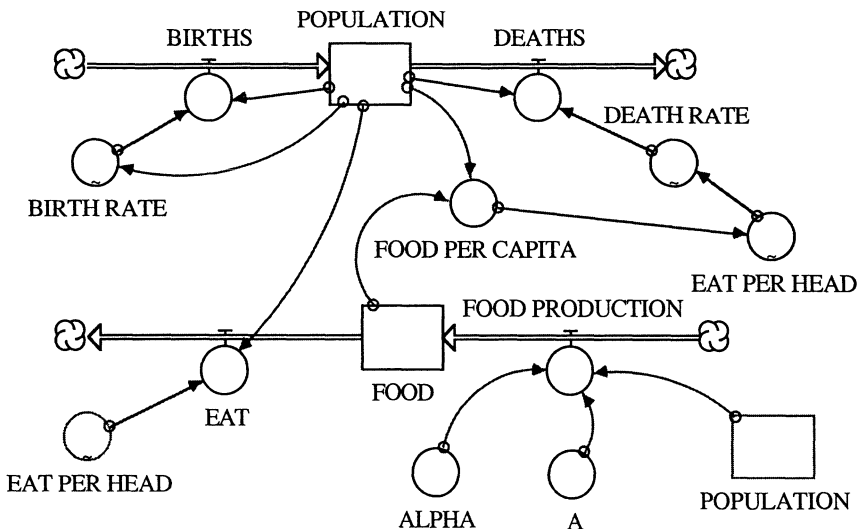

FIGURE 3.10

from our society you must draw an outflow from the original POPULATION stock, not the ghost. Similarly, information arrows can originate from the ghosts to connect them to other parts of the model, but it is not possible that ghosts receive information arrows.

Our model is again ready to run and it is time to stop and guess the form of the population response. About all that we can say now is that the population will reach some sort of steady state, commensurate with the rate of food production, after the effect of the initial food cache is over. We are hard pressed to know whether the steady state level will be higher than the previous peak population or lower. Try changing the initial level of the food cache to make the steady state level the historical high. In this way we can avoid the trauma of population overshoot and possibly chaos. Try varying the amount of the population available as labor to see how sensitive it is. Can you add food spoilage to the model (say 5% is lost per year)?

The results in Figure 3.11 are plotted for the same graphical relationship EAT PER HEAD as in the previous model. As the model gets more complicated, our ability to correctly guess the form of the answer decreases. Nevertheless, we should always try. By trying, we improve our intuition about such complex behavior. Wrong guesses should always be dissected to find any flaws in reasoning. Upon more careful thinking you may perhaps find that your reasoning was correct, and that your model has a flaw. In either case, making educated guesses can help you significantly improve your understanding and model of a system.

Our model is still incomplete. Surely, our people would want hoes, planting sticks, and scythes. At first they would make them from scrap materials but before long, they would organize a factory and begin to make their agricultural implements. This is the subject of the next and final part of the model of the industrialization of a simple agrarian society.
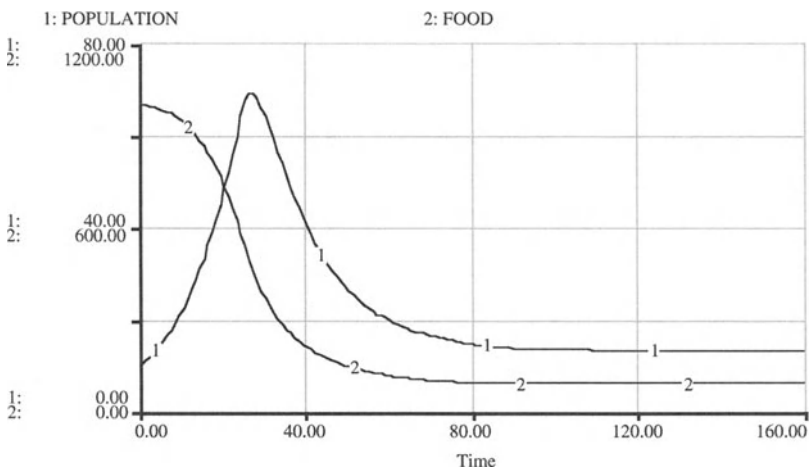

FIGURE 3.11

ADDING AGRICULTURE

```
FOOD(t) = FOOD(t - dt) + (FOOD_PRODUCTION - EAT) * dt
INIT FOOD = 1000 {Kilograms; reduce to eliminate the
first peak of the population.}
INFLOWS:
FOOD_PRODUCTION = A*(POPULATION/2)^ALPHA {Kilograms
per Time Period}
OUTFLOWS:
EAT = EAT_PER_HEAD*POPULATION {Kilograms per Time
Period}

POPULATION(t) = POPULATION(t - dt) + (BIRTHS - DEATHS)
* dt
INIT POPULATION = 10 {Individuals}
INFLOWS:
BIRTHS = BIRTH_RATE*POPULATION {Individuals per Time
Period}
OUTFLOWS:
DEATHS = DEATH_RATE*POPULATION {Individuals per Time
Period}

A = 5 {1/Individuals}
ALPHA = .3
FOOD_PER_CAPITA = FOOD/POPULATION {Kilograms per
Individual}
BIRTH_RATE = GRAPH(POPULATION)
(0.00, 0.0995), (20.0, 0.0975), (40.0, 0.091), (60.0,
0.0855), (80.0, 0.083), (100, 0.075), (120, 0.062),
(140, 0.053), (160, 0.035), (180, 0.015), (200, 0.00)
DEATH_RATE = GRAPH(EAT_PER_HEAD)
(0.00, 0.197), (0.1, 0.19), (0.2, 0.184), (0.3, 0.173),
(0.4, 0.163), (0.5, 0.147), (0.6, 0.121), (0.7, 0.095),
(0.8, 0.061), (0.9, 0.028), (1.00, 0.013)
EAT_PER_HEAD = GRAPH(FOOD_PER_CAPITA)
(0.00, 0.00), (1.00, 0.04), (2.00, 0.155), (3.00,
0.27), (4.00, 0.385), (5.00, 0.52), (6.00, 0.63),
(7.00, 0.72), (8.00, 0.82), (9.00, 0.92), (10.0, 0.995)
```

3.4 Adding Industry

To the basic model, we have added agriculture, which absorbs labor and capital through a production function, and capital production, which absorbs labor and produces implements used in agriculture. Because we have

both labor and implements contribute to the production process, we need to replace the production function of the previous section with one that shows both inputs as having influence on the food output. Our new production function is

FOOD PRODUCTION = A * (POPULATION/4)^.3 * IMPLEMENTS^.6   (4)

and it assumes that one quarter of the population participates in agriculture.

Note this form of converting labor and implements into food. It is again called a production function. There are many alternative forms of the production function used in economics, depending on the type of production process, the historical data, and statistical criteria. For the type of production function used here, there are restrictions on the exponents. These exponents should sum to less than or equal to 1.

Analogously to the previous section we specify a production function for the manufacturing of implements as a function of labor only:

MANUFACTURING = B*(POPULATION/4)^.2   (5)

The labor used in manufacturing is here simply one quarter of the population.

The implements wear out through use and so the number of them eventually reaches a steady state. The wear and tear is modeled as an outflow from the stock of implements, very much like we model the death of people as an outflow from a population stock.

DETERIORATION = .04*IMPLEMENTS   (6)

The rate at which implements deteriorate—the "death rate of the stock of tools"—is for simplicity set constant, but it could depend on the rate of their use in agriculture. The model could be further elaborated upon by finding the optimum disposition of labor between direct agriculture and implement production, where "optimum" is defined as the maximum food production per capita. Without these refinements, the model looks as in Figure 3.12.

Before you run the model, guess the shape of the FOOD and POPULATION stock curves. We cannot overemphasize that this sort of guessing tends to develop your intuition about the behavior of complex dynamic systems, so it is something that you should always do before running your own models. The results of our model assumptions are shown in Figure 3.13.

In subsequent runs of the simulation model, use at most one half of the population for the agricultural and industrial workforce. Find the division of labor between agriculture and industry that yields the largest steady state population at a level larger than the initial population. Similarly, try to find that division of labor that yields a time path for the population that is never declining and ultimately reaches a steady state.

The model of the simple society is becoming increasingly complicated. Many feedback processes of various strengths interact and determine the dynamics of the system. The relative importance or "strength" of these feedback processes depends to a large degree on the parameter values that we chose for our model. It is therefore very useful to investigate the sensitivity of our model result to small changes in the parameter values.
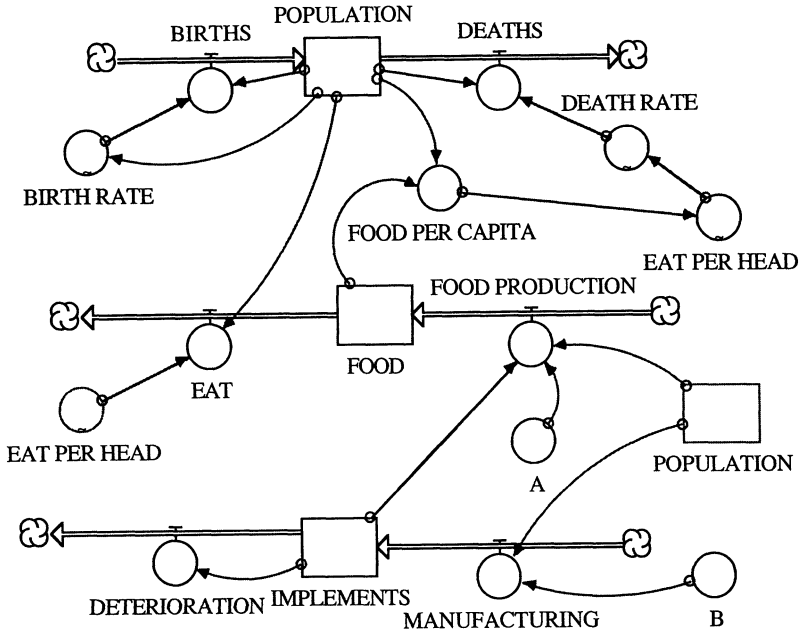
FIGURE 3.12

We can perform such a sensitivity analysis by choosing sensitivity runs, a convenient method provided by the STELLA software. Go to the "Run" pull-down menu, select "Sensi Specs," and choose the parameter B of our manufacturing production function as the one on which we want to perform a sensitivity analysis. Do three sensitivity runs, changing B between 0.05 and 0.15. Plot the three resulting curves for manufacturing production in the

FIGURE 3.13

same graph by choosing the Graph option in STELLA's Sensitivity Specs menu.

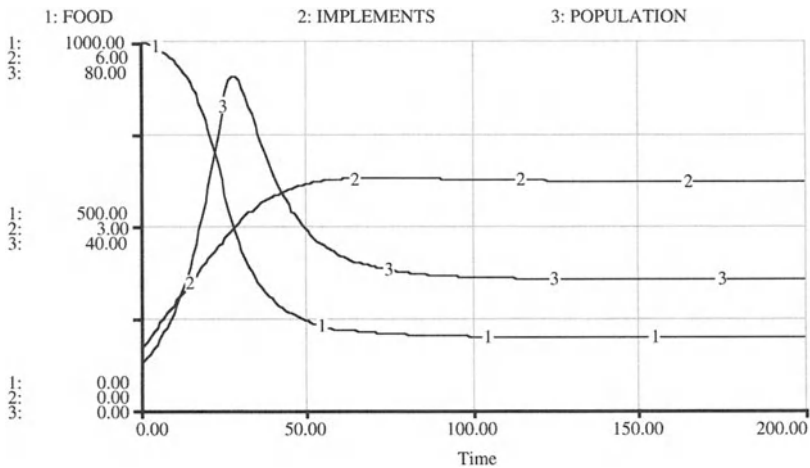How will MANUFACTURING change for different values of B? Run the model with the S-Run command and observe the resulting graphs. What will happen to MANUFACTURING if B is equal to 1, and what if it is greater than 1? Suppose that instead of maximizing the long-run steady state population, we maximized food available per capita or average lifespan. Can you do this by modifying this model?

ADDING INDUSTRY

```
FOOD(t) = FOOD(t - dt) + (FOOD_PRODUCTION - EAT) * dt
INIT FOOD = 1000 {Kilograms; reduce to eliminate the
first peak of the population.}
INFLOWS:
FOOD_PRODUCTION = A*(POPULATION/4)^.3*IMPLEMENTS^.6
{Kilograms per Time Period}
OUTFLOWS:
EAT = EAT_PER_HEAD*POPULATION {Kilograms per Time
Period}

IMPLEMENTS(t) = IMPLEMENTS(t - dt) + (MANUFACTURING -
DETERIORATION) * dt
INIT IMPLEMENTS = 1 {Number of Objects}
INFLOWS:
MANUFACTURING = B*(POPULATION/4)^.2 {Number of Objects
per Time Period}
OUTFLOWS:
DETERIORATION = .04*IMPLEMENTS {Number of Objects per
Time Period}

POPULATION(t) = POPULATION(t - dt) + (BIRTHS - DEATHS)
* dt
INIT POPULATION = 10 {Number of Individuals}
INFLOWS:
BIRTHS = BIRTH_RATE*POPULATION {Number of Individuals
per Time Period}
OUTFLOWS:
DEATHS = DEATH_RATE*POPULATION {Number of Individuals
per Time Period}

A = 5 {1/(Individuals*Objects) per Time Period}
B = .1 {1/Objects per Time Period}
```

```
FOOD_PER_CAPITA = FOOD/POPULATION {Kilogram per
Individual}
BIRTH_RATE = GRAPH(POPULATION)
(0.00, 0.0995), (20.0, 0.0975), (40.0, 0.091), (60.0,
0.0855), (80.0, 0.083), (100, 0.075), (120, 0.062),
(140, 0.053), (160, 0.035), (180, 0.015), (200, 0.00)
DEATH_RATE = GRAPH(EAT_PER_HEAD)
(0.00, 0.197), (0.1, 0.19), (0.2, 0.184), (0.3, 0.173),
(0.4, 0.163), (0.5, 0.147), (0.6, 0.121), (0.7, 0.095),
(0.8, 0.061), (0.9, 0.028), (1.00, 0.013)
EAT_PER_HEAD = GRAPH(FOOD_PER_CAPITA)
(0.00, 0.00), (1.00, 0.04), (2.00, 0.155), (3.00,
0.27), (4.00, 0.385), (5.00, 0.52), (6.00, 0.63),
(7.00, 0.72), (8.00, 0.82), (9.00, 0.92), (10.0, 0.995)
```

4

Two Independent Variables

> Might it not have been better to have left the death side of nature's population-control to itself until we have some future policy for dealing simultaneously with birth?
>
> —J. S. Huxley, *What Dare I Think?*

4.1 Population Cohorts

4.1.1 Basic Cohort Model

The goal of this chapter is to learn how to handle two independent variables in a STELLA model. Time is our usual independent variable and we use it to cause the change in our models. In the first section of this chapter, we introduce several stock variables to indicate various temporal aspects of the state variable of the system. Such a disaggregation can be done, for example, for a population into different age classes. Thus, so far we deal only with a more sophisticated representation of essentially a single variable, population.

Disaggregation can lead to a rather complicated model. Differentiating among people in a population by their exact age could lead to more than 100 state variables, if we model age in yearly increments, and more if we choose smaller increments. Not only could the resulting STELLA model be quite messy—consisting of many stocks and flows whose structure and functioning are virtually the same—but such a model may also be far too detailed to be of much use.

While the first section of this chapter maintains a high level of aggregation—people in the population are distinguished by only seven age classes—Section 4.1.2 shows how the STELLA representation of that system can be significantly simplified. Section 4.1.3 then presents an application of the age cohort model to the United States, and we expect that what you can learn there about population dynamics is of great use for applications to many other countries.

In the last section of this chapter we use the same modeling method as in the population age cohort model of this section to not only disaggregate a system with respect to the temporal dimension of the system but also with respect to its spatial dimension. By doing so, we model two independent variables—space and time. First, however, let's disaggregate a state variable with respect to the temporal dimension.

Let us develop a population model in which the state variable population is disaggregated into seven age cohorts. The model begins with 100 persons in the cohorts of 0- to 9- and 10- to 19-year-olds. Further, there are initially 75 individuals who are between 20 and 29 years old, 60 between 30 and 39 years, 50 between 40 and 49 years, 40 between 50 and 59 years, and

60

30 individuals of at least 60 years of age. Let us also assume that only members in the cohort of 20- to 29-year-olds reproduce. The coefficient for the birth rate is 0.12. We now need to model the reproductive rule such that members in the second cohort reproduce at a rate of 0.12 persons born per year per person of age 20 to 29 once they reach the third cohort. Also, we need to make sure that, during the first period, members in the third cohort move up into the next cohort where they are no longer able to reproduce. We model the reproductive process by using the built-in step function to change the birth rate accordingly. Then, the birth rate coefficient of 0.12 is multiplied by the size of the 20- to 29-year-old population. The population is moved to the next cohort by the "out-" transformers. In 10 years, nearly the whole cohort (why not the whole cohort?) is moved to the next stage. Finally, the last cohort is moved out of the system (death) at about age 70. The death rate varies across the cohorts. The resulting model is shown in Figure 4.1

The sizes of the age cohorts through time are given in Figures 4.2 and 4.3. Each curve represents the number of individuals in a given age cohort. See how long it takes to reach a steady state. A slightly modified version of this model is provided in Chapter 14 where we model age-specific population dynamics for the American Robin based on actual data describing age-specific survival rates.

Total population is nearly stable after about 150 time steps. The age distributions of the cohorts at the steady state are shown in the form of bar charts in Figures 4.4 and 4.5. To generate a bar chart, simply double-click on an open graph pad and select "Bar." Only a maximum of five variables can be plotted.

Experiment with changes in the fecundity of the reproductive age group in your model. Note the extreme sensitivity of the total with respect to the initial birth rate coefficient. Finding such sensitivity is one of the main targets of any good modeler.

Use the cycle timing concept in STELLA. Time-stamp the variable BIRTH RATE by checking the box in the upper right corner of its dialog box (Fig. 4.6). Then place a transformer receiving the signal from variable OUT 6P. Call this variable LIFESPAN and define it with the CTMEAN built-in function as

$$\text{LIFESPAN} = \text{CTMEAN(OUT 6P)} \tag{1}$$

This variable now gives the mean lifespan of the people in this model. Plot LIFESPAN and see if you can adjust the variable TIME 60 to give you a mean lifespan of 75 years.

What are the criteria for cohort size choice? Note how the transfer coefficient is 1/cohort size. What if the age distribution within the cohort is very uneven? This unevenness will produce errors in the growth or decline rates for that cohort. So, cohort choice is concerned with potential age distribution within a cohort. Variation of the birth and death rates with age will also influence the appropriate cohort choice.
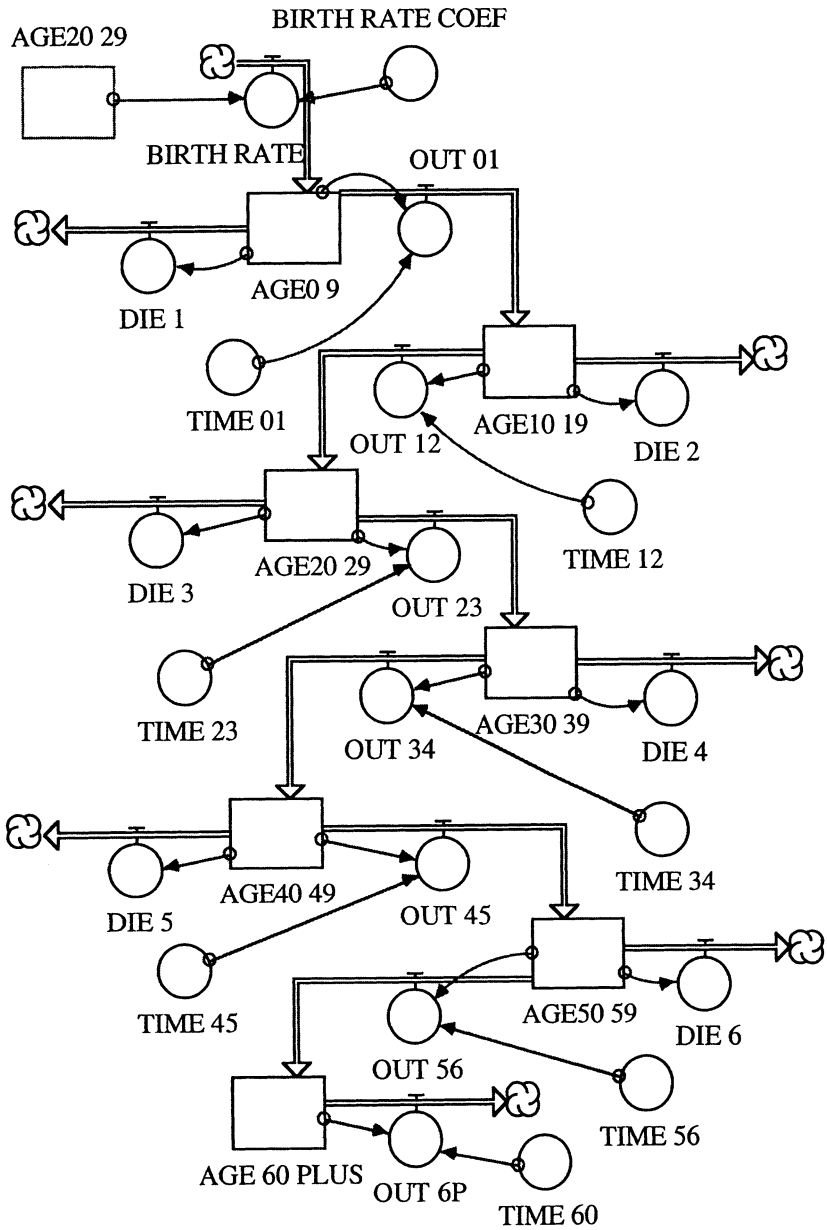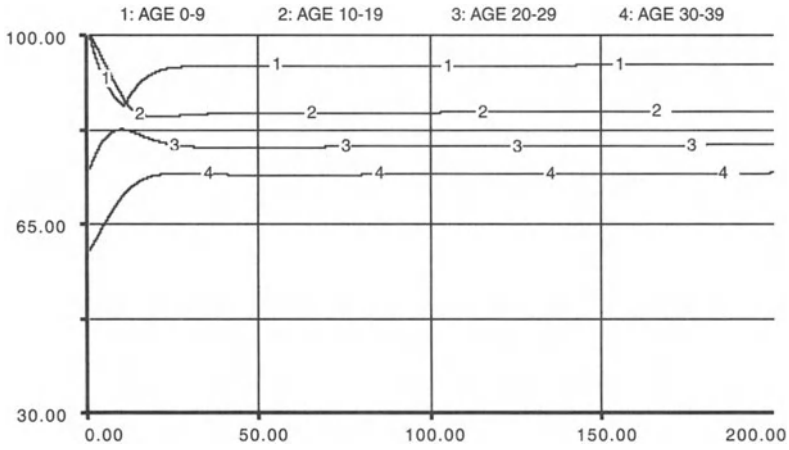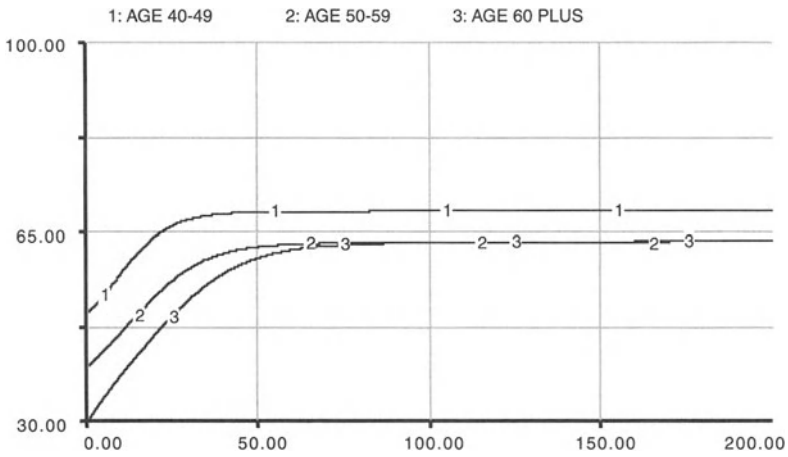
FIGURE 4.1

FIGURE 4.2

FIGURE 4.3
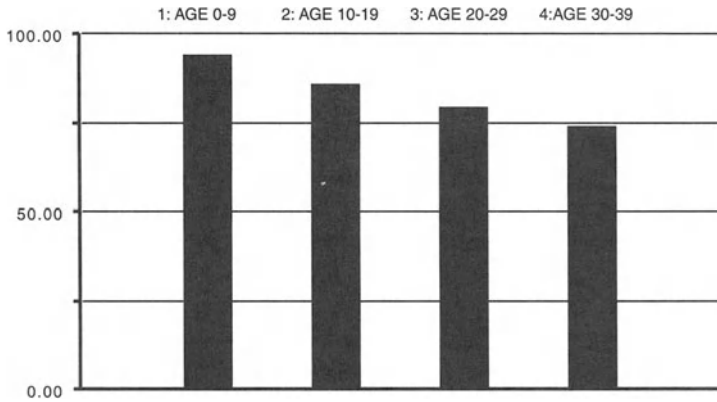
FIGURE 4.4
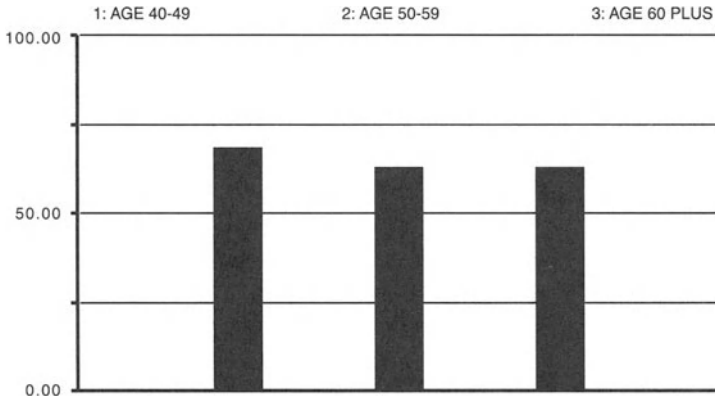
FIGURE 4.5

FIGURE 4.6

POPULATION COHORTS

```
AGE0_9(t) = AGE0_9(t - dt) + (BIRTH_RATE - OUT_01 -
DIE_1) * dt
INIT AGE0_9 = 100 {Individuals}
INFLOWS:
BIRTH_RATE = AGE20_29*BIRTH_RATE_COEF
OUTFLOWS:
OUT_01 = AGE0_9/TIME_01  {Individuals per Time Period}
DIE_1 = .011*AGE0_9 {Individuals per Time Period}

AGE10_19(t) = AGE10_19(t - dt) + (OUT_01 - OUT_12 -
DIE_2) * dt
INIT AGE10_19 = 100 {Individuals}
INFLOWS:
OUT_01 = AGE0_9/TIME_01  {Individuals per Time Period}
OUTFLOWS:
OUT_12 = AGE10_19/TIME_12  {Individuals per Time
Period}
DIE_2 = .01*AGE10_19 {Individuals per Time Period}
```

```
AGE20_29(t) = AGE20_29(t - dt) + (OUT_12 - OUT_23 -
DIE_3) * dt
INIT AGE20_29 = 75 {Individuals}
INFLOWS:
OUT_12 = AGE10_19/TIME_12  {Individuals per Time
Period}
OUTFLOWS:
OUT_23 = AGE20_29/TIME_23  {Individuals per Time
Period}
DIE_3 = .008*AGE20_29 {Individuals per Time Period}

AGE30_39(t) = AGE30_39(t - dt) + (OUT_23 - OUT_34 -
DIE_4) * dt
INIT AGE30_39 = 60 {Individuals}
INFLOWS:
OUT_23 = AGE20_29/TIME_23  {Individuals per Time
Period}
OUTFLOWS:
OUT_34 = AGE30_39/TIME_34  {Individuals per Time
Period}
DIE_4 = .007*AGE30_39  {Individuals per Time Period}

AGE40_49(t) = AGE40_49(t - dt) + (OUT_34 - OUT_45 -
DIE_5) * dt
INIT AGE40_49 = 50 {Individuals}
INFLOWS:
OUT_34 = AGE30_39/TIME_34  {Individuals per Time
Period}
OUTFLOWS:
OUT_45 = AGE40_49/TIME_45  {Individuals per Time
Period}
DIE_5 = .008*AGE40_49

AGE50_59(t) = AGE50_59(t - dt) + (OUT_45 - OUT_56 -
DIE_6) * dt
INIT AGE50_59 = 40 {Individuals}
INFLOWS:
OUT_45 = AGE40_49/TIME_45  {Individuals per Time
Period}
OUTFLOWS:
OUT_56 = AGE50_59/ TIME_56  {Individuals per Time
Period}
DIE_6 = .009*AGE50_59  {Individuals per Time Period}
```

```
AGE_60_PLUS(t) = AGE_60_PLUS(t - dt) + (OUT_56 -
OUT_6P) * dt
INIT AGE_60_PLUS = 30 {Individuals}
LINFLOWS:
OUT_56 = AGE50_59/ TIME_56  {Individuals per Time
Period}
OUTFLOWS:
OUT_6P = AGE_60_PLUS/TIME_60  {Individuals per Time
Period}

BIRTH_RATE_COEF = (11/100)*(1+STEP(0.2,10)) {Births
per Individuals in Age Cohort 20-30 per Time Period}
TIME_01 = 10
TIME_12 = 10
TIME_23 = 10
TIME_34 = 10
TIME_45 = 10
TIME_56 = 10
TIME_60 = 10
```

4.1.2 Population Cohort Array

Note how the structure of the model in the previous section is repeated for each cohort. The stock of an age cohort has potentially two inflows—one for the births and one for the aged individuals of a previous cohort. There are also two outflows—one for the aging individuals and one for deaths. The only substantive difference between cohorts is in the different numbers that are used to initialize the stocks and in the different birth and death rates. For example, the birth rates are, by assumption, all zero for all population cohorts other than that of the 20- to 29-year-olds.

To capitalize on this observation of the identical structure of the model of different age cohorts, and to economize on our modeling effort, we make use of STELLA's array functions. First, sketch out the stock-flow-converter relationships that underly our model. They are shown in Figure 4.7. Then double-click on the POPULATION stock and click in the box "Array" at the upper left portion of the dialog box. This will specify your model as a one- or two-dimensional array. A one-dimensional (1-D) array has stocks that form a single row. Our population cohort model in which individuals move unidirectionally from one cohort to another can be interpreted as such a 1-D array. In contrast, a two-dimensional (2-D) array has multiple rows and columns of stocks that are connected with each other with movement in two dimensions (Fig. 4.8).

Open the POPULATION stock, and specify it as an array. By default, STELLA assumes a 1-D Array. Next, go to the Editor and give a name to the
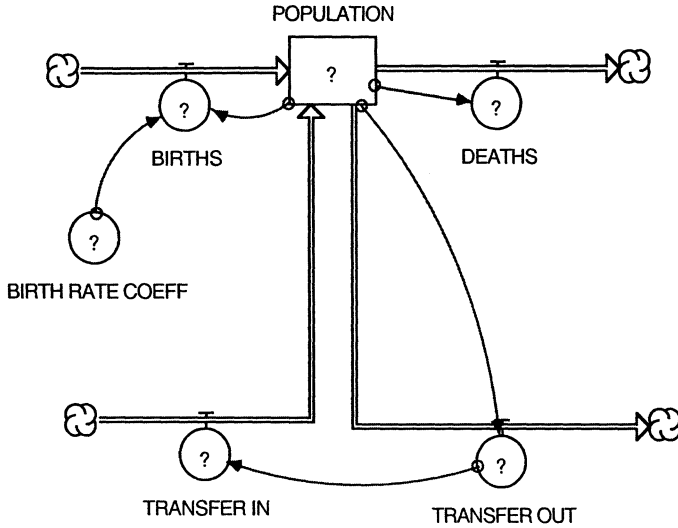
FIGURE 4.7

first "dimension" of your array—in our case, there is only one dimension. Call it COHORT.

You need to name the elements in the dimension of your array. Let's give the first element the name AGE0 9 by typing AGE0 9 in the "Element Name/#" field in the dialog box. Then use the arrow to navigate to a "New" element and then type in its name and continue to do this until you named the remaining six stocks of the model: AGE10 19, AGE20 29, AGE30 39, AGE40 49, AGE50 59, and AGE60 PLUS. Once you have done that, click OK to return to the dialog in which you can specify each stock's initial conditions. Click off
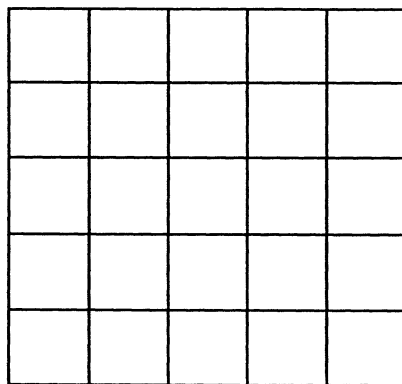
1-D Array   2-D Array

FIGURE 4.8

the "Apply to All" check box because the initial conditions are different for the seven stocks—one condition does not "apply to all." Now you will notice that you are promted with the message:

INITIAL(POPULATION[AGE0 9]) = ...{Place initial value here . . . }   (2)

Type in 100 to specify the initial stock of 0- to 9-year-olds. Next, navigate with the arrows below "Apply to All" to the second element of your 1-D array—POPULATION[AGE10 19]—and provide an initial value for POPU-LATION[AGE10 19]. Choose the same initial conditions as in the previous section of this chapter. Continue this routine until all elements in the array are named and have their proper initial conditions. Click OK and you should have a model that now looks like Figure 4.9. The stocks and flows are all stacked to indicate the use of arrays.

To specify the BIRTHS flow, double-click on the flow pipe and unclick the "Apply to All" check box. Then specify for each row the proper equations. For example,

BIRTHS AGE[0 9] = BIRTH RATE COEFF*POPULATION[AGE20 29]   (3)

Multiply in the other equations by zero so that no births are generated into stocks other than that for the first cohort. For example,

BIRTHS[AGE10 19] = 0*BIRTH RATE COEF*POPULATION[AGE10 19]   (4)

Similarly, specify the outflows for the deaths:

$$\text{DEATH[AGE0 9]} = .011\text{*POPULATION[AGE0 9]}   (5)$$

$$\text{DEATH[AGE10 19]} = .01\text{*POPULATION[AGE10 19]}   (6)$$

$$\text{DEATH[AGE20 29]} = .008\text{*POPULATION[AGE20 29]}   (7)$$
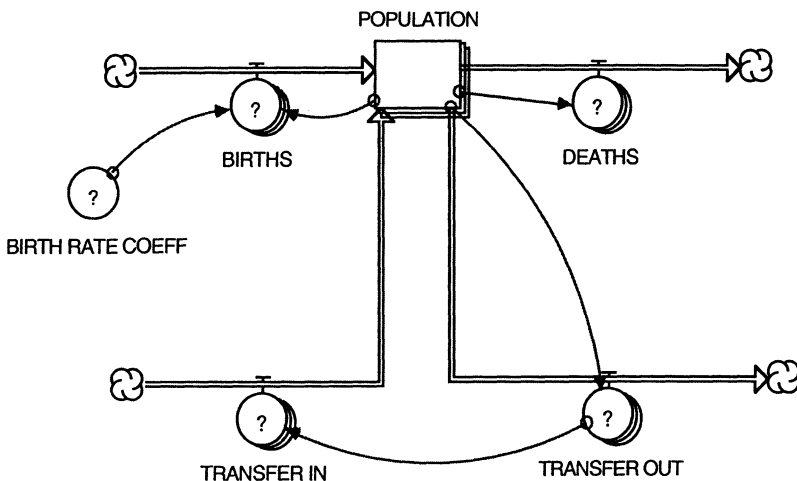

FIGURE 4.9

$$\text{DEATH[AGE30 39]} = .007*\text{POPULATION[AGE30 39]} \qquad (8)$$

$$\text{DEATH[AGE40 49]} = .008*\text{POPULATION[AGE40 49]} \qquad (9)$$

$$\text{DEATH[AGE50 59]} = .009*\text{POPULATION[AGE50 59]} \qquad (10)$$

$$\text{DEATH[AGE60 PLUS]} = .1*\text{POPULATION[AGE60 PLUS]} \qquad (11)$$

Once the births and deaths are specified, we need to model the transition of people from stock to stock as they age. Note that the outflow from one age cohort is equal to the inflow into the next cohort. For example,

$$\text{TRANSFER OUT[AGE0 9]} = \text{POPULATION[AGE0 9]/10} \qquad (12)$$

$$\text{TRANSFER IN[AGE10 19]} = \text{TRANSFER OUT[AGE0 9]} \qquad (13)$$

and that there is no inflow into the AGE0 9 cohort from other cohorts, i.e.,

$$\text{TRANSFER IN[AGE0 9]} = 0*\text{TRANSFER OUT[AGE0 9]} \qquad (14)$$

As in the model of the previous section, the birth rate coefficient is

$$\text{BIRTH RATE COEFF} = 11/100*(1+\text{STEP}(.2,10)) \qquad (15)$$

Calculate the total population with the built-in function ARRAYSUM, which sums up the values that the elements in an array take on. Generate a variable called TOTAL POPULATION, and draw an information arrow from POPULATION to TOTAL POPULATION. Then double-click on TOTAL POPULATION, select ARRAYSUM from the list of built-ins, and enter POPULATION as the argument of that function. A dialog window will appear (as shown in Figure 4.10).

The dialog window contains an asterisk, denoting "all elements in the array," the name of the dimension of the array, and the names of each element. Since we want to sum up the entries of all elements, simply click on the asterisk. The specification of TOTAL POPULATION becomes

$$\text{TOTAL POPULATION} = \text{ARRAYSUM(POPULATION[*])} \qquad (16)$$
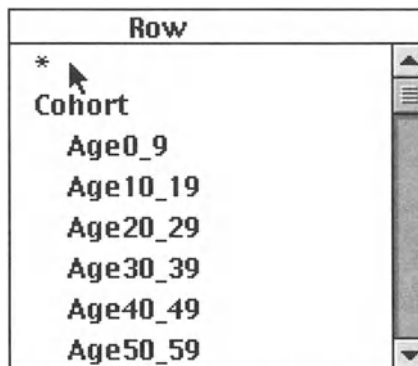
FIGURE 4.10
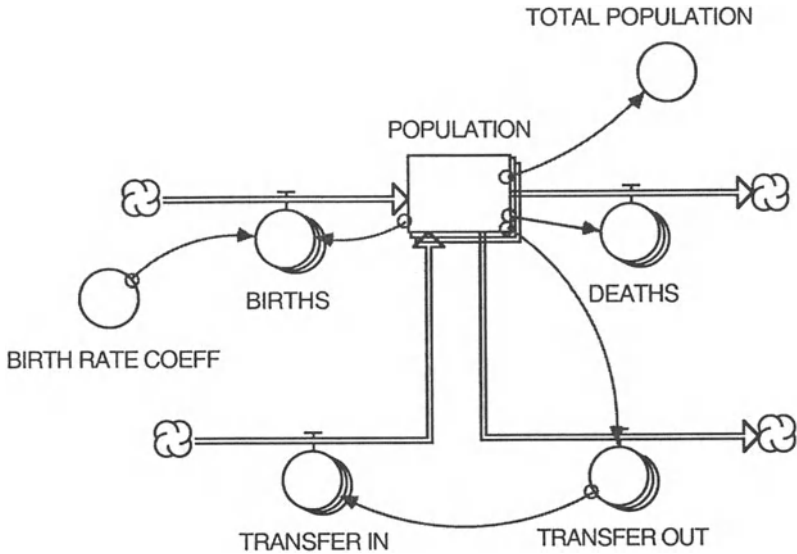
FIGURE 4.11

and your STELLA model should look like the one in Figure 4.11. Create bar charts and time series plots for each population cohort. When you define those graphs, you will encounter the dialog window as in Figure 4.10. Select up to five individual stocks for your plots. Also, generate a plot for the total population. This last is shown in Figure 4.12. Compare your model output to the results of Section 4.1.1.

This graph is identical to the population growth pattern from section 4.1.1.

FIGURE 4.12

POPULATION COHORT ARRAY

```
POPULATION[Age0_9](t) = POPULATION[Age0_9](t - dt) +
(BIRTHS[Age0_9] + TRANSFER_IN[Age0_9] - DEATHS[Age0_9]
- TRANSFER_OUT[Age0_9]) * dt
INIT POPULATION[Age0_9] = 100
POPULATION[Age10_19](t) = POPULATION[Age10_19](t - dt)
+ (BIRTHS[Age10_19] + TRANSFER_IN[Age10_19] -
DEATHS[Age10_19] - TRANSFER_OUT[Age10_19]) * dt
INIT POPULATION[Age10_19] = 100
POPULATION[Age20_29](t) = POPULATION[Age20_29](t - dt)
+ (BIRTHS[Age20_29] + TRANSFER_IN[Age20_29] -
DEATHS[Age20_29] - TRANSFER_OUT[Age20_29]) * dt
INIT POPULATION[Age20_29] = 75
POPULATION[Age30_39](t) = POPULATION[Age30_39](t - dt)
+ (BIRTHS[Age30_39] +
TRANSFER_IN[Age30_39] - DEATHS[Age30_39] -
TRANSFER_OUT[Age30_39]) * dt
INIT POPULATION[Age30_39] = 60
POPULATION[Age40_49](t) = POPULATION[Age40_49](t - dt)
+ (BIRTHS[Age40_49] + TRANSFER_IN[Age40_49] -
DEATHS[Age40_49] - TRANSFER_OUT[Age40_49]) * dt
INIT POPULATION[Age40_49] = 50
POPULATION[Age50_59](t) = POPULATION[Age50_59](t - dt)
+ (BIRTHS[Age50_59] + TRANSFER_IN[Age50_59] -
DEATHS[Age50_59] - TRANSFER_OUT[Age50_59]) * dt
INIT POPULATION[Age50_59] = 40
POPULATION[Age_60_Plus](t) = POPULATION[Age_60_Plus](t
- dt) + (BIRTHS[Age_60_Plus] +
TRANSFER_IN[Age_60_Plus] - DEATHS[Age_60_Plus] -
TRANSFER_OUT[Age_60_Plus]) * dt
INIT POPULATION[Age_60_Plus] = 30

INFLOWS:
BIRTHS[Age0_9] = BIRTH_RATE_COEFF*POPULATION[Age20_29]
BIRTHS[Age10_19] =
0*BIRTH_RATE_COEFF*POPULATION[Age10_19]
BIRTHS[Age20_29] =
0*BIRTH_RATE_COEFF*POPULATION[Age10_19]
BIRTHS[Age30_39] =
0*BIRTH_RATE_COEFF*POPULATION[Age20_29]
BIRTHS[Age40_49] =
0*BIRTH_RATE_COEFF*POPULATION[Age20_29]
BIRTHS[Age50_59] =
0*BIRTH_RATE_COEFF*POPULATION[Age20_29]
```
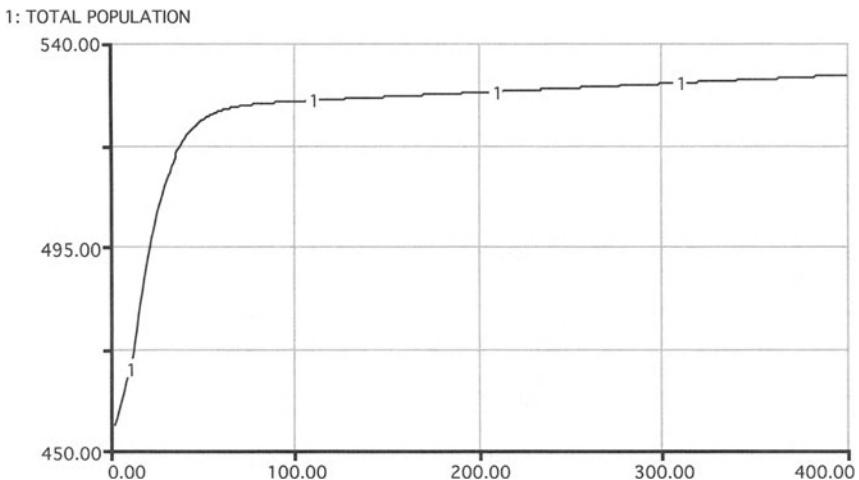
```
BIRTHS[Age_60_Plus] =
0*BIRTH_RATE_COEFF*POPULATION[Age20_29]
TRANSFER_IN[Age0_9] = 0*TRANSFER_OUT[Age0_9]
TRANSFER_IN[Age10_19] = TRANSFER_OUT[Age0_9]
TRANSFER_IN[Age20_29] = TRANSFER_OUT[Age10_19]
TRANSFER_IN[Age30_39] = TRANSFER_OUT[Age20_29]
TRANSFER_IN[Age40_49] = TRANSFER_OUT[Age30_39]
TRANSFER_IN[Age50_59] = TRANSFER_OUT[Age40_49]
TRANSFER_IN[Age_60_Plus] = TRANSFER_OUT[Age50_59]

OUTFLOWS:
DEATHS[Age0_9] = .011*POPULATION[Age0_9]
DEATHS[Age10_19] = .01*POPULATION[Age10_19]
DEATHS[Age20_29] = .008*POPULATION[Age20_29]
DEATHS[Age30_39] = .007*POPULATION[Age30_39]
DEATHS[Age40_49] = .008*POPULATION[Age40_49]
DEATHS[Age50_59] = .009*POPULATION[Age50_59]
DEATHS[Age_60_Plus] = .1*POPULATION[Age_60_Plus]
TRANSFER_OUT[Age0_9] = POPULATION[Age0_9]/10
TRANSFER_OUT[Age10_19] = POPULATION[Age10_19]/10
TRANSFER_OUT[Age20_29] = POPULATION[Age20_29]/10
TRANSFER_OUT[Age30_39] = POPULATION[Age30_39]/10
TRANSFER_OUT[Age40_49] = POPULATION[Age40_49]/10
TRANSFER_OUT[Age50_59] = POPULATION[Age50_59]/10
TRANSFER_OUT[Age_60_Plus] = 0*POPULATION[Age_60_Plus]

BIRTH_RATE_COEFF = 11/100*(1+STEP(.2,10))
TOTAL_POPULATION = ARRAYSUM(POPULATION[*])
```

4.1.3 U.S. Population Growth

Whole courses in demographics are used to convey the ideas behind the model of this section. In one simple model, we address the questions of birth, death, and imigration rates and their effects on the numbers of people in the various age groups. We discuss the ways to calibrate a model using both theory and real data and how to project existing data on these rates to calculate first the known population levels of the latest census year, and then to project these rates to allow an estimate of the future level of the U.S. population.[1] We realize that we skim over some of the sophisticated statistical models that allow a consideration of the many factors that enter into the projection of these rates, and other such variables as census count-

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[1]All the references are given in the model in the variable for which they provided the data.

ing errors and omissions. But we are sure that a major lesson in model building and demographics is conveyed by this model.

The issue is to estimate the population of the United States in the year 2020 and compare it with the official estimates by the U.S. Census Bureau. Before we can do that, we need to derive at least some perspective on the projection of future birth, death, and imigration rates. But before we can do that, we need to build a model of the historical population to ensure that we have the right model structure and that we are using the historical data correctly. So, let's develop the model, initialize it with historical data, and see how well the model does in "projecting" other historical data.

The modeling effort is divided into three parts. First, construct the model with an arbitrarily early starting point. We choose the year 1900. The numbers of people in the United States is rather well known for that year. Then we need the historical birth, death, and imigration rates for the decadal years beginning with 1900 and ending at 1960. This is the validation step. If something is wrong with our model structure or the way we have portrayed the data, this step will reveal it.

We know that the census data exist for the year 1990 but we will use those data to examine our projection skill, begining with the year 1960 and seeing how well we can project the three rates across these three decades. Finally, we will set up the model with actual rate data up to the year 1990 and project the three rates to the year 2020, allowing us to calculate the cohort and total population in that year and compare it with Census Bureau projections.

The most elementary step of such modeling is to realize that the stocks of populations, while based on age, are not divided up by years. This would result in over 100 stocks, each with its own set of birth, death, and imigration rates. Instead, we note that the census takers examine the death rates for each age and observe that the death rates are nearly the same for groups of ages. This observation defines the cohort sizes, that is, how many years are to be included in each grouping or cohort of the population. Our model will contain a stock for each of these cohorts.

The first of these stocks, the AGE 0 1 stock has a major input of new births and its own death and imigration rates. Let's deal with the BIRTHS first. Here we meet the first of data-model misfits. We have decided to model both sexes together but the birth data are (logically) reported as a number of births per 1000 women for the given year. We assume that the sex ratio is one and so we must divide the product of the BF (Birth Fraction) by 2 and by 1000.

Secondly we notice that the population is reported differently before and after the year 1940. After 1940 the reporting is more detailed, which causes a second problem. While the cohorts are based on the nearly constant death rate concept, the BF categories do not match the chosen cohorts. They are for cohorts at half the chosen size. This means that after 1940 we must divide in the BIRTHS calculation by 2 for the sex ratio correction, by 1000 for the data correction and by 2 again to allow us to combine the the BF into the larger cohorts.

Note that the population stocks are reported in whole numbers. Our only data on imigration rates (per 1000 people in the United States at the time) was for the entire population so we simply multiplied the imigration rate by the cohort population size and divided by 1000. In our simple model we assume that the number of people in each year of age in a typical cohort is the same and that each year, 1/cohort size ages into the next cohort. For the first cohort whose size is a single year, all the surviving babies move to the next cohort.

The same process is repeated for each cohort, until the final one. For the 70+ category, we used the census data on the life expectancy for this category (the inverse of the life expectancy is the death rate). The main parts of the final model are shown in Figure 4.13. We also calculate total population as shown in Figure 4.14.

Actual and projected data used in this model are contained in graphical form in two "sectors" (Fig. 4.15). These sectors are created with STELLA's sector symbol shown as sixth tool of your tool bar (Fig. 4.16). To create a sector, click on the symbol, place it in the diagram and adjust its size to cover the model components that you wish to enclose in it. To name a sector, simply overwrite the default name STELLA provides for it.

Once a sector is created, the model can be run holding the elements in a secctor constant or updating them. Go to the Run pull-down menu and you will see that you can specify the sectors that you wish to run along in your model. This tool comes handy when you want to switch on or off individual components of your model. We use sectors here simply to organize our model and keep like things together in the same sector rather than run them in isolation from each other.

When we run this model to "predict" the 1960 population level, we overshoot by about 1.25 %, not bad for amateurs (Fig. 4.17).

The second step is to extend the three rates from 1960 to 1990 and see how well we can "predict" the 1990 level. Were we real demographers, we would develop statistical models for these rates with the hope of revealing their dependency on population size, on the level of real income, the degree of uncertainty created by wars, the spread of diseases, the rate of improvement in health care, and so on. For our purposes, we simply extended by eye what appeared to be trends in these rates. And yet we did rather well at such extension, overstating the 1990 population by only 1.6% (Fig. 4.18).

Finally, we updated the three rates for the 1960 to 1990 period with the real data, and extended the run of the model 30 more years, to the year 2020, missing the Census Bureau's middle-range projections by only 0.4% (Fig. 4.19).

Of course, these projections are based on the idea that the future is like the recent past. A view of the history of the population fluctuations shows that the population level is influenced by a depression and a major war.

Without taking a position in the debate on imigration policy, we can examine the simple effect on the U.S. population of the absence of imigration from the year 1900 on. Simply shut off the imigration fraction effect and we
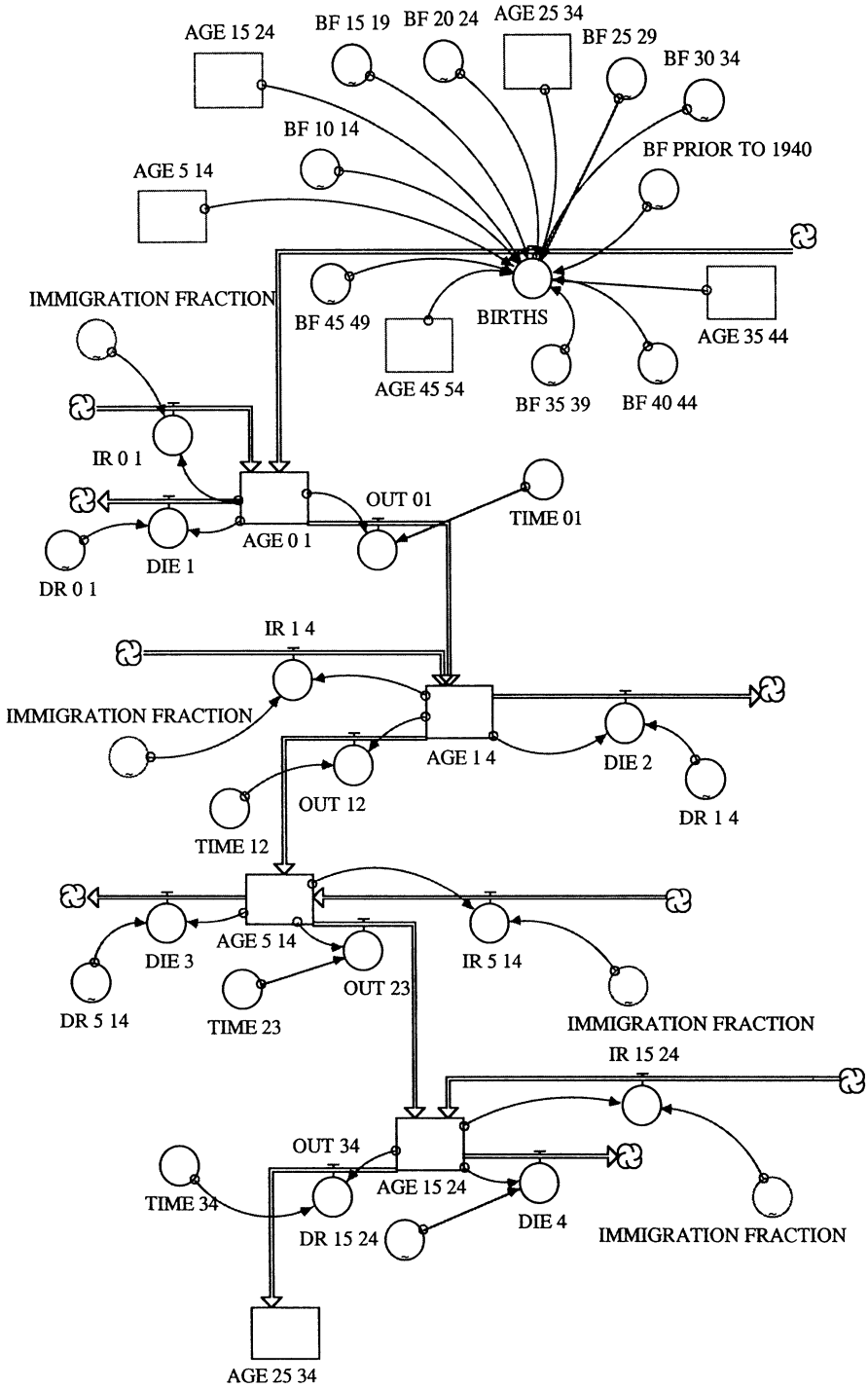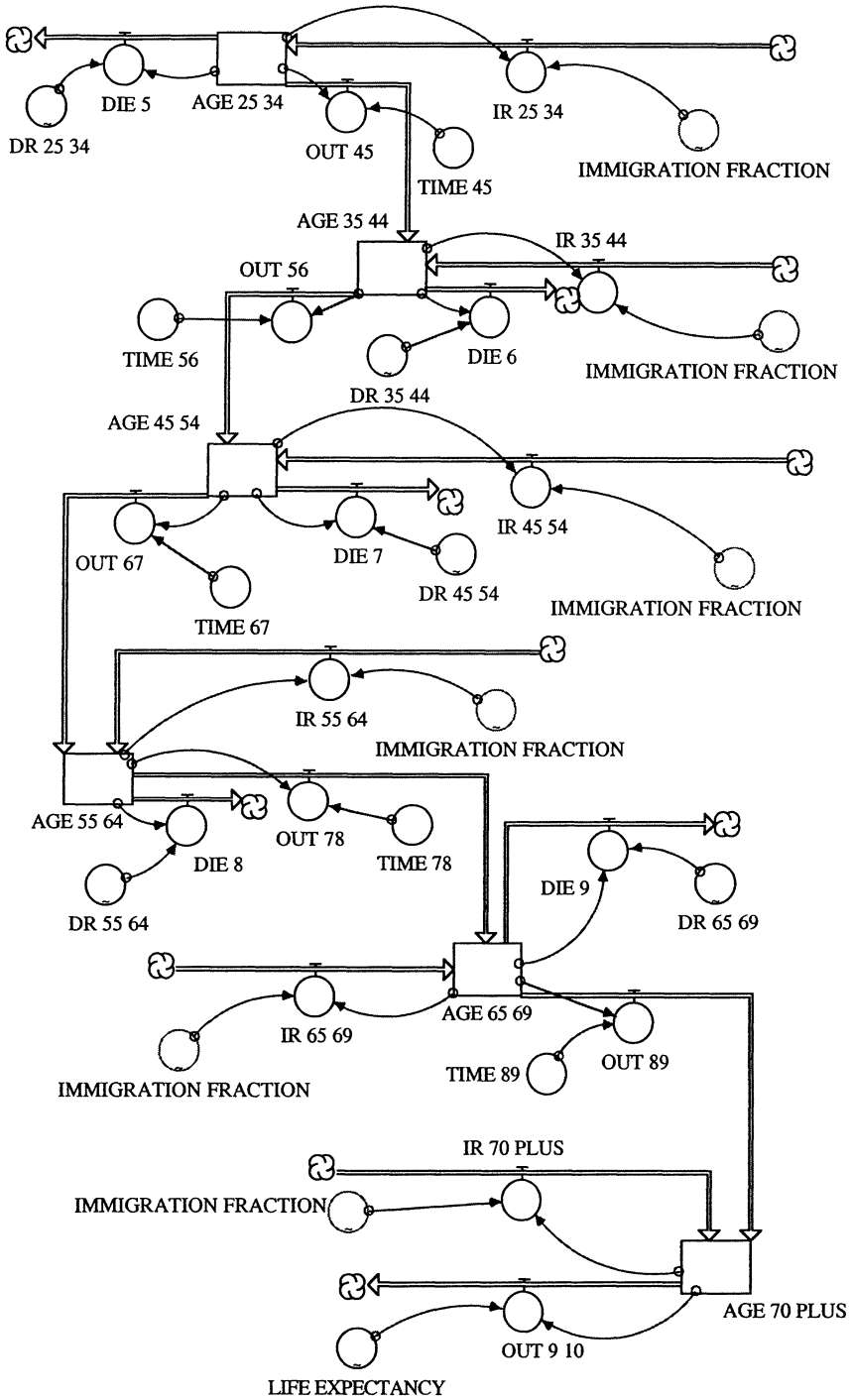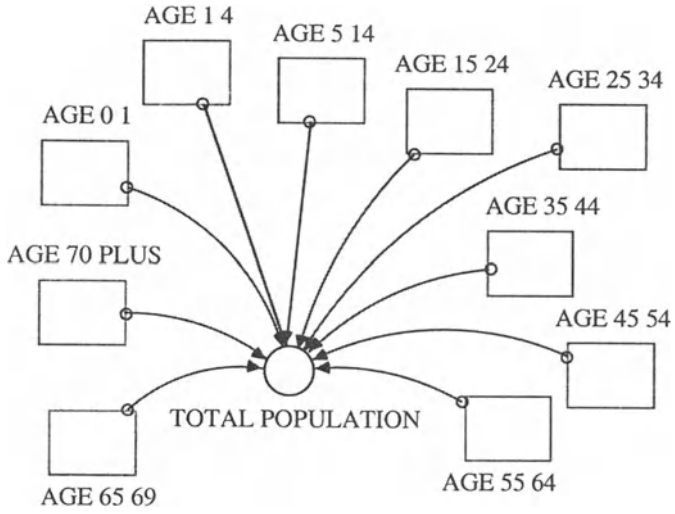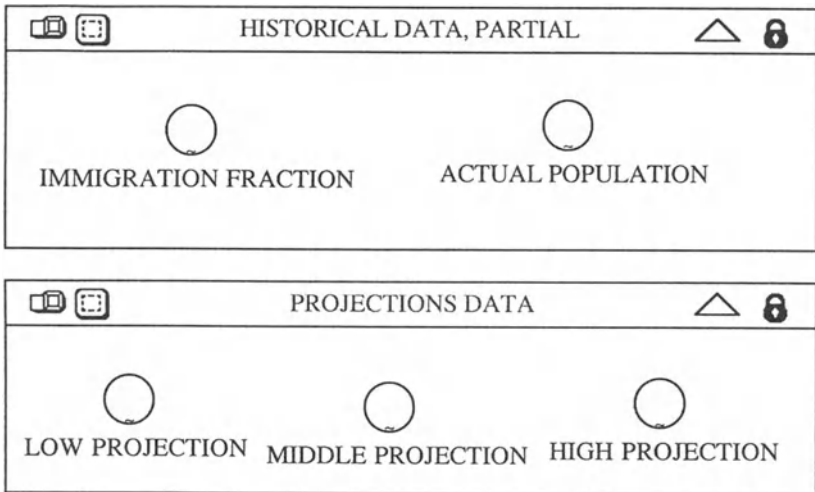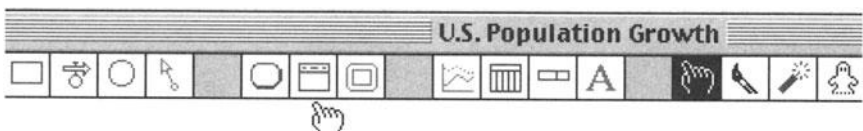
FIGURE 4.13
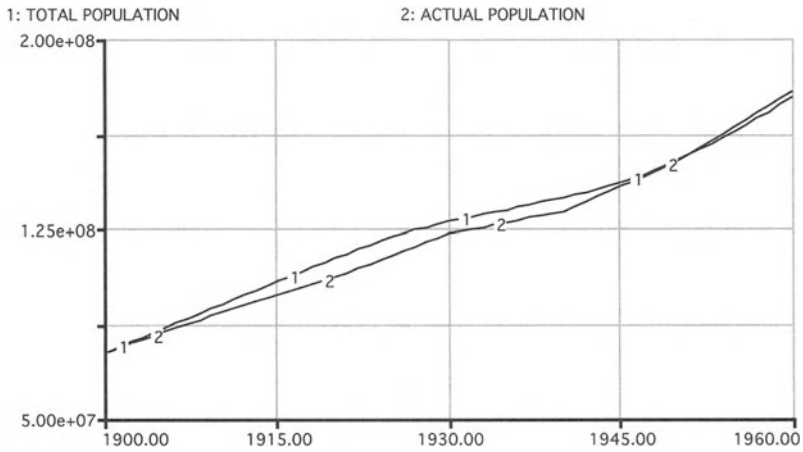
FIGURE 4.13 (*continued*)

FIGURE 4.14

FIGURE 4.15

FIGURE 4.16

1: TOTAL POPULATION   2: ACTUAL POPULATION

FIGURE 4.17

1: TOTAL POPULATION   2: ACTUAL POPULATION

FIGURE 4.18

1: TOTAL POPULATION   2: HIGH PROJECTION   3: MIDDLE PROJECTION   4: LOW PROJECTION
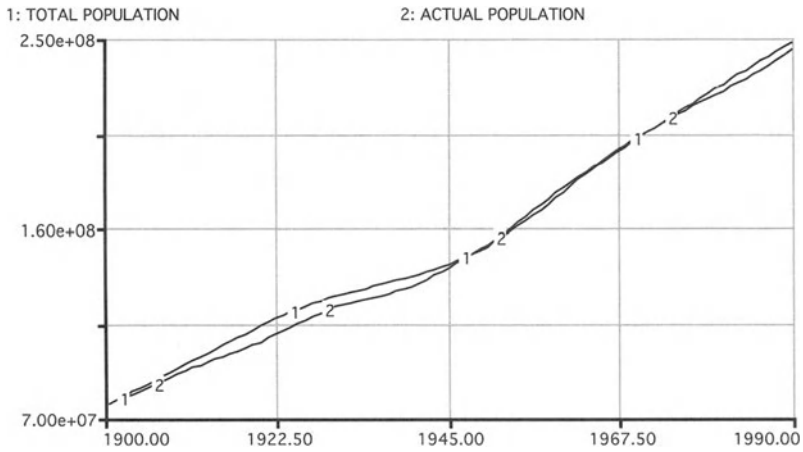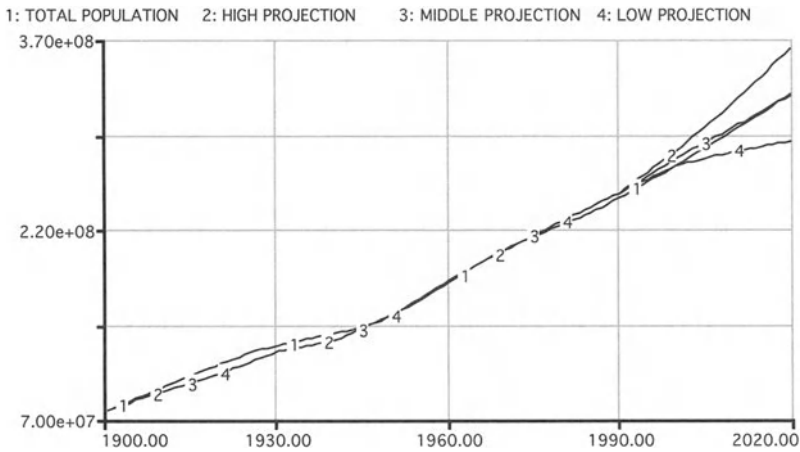
FIGURE 4.19

find that legal imigrants over the 90-year period from 1900 to 1990 have added 60 million people to the population. These 60 million people have of course added much more than population to the United States by way of their unusual innovation and productivity. We show the effect of migration on the population to demonstrate how facile such models can be.

You can use the method of step-wise forecasting and parameter adjustments not only in modeling the dyanamics of a population of your choice. The method also provides a powerful means to ensure that a dynamic model of a system at least replicates the data for that system. An inability to come with the model at least close to the historical system behavior is a good sign that you missed some important driver behind the observed dynamics. How closely you wish to replicate past system behavior depends on the purpose of the model and your motivation.

In the process of model calibration you will learn more about the data of your system. You will also encounter "anomalies" that your model does not capture. In our model, two such anomalies were the wars. Ex post we can correct our model for their effects on population dynamics. However, it would be virtually impossible to predict at the outset the occurrance of such anomalies. Therefore, we often disregard them. But remember, structural changes, such as those caused by wars, changes in the political system of a country, or major technological changes, can place bounds on the reliability of virtually any model of a social or economic system.

Would it improve matters to divide the model by gender? Could we have started this model off in 1940 instead of 1900 and accomplished the same goals? Can you explain why this model does not fit as well in the pre-1940 portion and how you would improve that fit? Can you change the birth rates and immigration policy in an attempt to bring the U.S. population to a steady state by 2030, while allowing the trends in death rates to continue?

```
                    US POPULATION GROWTH

HISTORICAL DATA, PARTIAL
ACTUAL_POPULATION = GRAPH(TIME)
(1900, 7.6e+07), (1910, 9.2e+07), (1920, 1.1e+08),
(1930, 1.2e+08), (1940, 1.3e+08), (1950, 1.5e+08),
(1960, 1.8e+08), (1970, 2e+08), (1980, 2.3e+08),
(1990, 2.5e+08), (2000, 0.00), (2010, 0.00), (2020,
0.00)
DOCUMENT: Actual population of the U.S. (1900 - 1990)
Data taken from: Bureau of the Census, Statistical
Abstracts of the U.S. (1994)

IMMIGRATION_FACTOR = GRAPH(TIME)
(1900, 10.4), (1910, 5.70), (1920, 3.50), (1930, 0.4),
(1940, 0.7), (1950, 1.50), (1960, 1.70), (1970, 2.10),
```

(1980, 3.10), (1990, 4.10), (2000, 4.32), (2010, 4.38),
(2020, 4.62)
DOCUMENT: Immigration fraction is the number of
immigrants per 1000 U.S. persons per year. Data was
taken from: U.S. Bureau of the Census, Statistical
Abstracts of the U.S. (1994)

PROJECTIONS DATA
HIGH_PROJECTION = GRAPH(TIME)
(1900, 7.6e+07), (1910, 9.2e+07), (1920, 1.1e+08),
(1930, 1.2e+08), (1940, 1.3e+08), (1950, 1.5e+08),
(1960, 1.8e+08), (1970, 2e+08), (1980, 2.3e+08),
(1990, 2.5e+08), (2000, 2.8e+08), (2010, 3.2e+08),
(2020, 3.6e+08)
DOCUMENT: This is the highest series for projections of
the U.S. population through 2020. The populations from
1900 to 1990 are the actual U.S. population values.
Data taken from: Bureau of the Census, Statistical
Abstracts of the U.S. (1994)

LOW_PROJECTION = GRAPH(TIME)
(1900, 7.6e+07), (1910, 9.2e+07), (1920, 1.1e+08),
(1930, 1.2e+08), (1940, 1.3e+08), (1950, 1.5e+08),
(1960, 1.8e+08), (1970, 2e+08), (1980, 2.3e+08),
(1990, 2.5e+08), (2000, 2.7e+08), (2010, 2.8e+08),
(2020, 2.9e+08)
DOCUMENT: This is the lowest series for projections of
the U.S. population through 2020. The populations from
1900 to 1990 are the actual U.S. population values.
Data taken from: Bureau of the Census, Statistical
Abstracts of the U.S. (1994)

MIDDLE_PROJECTION = GRAPH(TIME)
(1900, 7.6e+07), (1910, 9.2e+07), (1920, 1.1e+08),
(1930, 1.2e+08), (1940, 1.3e+08), (1950, 1.5e+08),
(1960, 1.8e+08), (1970, 2e+08), (1980, 2.3e+08),
(1990, 2.5e+08), (2000, 2.8e+08), (2010, 3e+08),
(2020, 3.3e+08)
DOCUMENT: This is the middle series for projections of
the U.S. population through 2020. The populations from
1900 to 1990 are the actual U.S. population values.
Data taken from: Bureau of the Census, Statistical
Abstracts of the U.S. (1994)

```
Not in a sector
AGE_0_1(t) = AGE_0_1(t - dt) + (BIRTHS + IR_0_1 -
OUT_01 - DIE_1) * dt
INIT AGE_0_1 = 1836200 {Individuals}
DOCUMENT: Number of 0 - 1 year olds in 1900. Data taken
from: Bureau of the Census, Historical Statistics of
the U.S. (Colonial Times to 1970)
INFLOWS:
BIRTHS = IF(TIME<1940)
THEN((AGE_15_24+AGE_25_34+AGE_35_44)*BF_PRIOR_TO_1940/
2000)
ELSE((AGE_5_14*BF_10_14+AGE_15_24*BF_15_19+AGE_15_24*BF
_20_24+AGE_25_34*BF_25_29+AGE_25_34*BF_30_34+AGE_35_44
*BF_35_39+AGE_35_44*BF_40_44+AGE_45_54*BF_45_49)/4000)
DOCUMENT: BIRTHS is the sum of the number of births per
year for each cohort.
IR_0_1 = IMMIGRATION_FACTOR*AGE_0_1/1000 {Number of
immigrants per year per cohort}
OUTFLOWS:
OUT_01 = AGE_0_1/TIME_01   {Individuals per Time
Period}
DIE_1 = (AGE_0_1/1000)*DR_0_1 {Deaths per year}

AGE_15_24(t) = AGE_15_24(t - dt) + (OUT_23 + IR_15_24
- OUT_34 - DIE_4) * dt
INIT AGE_15_24 = 14951000 {Individuals}
DOCUMENT: Number of 15 - 24 year olds in 1900. Data
taken from: Bureau of the Census, Historical Statistics
of the U.S. (Colonial Times to 1970)
INFLOWS:
OUT_23 = AGE_5_14/TIME_23   {Individuals per Time Period}
IR_15_24 = IMMIGRATION_FACTOR*AGE_15_24/1000*0 {Number
of immigrants per year per cohort}
OUTFLOWS:
OUT_34 = AGE_15_24/TIME_34   {Individuals per Time
Period}
DIE_4 = (AGE_15_24/1000)*DR_15_24 {Deaths per year}

AGE_1_4(t) = AGE_1_4(t - dt) + (OUT_01 + IR_1_4 -
OUT_12 - DIE_2) * dt
INIT AGE_1_4 = 7344800 {Individuals}
DOCUMENT: Number of 1 - 4 year olds in 1900. Data taken
from: Bureau of the Census, Historical Statistics of
the U.S. (Colonial Times to 1970)
```

```
INFLOWS:
OUT_01 = AGE_0_1/TIME_01  {Individuals per Time
Period}
IR_1_4 = IMMIGRATION_FACTOR*AGE_1_4/1000*0 {Number of
immigrants per year per cohort}
OUTFLOWS:
OUT_12 = AGE_1_4/TIME_12  {Individuals per Time
Period}
DIE_2 = (AGE_1_4/1000)*DR_1_4 {Deaths per year}

AGE_25_34(t) = AGE_25_34(t - dt) + (OUT_34 + IR_25_34
- OUT_45 - DIE_5) * dt
INIT AGE_25_34 = 12161000 {Individuals}
DOCUMENT: Number of 25 - 34 year olds in 1900. Data
taken from: Bureau of the Census, Historical Statistics
of the U.S. (Colonial Times to 1970)
INFLOWS:
OUT_34 = AGE_15_24/TIME_34  {Individuals per Time
Period}
IR_25_34 = IMMIGRATION_FACTOR*AGE_25_34/1000*0 {Number
of immigrants per year per cohort}
OUTFLOWS:
OUT_45 = AGE_25_34/TIME_45  {Individuals per Time
Period}
DIE_5 = (AGE_25_34/1000)*DR_25_34 {Deaths per year}

AGE_35_44(t) = AGE_35_44(t - dt) + (OUT_45 + IR_35_44
- OUT_56 - DIE_6) * dt
INIT AGE_35_44 = 9273000 {Individuals}
DOCUMENT: Number of 35 - 44 year olds in 1900. Data
taken from: Bureau of the Census, Historical Statistics
of the U.S. (Colonial Times to 1970)
INFLOWS:
OUT_45 = AGE_25_34/TIME_45  {Individuals per Time
Period}
IR_35_44 = IMMIGRATION_FACTOR*AGE_35_44/1000*0
{Number of immigrants per year per cohort}
OUTFLOWS:
OUT_56 = AGE_35_44/ TIME_56  {Individuals per Time
Period}
DIE_6 = (AGE_35_44/1000)*DR_35_44 {Deaths per year}

AGE_45_54(t) = AGE_45_54(t - dt) + (OUT_56 + IR_45_54
- OUT_67 - DIE_7) * dt
```

```
INIT AGE_45_54 = 6437000 {Individuals}
DOCUMENT: Number of 45 - 54 year olds in 1900. Data
taken from: Bureau of the Census, Historical Statistics
of the U.S. (Colonial Times to 1970)
INFLOWS:
OUT_56 = AGE_35_44/ TIME_56  {Individuals per Time
Period}
IR_45_54 = IMMIGRATION_FACTOR*AGE_45_54/1000*0 {Number
of immigrants per year per cohort}
OUTFLOWS:
OUT_67 = AGE_45_54/ TIME_67  {Individuals per Time
Period}
DIE_7 = (AGE_45_54/1000)*DR_45_54 {Deaths per year}

AGE_55_64(t) = AGE_55_64(t - dt) + (OUT_67 + IR_55_64
- OUT_78 - DIE_8) * dt
INIT AGE_55_64 = 4026000 {Individuals}
DOCUMENT: Number of 55 - 64 year olds in 1900. Data
taken from: Bureau of the Census, Historical Statistics
of the U.S. (Colonial Times to 1970)
INFLOWS:
OUT_67 = AGE_45_54/ TIME_67  {Individuals per Time
Period}
IR_55_64 = IMMIGRATION_FACTOR*AGE_55_64/1000*0 {Number
of immigrants per year per cohort}
OUTFLOWS:
OUT_78 = AGE_55_64/ TIME_78  {Individuals per Time
Period}
DIE_8 = (AGE_55_64/1000)*DR_55_64 {Deaths per year}

AGE_5_14(t) = AGE_5_14(t - dt) + (OUT_12 + IR_5_14 -
OUT_23 - DIE_3) * dt
INIT AGE_5_14 = 16966000 {Individuals}
DOCUMENT: Number of 5 - 14 year olds in 1900. Data
taken from: Bureau of the Census, Historical Statistics
of the U.S. (Colonial Times to 1970)
INFLOWS:
OUT_12 = AGE_1_4/TIME_12  {Individuals per Time Period}
IR_5_14 = IMMIGRATION_FACTOR*AGE_5_14/1000*0 {Number
of immigrants per year per cohort}
OUTFLOWS:
OUT_23 = AGE_5_14/TIME_23  {Individuals per Time
Period}
DIE_3 = (AGE_5_14/1000)*DR_5_14 {Deaths per year}
```

```
AGE_65_69(t) = AGE_65_69(t - dt) + (OUT_78 + IR_65_69
- DIE_9 - OUT_89) * dt
INIT AGE_65_69 = 1291250 {Individuals}
DOCUMENT: Number of 65 - 69 year olds in 1900. Data
taken from: Bureau of the Census, Historical Statistics
of the U.S. (Colonial Times to 1970)
INFLOWS:
OUT_78 = AGE_55_64/ TIME_78  {Individuals per Time
Period}
IR_65_69 = IMMIGRATION_FACTOR*AGE_65_69/1000*0 {Number
of immigrants per year per cohort}
OUTFLOWS:
DIE_9 = (AGE_65_69/1000)*DR_65_69 {Deaths per year}
OUT_89 = AGE_65_69/ TIME_89  {Individuals per Time
Period}

AGE_70_PLUS(t) = AGE_70_PLUS(t - dt) + (IR_70_PLUS +
OUT_89 - OUT_9_10) * dt
INIT AGE_70_PLUS = 1807750 {Individuals}
DOCUMENT: Number of 70+ year olds in 1900. Data taken
from: Bureau of the Census, Historical Statistics of
the U.S. (Colonial Times to 1970)
INFLOWS:
IR_70_PLUS = IMMIGRATION_FACTOR*AGE_70_PLUS/1000*0
{Number of immigrants per year per cohort}
OUT_89 = AGE_65_69/ TIME_89  {Individuals per Time
Period}
OUTFLOWS:
OUT_9_10 = AGE_70_PLUS/LIFE_EXPECTANCY {Individuals
per Time Period}
TIME_01 = 1
DOCUMENT: Dwell time = 1 year.
TIME_12 = 4
DOCUMENT: Dwell time = 4 years.
TIME_23 = 10
DOCUMENT: Dwell time = 10 years.
TIME_34 = 10
DOCUMENT: Dwell time = 10 years.
TIME_45 = 10
DOCUMENT: Dwell time = 10 years.
TIME_56 = 10
DOCUMENT: Dwell time = 10 years.
TIME_67 = 10
DOCUMENT: Dwell time = 10 years.
TIME_78 = 10
```

```
DOCUMENT: Dwell time = 10 years.
TIME_89 = 5
DOCUMENT: Dwell time = 5 years.
TOTAL_POPULATION =
AGE_0_1+AGE_1_4+AGE_5_14+AGE_15_24+AGE_25_34+AGE_35_4
4+AGE_45_54+AGE_55_64+AGE_65_69+AGE_70_PLUS
DOCUMENT: Total population = sum of population in each
cohort.
BF_10_14 = GRAPH(TIME)
(1900, 0.00), (1910, 0.00), (1920, 0.00), (1930, 0.00),
(1940, 0.7), (1950, 1.00), (1960, 0.8), (1970, 1.20),
(1980, 1.10), (1990, 1.40), (2000, 1.40), (2010, 1.23),
(2020, 1.45)
DOCUMENT: Birth fraction is the number of births per
1000 women per year in each cohort. Data taken from:
Bureau of the Census, Statistical Abstracts of the U.S.
(1963)


BF_15_19 = GRAPH(TIME)
(1900, 0.00), (1910, 0.00), (1920, 0.00), (1930, 0.00),
(1940, 54.1), (1950, 81.6), (1960, 89.1), (1970, 68.3),
(1980, 53.0), (1990, 59.9), (2000, 61.6), (2010, 64.8),
(2020, 65.2)
DOCUMENT: Birth fraction is the number of births per
1000 women per year in each cohort. Data taken from:
Bureau of the Census, Statistical Abstracts of the U.S.
(1963)


BF_20_24 = GRAPH(TIME)
(1900, 0.00), (1910, 0.00), (1920, 0.00), (1930, 0.00),
(1940, 136), (1950, 197), (1960, 258), (1970, 168),
(1980, 115), (1990, 116), (2000, 121), (2010, 124),
(2020, 126)
DOCUMENT: Birth fraction is the number of births per
1000 women per year in each cohort. Data taken from:
Bureau of the Census, Statistical Abstracts of the U.S.
(1963)


BF_25_29 = GRAPH(TIME)
(1900, 0.00), (1910, 0.00), (1920, 0.00), (1930, 0.00),
(1940, 123), (1950, 166), (1960, 197), (1970, 145),
(1980, 113), (1990, 120), (2000, 123), (2010, 126),
(2020, 128)
DOCUMENT: Birth fraction is the number of births per
1000 women per year in each cohort. Data taken from:
```

```
Bureau of the Census, Statistical Abstracts of the U.S.
(1963)


BF_30_34 = GRAPH(TIME)
(1900, 0.00), (1910, 0.00), (1920, 0.00), (1930, 0.00),
(1940, 83.4), (1950, 104), (1960, 113), (1970, 73.3),
(1980, 61.9), (1990, 80.8), (2000, 82.2), (2010, 84.5),
(2020, 90.8)
DOCUMENT: Birth fraction is the number of births per 1000
women per year in each cohort. Data taken from: Bureau of
the Census, Statistical Abstracts of the U.S. (1963)


BF_35_39 = GRAPH(TIME)
(1900, 0.00), (1910, 0.00), (1920, 0.00), (1930, 0.00),
(1940, 46.3), (1950, 52.9), (1960, 56.2), (1970, 31.7),
(1980, 19.8), (1990, 31.7), (2000, 35.9), (2010, 38.2),
(2020, 40.2)
DOCUMENT: Birth fraction is the number of births per 1000
women per year in each cohort. Data taken from: Bureau of
the Census, Statistical Abstracts of the U.S. (1963)


BF_40_44 = GRAPH(TIME)
(1900, 0.00), (1910, 0.00), (1920, 0.00), (1930, 0.00),
(1940, 15.6), (1950, 15.1), (1960, 15.5), (1970, 8.10),
(1980, 3.90), (1990, 5.50), (2000, 6.32), (2010, 6.96),
(2020, 7.44)
DOCUMENT: Birth fraction is the number of births per 1000
women per year in each cohort. Data taken from: Bureau of
the Census, Statistical Abstracts of the U.S. (1963)


BF_45_49 = GRAPH(TIME)
(1900, 0.00), (1910, 0.00), (1920, 0.00), (1930, 0.00),
(1940, 1.90), (1950, 1.20), (1960, 0.9), (1970, 0.5),
(1980, 0.2), (1990, 0.2), (2000, 0.2), (2010, 0.18),
(2020, 0.19)
DOCUMENT: Birth fraction is the number of births per 1000
women per year in each cohort. Data taken from: Bureau of
the Census, Statistical Abstracts of the U.S. (1963)


BF_PRIOR_TO_1940 = GRAPH(TIME)
(1900, 130), (1910, 127), (1920, 118), (1930, 89.2),
(1940, 79.9), (1950, 0.00), (1960, 0.00), (1970, 0.00),
(1980, 0.00), (1990, 0.00), (2000, 0.00), (2010, 0.00),
(2020, 0.00)
```

DOCUMENT: Before 1940, data could only be found for the
number of births per 1000 women between the ages of 15
and 44. Data taken from: Bureau of the Census, Statis-
tical Abstracts of the U.S. (Colonial Times to 1970)

DR_0_1 = GRAPH(TIME)
(1900, 162), (1910, 132), (1920, 92.3), (1930, 69.0),
(1940, 54.9), (1950, 33.0), (1960, 27.0), (1970, 21.4),
(1980, 12.9), (1990, 9.70), (2000, 6.60), (2010, 4.12),
(2020, 2.48)
DOCUMENT: Death rates are the number of deaths per year
per 1000 population.
Data is taken from: Bureau of the Census, Statistical
Abstracts of the U.S. (1963)

DR_15_24 = GRAPH(TIME)
(1900, 5.90), (1910, 4.50), (1920, 4.90), (1930, 3.30),
(1940, 2.00), (1950, 1.30), (1960, 1.10), (1970, 1.29),
(1980, 1.15), (1990, 0.98), (2000, 0.87), (2010, 0.75),
(2020, 0.69)
DOCUMENT: Death rates are the number of deaths per year
per 1000 population.
Data is taken from: Bureau of the Census, Statistical
Abstracts of the U.S. (1963)

DR_1_4 = GRAPH(TIME)
(1900, 19.8), (1910, 14.0), (1920, 9.90), (1930, 5.60),
(1940, 2.90), (1950, 1.40), (1960, 1.10), (1970, 0.84),
(1980, 0.64), (1990, 0.47), (2000, 0.3), (2010, 0.2),
(2020, 0.2)
DOCUMENT: Death rates are the number of deaths per year
per 1000 population.
Data is taken from: Bureau of the Census, Statistical
Abstracts of the U.S. (1963)

DR_25_34 = GRAPH(TIME)
(1900, 8.20), (1910, 6.50), (1920, 6.80), (1930, 4.70),
(1940, 3.10), (1950, 1.80), (1960, 1.50), (1970, 1.59),
(1980, 1.36), (1990, 1.39), (2000, 1.17), (2010, 1.08),
(2020, 0.855)
DOCUMENT: Death rates are the number of deaths per year
per 1000 population.
Data is taken from: Bureau of the Census, Statistical
Abstracts of the U.S. (1963)

```
DR_35_44 = GRAPH(TIME)
(1900, 10.2), (1910, 9.00), (1920, 8.10), (1930, 6.80),
(1940, 5.20), (1950, 3.60), (1960, 3.00), (1970, 3.17),
(1980, 2.29), (1990, 2.24), (2000, 1.98), (2010, 1.65),
(2020, 1.65)
DOCUMENT: Death rates are the number of deaths per year
per 1000 population.
Data is taken from: Bureau of the Census, Statistical
Abstracts of the U.S. (1963)


DR_45_54 = GRAPH(TIME)
(1900, 15.0), (1910, 13.7), (1920, 12.2), (1930, 12.2),
(1940, 10.6), (1950, 8.50), (1960, 7.60), (1970, 7.38),
(1980, 5.90), (1990, 4.77), (2000, 3.90), (2010, 3.38),
(2020, 2.55)
DOCUMENT: Death rates are the number of deaths per year
per 1000 population.
Data is taken from: Bureau of the Census, Statistical
Abstracts of the U.S. (1963)


DR_55_64 = GRAPH(TIME)
(1900, 27.2), (1910, 26.2), (1920, 23.6), (1930, 24.0),
(1940, 22.2), (1950, 19.0), (1960, 17.4), (1970, 16.9),
(1980, 13.8), (1990, 12.2), (2000, 10.8), (2010, 9.24),
(2020, 8.54)
DOCUMENT: Death rates are the number of deaths per year
per 1000 population.
Data is taken from: Bureau of the Census, Statistical
Abstracts of the U.S. (1963)


DR_5_14 = GRAPH(TIME)
(1900, 3.90), (1910, 2.90), (1920, 2.60), (1930, 1.70),
(1940, 1.00), (1950, 0.6), (1960, 0.5), (1970, 0.42),
(1980, 0.31), (1990, 0.24), (2000, 0.14), (2010, 0.12),
(2020, 0.1)
DOCUMENT: Death rates are the number of deaths per year
per 1000 population.
Data is taken from: Bureau of the Census, Statistical
Abstracts of the U.S. (1963)


DR_65_69 = GRAPH(TIME)
(1900, 56.4), (1910, 55.6), (1920, 52.5), (1930, 51.4),
(1940, 48.4), (1950, 41.0), (1960, 38.2), (1970, 37.3),
(1980, 31.2), (1990, 27.4), (2000, 25.0), (2010, 22.0),
(2020, 21.0)
```

```
DOCUMENT: Death rates are the number of deaths per year
per 1000 population.
Data is taken from: Bureau of the Census, Statistical
Abstracts of the U.S. (1963)

LIFE_EXPECTANCY = GRAPH(TIME)
(1900, 9.30), (1910, 9.10), (1920, 9.70), (1930, 9.60),
(1940, 9.95), (1950, 10.9), (1960, 11.3), (1970, 12.1),
(1980, 12.8), (1990, 13.9), (2000, 14.6), (2010, 15.2),
(2020, 16.2)
DOCUMENT: Dwell time = life expectancy for the average
70 year old.
Data taken from: Bureau of the Census, Statistical
Abstracts of the U.S. (Colonial Times to 1970)
```

4.2 River Toxins

In this section of our chapter on modeling two independent variables, we apply the methods outlined in Section 4.1.1 to model not only the change of a system over time but also how these changes manifest themselves in space. For example, suppose that a pollutant is released into a river at one point and we want to know how the concentration of that pollutant changes with time AND at different points downstream of the pollution point. Time and the distance downstream are the independent variables. How can we design a model to keep track of distance and time simultaneously?

To solve such a problem we establish a chain of stocks of the pollutant that represent connected sections of the river, 1, 2, 3, ..., etc., and we connect each stock to the next one with a controlled transfer variable, F1, F2, F3, ..., etc. The stocks represent the volume of pollution in each section and are measured, for example, in cubic meters of pollutants. Each of the transfer vaiables F1, F2, ..., etc. are controlled by a dwell time, T1, T2, T3, ..., etc., which represents how long a molecule of pollutant and of river water stay in that particular section. The chain continues until the length of river is described with sufficient accuracy. The stocks give the amount of pollutant in each section at any time.

Let us assume, without any loss of generality, that the pollutant is released at an initial injection point F0 and that there are six river sections of interest. The purpose of the model is to find the concentration in any subsequent section at any time. We can find the concentrations at different distances from the point of release by dividing the amount of pollutant in a section by the volume of water in that section. The volumes of the sections V1 and V2 are determined by the flow rate, Q1, which is the same for sections 1 and 2. After section 2, this flow rate changes. This change may be caused, for

example, by changes in the river bed or the presence of some levies. The volumes of the sections V3, V4, V5, and V6 are determined by the flow rate Q2.

The dwell or residence time, T, in any section is just the volume of that section divided by the flow in that section. These dwell time parameters are used in this model analogously to the TIME 01, TIME 12, etc. parameters of the age cohort model in the previous chapter that were required to move the population through the different age cohorts (Figure 4.20).

The volume-flow relations are assumed to be empirically determined and, in principle, vary over time in response to changes in the flow rates. For simplicity of the model, however, we assume fixed values for Q1 and Q2. Thus, the graphs that specify the volume-flow relationships are not essential in our model, because if Q1 and Q2 are constant, V1, V2, V3, etc. are also constant. Nevertheless, we included the volume-flow relations in the model to make it easier later on to investigate the implications of changing flow rates for pollution concentrations.

For the model, we specified volume as an increasing function of the flow rate. The graphical relationships between the volume, V, and the flow rate, Q, of the river employed in our model is shown in Figure 4.21 for the first
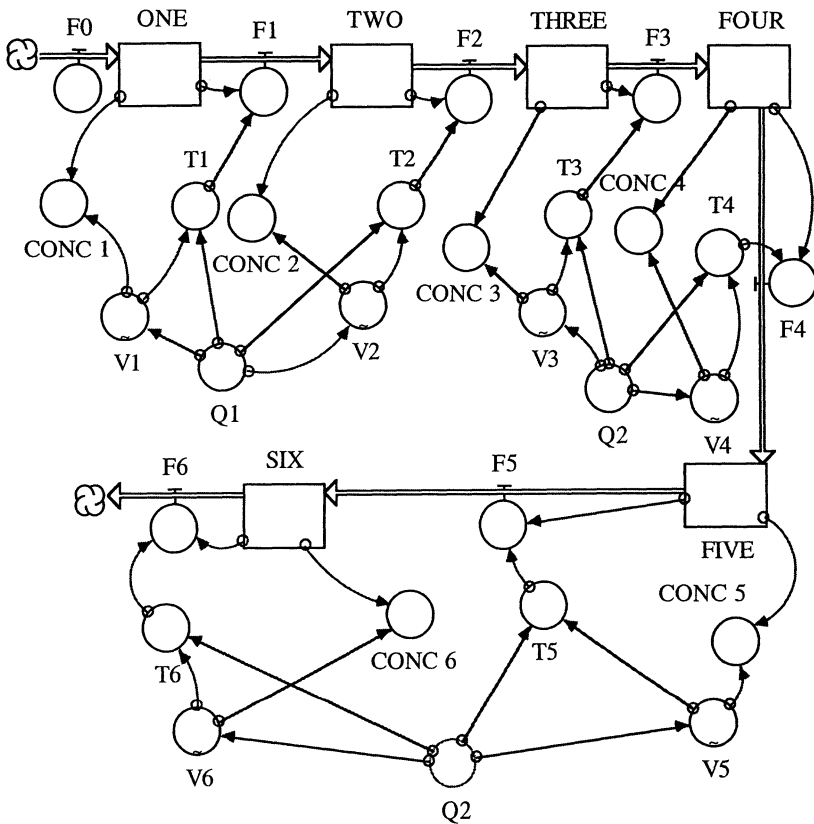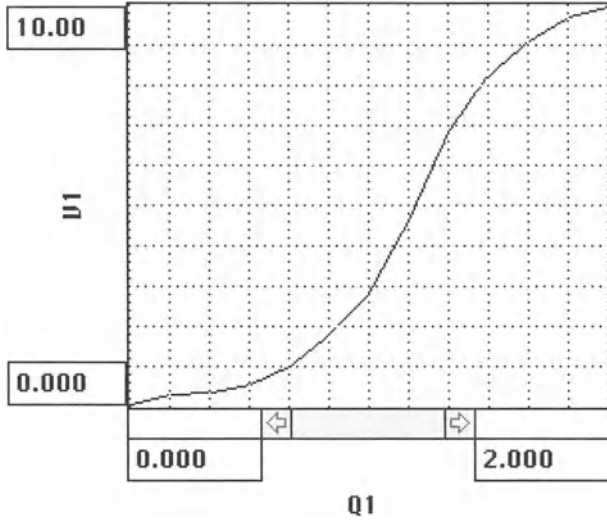

FIGURE 4.20

FIGURE 4.21

section, i.e., for V1 as a function of Q1. All other graphical relationships are very similar. You can find them on the CD ROM that accompanies this book.

Model results are shown in Figure 4.22. The concentration of pollutants in the first section of the river has a very pronounced spike due to the release into that section. Subsequent sections of the river show increases in concentration with time, up to a maximum, and a subsequent decline, though the concentration never reaches zero in the long run. The further downstream a section lies, the lower the highest concentration is, because
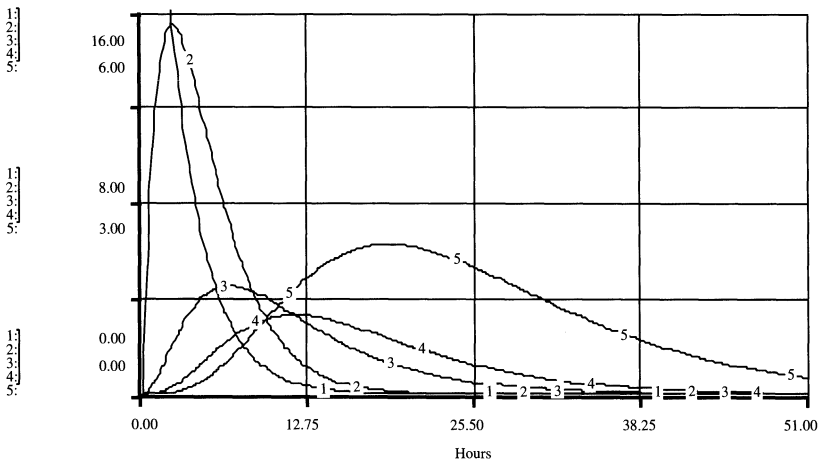

FIGURE 4.22

of the dillution effect. Note that the concentration of pollutants in the fifth section only appears at first glance to be larger than that of the previous three sections, simply because we chose to plot the concentration of that section over a different scale.

Now imagine that you are the owner of a chemical plant located in section 3 of this river. You are the owner of a plant that is identical to the one that is periodically polluting the river at station 1 and so you too have to rid yourself of the same quantity of pollution. Suppose that you learned that the Environmental Protection Agency (EPA) is setting up a pollution concentration measuring station at section 6 and that the EPA already has the model for stream flow that we just developed above. You learn that the EPA thinks that it is not your plant that is dumping pollutants in the stream but that all the pollution is released by the plant at section 1.

As plant owner, you know that your plant is dumping its pollutants into the river. However, the EPA does not suspect that you too release pollution into the river. The EPA's assumption is based on their observation that the firm at station 1 dumps some amount of pollutants into the river and on their measurements at section 6. You call your engineers into your office and they reveal their method that is fooling the EPA into thinking that all the pollution is coming from the plant at station 1. How did they do it? What are the implications for water quality control by the EPA in our model?

Redo the model above but add a side stream of water that enters the main channel at section 3. Be careful how you model the diluting effect of this sidestream.

In a further refinement of the model, you may want to include fluctuations of the flow rates over time to make use of the volume-flow relations specified in our model above. Choose fluctuations of the flow rates along a sine wave with a mean equal to the values of Q1 and Q2 chosen before. Such changes in flow rates can be easily integrated into the model where one firm tries to trick EPA. How do the results differ from the case with constant flow rates?

To make the model even more realistic, add a process of biological decay of the pollutant to the model. How should the biological decay activity depend on the pollution concentration? We would expect that the organisms in our river are better able to deccompose or assimilate pollutants when concentrations are low, and worse, or absent, for high pollution levels.

RIVER TOXINS

```
FIVE(t) = FIVE(t - dt) + (F4 - F5) * dt
INIT FIVE = 0 {Cubic Meters}
INFLOWS:
F4 = FOUR/T4 {Ejects 1/T1 th of its Volume into the
Next Station at Each Time Step; Cubic Meter per Minute}
```

```
OUTFLOWS:
F5 = FIVE/T5 {Ejects 1/T1 th of its Volume into the
Next Station at Each Time Step; Cubic Meter per Minute}


FOUR(t) = FOUR(t - dt) + (F3 - F4) * dt
INIT FOUR = 0 {Cubic Meters}
INFLOWS:
F3 = THREE/T3 {Ejects 1/T3 th of its Volume into the
Next Station at Each Time Step; Cubic Meter per Minute}
OUTFLOWS:
F4 = FOUR/T4 {Ejects 1/T1 th of its Volume into the
Next Station at Each Time Step; Cubic Meter per Minute}


ONE(t) = ONE(t - dt) + (F0 - F1) * dt
INIT ONE = 0 {Amount of toxin in the first station;
Cubic Meters}
INFLOWS:
F0 = PULSE(100,DT,1000) {Pulses toxin into the first
station at the start of the run. Injections can be made
at various stations at various times. Cubic Meters per
Minute}
OUTFLOWS:
F1 = ONE/T1 {Ejects 1/T1 th of its Volume into the
Next Station at Each Time Step; Cubic Meter per Minute}


SIX(t) = SIX(t - dt) + (F5 - F6) * dt
INIT SIX = 0 {Cubic Meters}
INFLOWS:
F5 = FIVE/T5 {Ejects 1/T1 th of its Volume into the
Next Station at Each Time Step; Cubic Meter per Minute}
OUTFLOWS:
F6 = SIX/T6 {Ejects 1/T6 th of its Volume into the
Next Station at Each Time Step; Cubic Meter per Minute}


THREE(t) = THREE(t - dt) + (F2 - F3) * dt
INIT THREE = 0 {Cubic Meters}
INFLOWS:
F2 = TWO/T2{Ejects 1/T2 th of its Volume into the Next
Station at Each Time Step; Cubic Meter per Minute}
OUTFLOWS:
F3 = THREE/T3 {Ejects 1/T3 th of its Volume into the
Next Station at Each Time Step; Cubic Meter per Minute}
```

```
TWO(t) = TWO(t - dt) + (F1 - F2) * dt
INIT TWO = 0 {Cubic Meters}
INFLOWS:
F1 = ONE/T1 {Ejects 1/T1 th of its Volume into the
Next Station at Each Time Step; Cubic Meter per Minute}
OUTFLOWS:
F2 = TWO/T2{Ejects 1/T2 th of its Volume into the Next
Station at Each Time Step; Cubic Meter per Minute}

CONC_1 = ONE/V1 {Cubic Meters of Pollutants per Cubic
Meter of Water}
CONC_2 = TWO/V2 {Cubic Meters of Pollutants per Cubic
Meter of Water}
CONC_3 = THREE/V3 {Cubic Meters of Pollutants per
Cubic Meter of Water}
CONC_4 = FOUR/V4 {Cubic Meters of Pollutants per Cubic
Meter of Water}
CONC_5 = FIVE/V5 {Cubic Meters of Pollutants per Cubic
Meter of Water}
CONC_6 = SIX/V6 {Cubic Meters of Pollutants per Cubic
Meter of Water}
Q1 = 1 {Cubic Meters per Minute}
Q2 = 1.6 {Cubic Meters per Minute}
T1 = V1/Q1    {Dwell Time in Each Section; Minutes}
T2 = V2/Q1    {Dwell Time in Each Section; Minutes}
T3 = V3/Q2    {Dwell Time in Each Section; Minutes}
T4 = V4/Q2    {Dwell Time in Each Section; Minutes}
T5 = V5/Q2    {Dwell Time in Each Section; Minutes}
T6 = V6/Q2    {Dwell Time in Each Section; Minutes}
V1 = GRAPH(Q1)                      {Cubic Meters}
(0.00, 0.05), (0.167, 0.3), (0.333, 0.35), (0.5, 0.55),
(0.667, 1.00), (0.833, 1.80), (1, 2.80), (1.17, 4.60),
(1.33, 6.80), (1.50, 8.25), (1.67, 9.05), (1.83, 9.70),
(2.00, 9.95)
V2 = GRAPH(Q1)                      {Cubic Meters}
(0.00, 0.00), (0.167, 0.7), (0.333, 1.15), (0.5, 1.50),
(0.667, 1.75), (0.833, 1.95), (1, 2.15), (1.17, 2.50),
(1.33, 2.85), (1.50, 3.50), (1.67, 4.85), (1.83, 7.90),
(2.00, 10.0)
V3 = GRAPH(Q2)                      {Cubic Meters}
(0.00, 0.075), (0.167, 1.35), (0.333, 2.10), (0.5,
2.48), (0.667, 2.92), (0.833, 3.30), (1, 3.90), (1.17,
4.88), (1.33, 6.83), (1.50, 9.07), (1.67, 11.6), (1.83,
13.3), (2.00, 14.8)
```

```
V4 = GRAPH(Q2)                          {Cubic Meters}
(1.00, 0.00), (1.08, 2.02), (1.17, 3.38), (1.25, 4.72),
(1.33, 5.92), (1.42, 6.67), (1.50, 7.80), (1.58, 8.55),
(1.67, 9.07), (1.75, 9.90), (1.83, 12.3), (1.92, 14.0),
(2.00, 15.0)
V5 = GRAPH(Q2)                          {Cubic Meters}
(0.00, 0.09), (0.167, 2.07), (0.333, 4.05), (0.5,
5.85), (0.667, 6.39), (0.833, 6.75), (1, 7.47), (1.17,
8.28), (1.33, 9.63), (1.50, 11.7), (1.67, 14.7), (1.83,
17.0), (2.00, 18.0)
V6 = GRAPH(Q2)                          {Cubic Meters}
(0.00, 0.1), (0.167, 1.20), (0.333, 1.40), (0.5, 2.10),
(0.667, 2.70), (0.833, 3.80), (1, 5.40), (1.17, 8.50),
(1.33, 11.7), (1.50, 15.4), (1.67, 17.4), (1.83, 18.2),
(2.00, 19.8)
```

5
Randomness

> To me our knowledge of the way things work, in society or in nature, comes trailing clouds of vagueness. Vast ills have followed a belief in certainty, whether historical inevitability, grand diplomatic designs, or extreme views on economic policy. When developing policy with wide effects for an individual society, caution is needed because we cannot predict the consequences.[1]

5.1 Flipping a Coin

In the previous model, we modeled the dynamic behavior of systems in an almost mechanical way. Once the relationships among system components were defined and initial conditions were specified within certain bounds, the dynamics of the system as a whole were predetermined and can be exactly repeated with each model run. Real systems are not necessarily that predictable. Rather, some random element may determine the direction of a system's behavior. This case is illustrated in the model developed in this chapter. Other sources of unpredictability of system behavior are discussed later in the book.

   If we flip a coin repeatedly several thousand times, we see a random tendency, similar to a drunk staggering right or left from a reference line, the standard setting for the discussion of the random walk. The head and tail rate are easy to generate and accumulate. For the generation of head and tail in a coin toss, we create a random number between 0 and 1, using STELLA's built-in function RANDOM:

$$\text{FLIPPER} = \text{RANDOM}(0,1) \tag{1}$$

We interpret numbers below .5 as head and above .5 as tail:

$$\text{HEAD RATE} = \text{IF FLIPPER} < .5 \text{ THEN 1 ELSE 0} \tag{2}$$

$$\text{TAIL RATE} = \text{IF FLIPPER} > .5 \text{ THEN 1 ELSE 0} \tag{3}$$

We can also generate the difference between the head and tail rates and sum that difference. The model is composed of a stock for the accumulation, CUMULATIVE NET, and the difference between the head and tail rates, NET:

$$\text{NET} = \text{HEAD RATE - TAIL RATE} \tag{4}$$

The model and its results are shown in Figures 5.1 and 5.2, respectively. Set up the model, run it several times and graph the CUMULATIVE NET. Note

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[1]Arrow, K.J. 1992. I Know a Hawk from a Handsaw, in M. Szenberg (ed.) *Eminent Economists: Their Life Philosophies*, Cambridge University Press, Cambridge, p. 45.
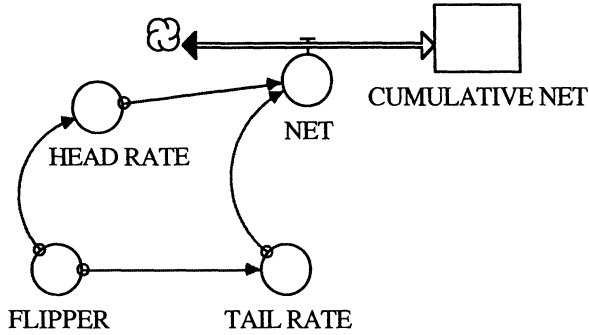
FIGURE 5.1

how different the graphs are from the one we have run. Run over a long enough time, any trend thought to exist in the short run will be reversed. Try it. Run the model for 500 flips, judge the trend and then resume the model for the remaining 1500 flips and see if your judgment holds.
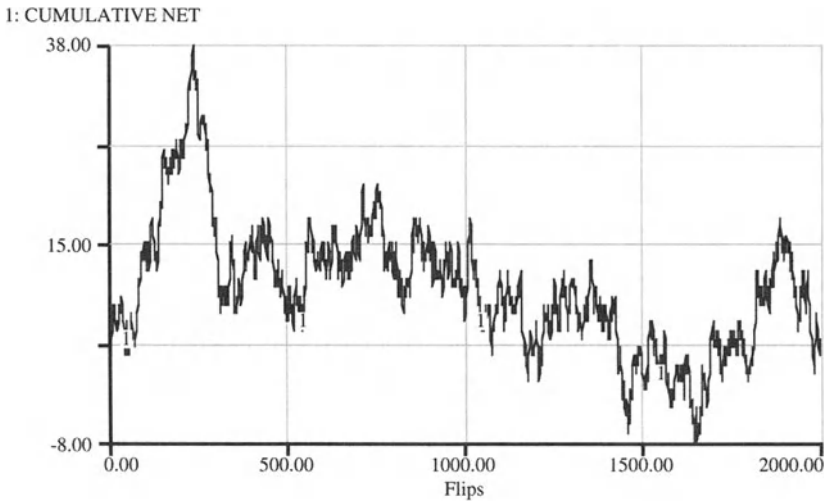

FIGURE 5.2

FLIPPING A COIN

CUMULATIVE_NET(t) = CUMULATIVE_NET(t - dt) + (NET) *
dt
INIT CUMULATIVE_NET = 0 {Cumulative Number of Net Head
Tossed}

```
INFLOWS:
NET = HEAD_RATE-TAIL_RATE {Number of Net Head Tossed}

FLIPPER = RANDOM(0,1)
HEAD_RATE = IF FLIPPER < .5 THEN 1 ELSE 0 {Number of
Head}
TAIL_RATE = IF FLIPPER > .5 THEN 1 ELSE 0 {Number of
Tail}
```

5.2 Intoxication Model

Let us form and demonstrate a model of the path of a person who is ine-briated to various degrees as she tries to walk from some initial position to-ward a destination 100 feet away. Let the nominal step size be 1 meter. After each step, this person resights the goal and takes another step. The di-rection should have a normally distributed variation around the desired di-rection. Analogously, the step size variation should have a normally distrib-uted variation around the normal step size of a meter. Assume that the standard deviation for the direction and the step size are the same value—call it the INTOXICATION INDEX: the smaller this index, the less intoxi-cated this person is. Show the travel patterns for various indices.

STEP SIZE can be generated by a random variable of normal distribution with mean = 1 and standard deviation = INTOXICATION INDEX:

$$\text{STEP SIZE} = \text{NORMAL}(1, \text{INTOXICATION INDEX}) \tag{5}$$

In order to consider the direction taken by the person, we may assume that the person is initially situated at a point (0,0) and her goal is in the point (0,100). The desired angle at which the person approaches her goal is

$$\begin{aligned}\text{DESIRED ANGLE} = \text{ IF } X< 0 \text{ THEN ARCTAN }((Y\text{-}100)/X) \\ \text{ELSE ARCTAN }((Y\text{-}100)/X) + \text{PI}\end{aligned} \tag{6}$$

If the person is intoxicated this will, of course, not necessarily be the angle at which the person approaches her goal. The chosen angle is a normally distributed random variable with mean = DESIRED ANGLE and standard deviation = INTOXICATION INDEX:

$$\begin{aligned}\text{CHOSEN ANGLE} = \text{NORMAL(DESIRED ANGLE,} \\ \text{INTOXICATION INDEX)}\end{aligned} \tag{7}$$

The actual movement towards the goal is

$$\text{MOVE Y} = \text{STEP SIZE} * \text{SIN(CHOSEN ANGLE)} \tag{8}$$

and movement away from a straight line is given by

$$\text{MOVE X} = \text{STEP SIZE} * \text{COS(CHOSEN ANGLE)} \tag{9}$$

The flows MOVE Y and MOVE X need to be defined as biflows because these moves can take on negative values, i.e., the person may stagger away from her goal. The total distance covered by the person is just the accumulation of all steps. Here, we have to be careful to sum up all step sizes independent of whether they lead to the goal (positive step sizes) or away from the goal (negative step sizes).

The final part in this model requires that the person does not continue to move if she is "close enough" to her goal. We let STELLA pause the simulation run if the person is within at least half a foot in the X direction and at least within a quarter of a foot from the goal in Y direction. Pausing STELLA is done by the translation variable CLOSE, which we define as

$$\text{CLOSE} = \text{IF } (X \leq .5) \text{ AND } (Y \leq .25) \text{ THEN PAUSE, ELSE } 0 \qquad (10)$$

The resulting model is shown in Figure 5.3.

Plot Y against X to show the location of the person at each step. You can set up such a plot by moving the graph icon to the STELLA diagram and then double-clicking on the graph. When the dialogue box appears, select "Scatter Plot" and specify X as the "X Axis" and Y as the "Y Axis."

When you run the model and the person is close enough to the goal, STELLA will pause the model. You can then choose to stop it or resume the mode run. If you resume the model, you will find that the person may temporarily move away from the goal until she again comes close.
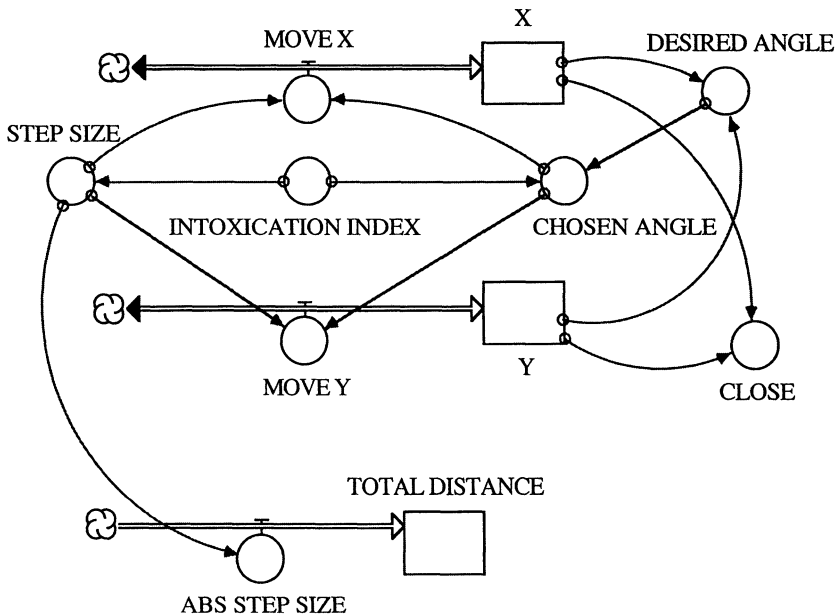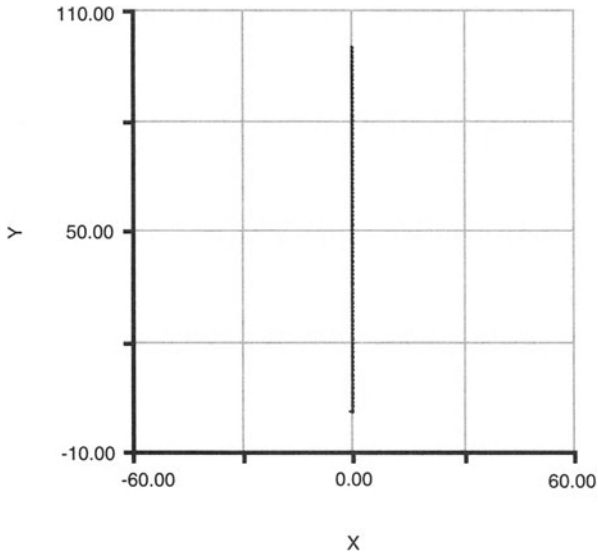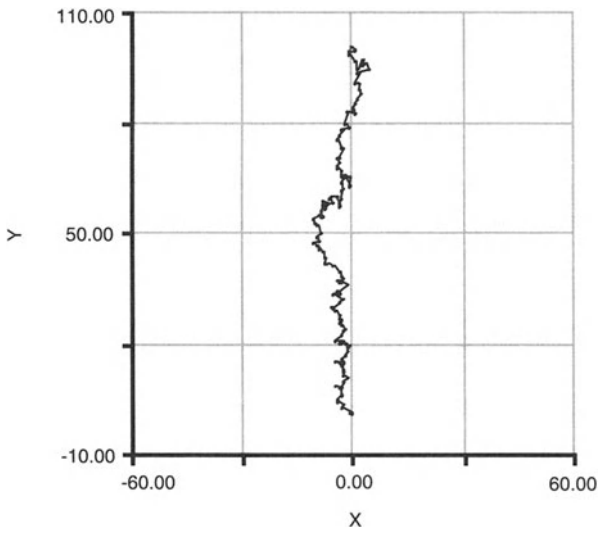

FIGURE 5.3

FIGURE 5.4

FIGURE 5.5

The model runs in Figures 5.4 and 5.5 of the Intoxication Model are done with an INTOXICATION INDEX of 0 and 1.0, respectively. Thus, each of these pictures represents the path for increasing intoxication. We find that if the person is not or only slightly intoxicated, i.e., INTOXICATION INDEX is 0 or close to 0, the path resembles a straight line. The higher the intoxication index, the more the person staggers around, even moving backwards at times, and the longer it takes to reach the destination. Run the model and you will find that for intoxication indices greater than 0, the path will always be different, and that with increasing intoxication the number of steps (time length of the simulation) must be increased in order to reach the goal.

Develop a model in which a person is intoxicated and attempts to approach a car. Assume in this case that the car rolls at a constant velocity to the right while the person approaches it. For that model, provide a sensitivity analysis of some key parameter, such as the intoxication index and the velocity, and plot a series of "walks" for alternative parameter values.

```
                    INTOXICATION MODEL

TOTAL_DISTANCE(t) = TOTAL_DISTANCE(t - dt) +
(ABS_STEP_SIZE) * dt
INIT TOTAL_DISTANCE = 0
INFLOWS:
ABS_STEP_SIZE = ABS(STEP_SIZE)


X(t) = X(t - dt) + (MOVE_X) * dt
INIT X = 0
INFLOWS:
MOVE_X = STEP_SIZE*COS(CHOSEN_ANGLE)


Y(t) = Y(t - dt) + (MOVE_Y) * dt
INIT Y = 0
INFLOWS:
MOVE_Y = STEP_SIZE*SIN(CHOSEN_ANGLE)


CHOSEN_ANGLE =
NORMAL(DESIRED_ANGLE,INTOXICATION_INDEX)
CLOSE = IF (X<=.5) AND (100-Y <= .25) THEN PAUSE
ELSE 0
DESIRED_ANGLE = IF X < 0 THEN ARCTAN((Y-100)/(X)) ELSE
ARCTAN((Y-100)/(X)) + PI
INTOXICATION_INDEX = 0
STEP_SIZE = NORMAL(1,INTOXICATION_INDEX)
```

6
Positive and Negative Feedback

> We are faced at every turn with the problem of Organic Unity, of Des-
> creteness, of Dicontinuity—the whole is not equal to the sum of the
> parts, comparisons of quantity fail us, small changes produce large ef-
> fects, and the assumptions of a uniform and homogeneous continuum
> are not satisfied.[1]

6.1 The Basic Model

At the beginning of the book we discussed the importance of positive and
negative feedback processes for the dynamic behavior of systems. Much at-
tention has been given by researchers to negative feedback processes, be-
cause negative feedback processes lead systems towards equilibrium states.
In this chapter, we turn our attention to positive feedback processes. Posi-
tive feedback reinforces a given tendency of a system and can lead a system
away from equilibrium states, possibly causing quite unexpected results.

Here is a simple model of a common process with a surprising result. To
our knowledge, the first to point out this model was the economist Brian
Arthur in his article on positive feedbacks in the economy[2]. Brian Arthur
cites positive feedback processes as driving forces in determining which of
two (or more) alternative new technologies will come to dominate a mar-
ket. A classic example is the development of the two video technologies
VHS and Betamax. Both systems entered the market at roughly the same
time and, early on, maintained almost the same market shares. It was not
clear which of the two technologies would ultimately dominate the market.
However, if one succeeded to increase its market share slightly over the
other, its increased presence would make it more attractive for households
to buy that technology. In response, video rental businesses and stores
would carry a slightly larger amount of tapes and recorders for the respec-
tive system, which, in turn, makes it more attractive for new buyers,
thereby increasing its attractiveness to prospective buyers even further.

The forces that determine what technology increases its market share
during the first periods are often small and may appear to be random. In-
itially small differences in market shares ultimately increase until one sys-
tem is almost entirely eradicated from the market. The video-technology,
indeed, followed that trend, with VHS being the market-dominating tech-

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[1]Keynes, J.M. 1933. In D. Moggridge (ed.) *The Collected Writings of John Maynard
Keynes*, St. Martin's, New York, Vol. X, p. 262.
[2]Arthur, B. 1990. Positive Feedbacks in the Economy, *Scientific American*, February
1990, pp. 92–99.

nology today. Similar developments can be observed for a number of other technologies or products that have close substitutes that enter the market at the same time.[3]

A potential application of positive feedback in ecological systems can be found in a study of the competition of two species for shoreline space.[4] Here we have the competition between two shore-water-oriented plants that spread through root growth. If either of the two are the only plants in an area, that plant will become the sole plant growing in that area, however if the plants are equally present after a clear cut disturbance, either plant may dominate but most likely, however, the plant that does the best in shallow water will take over that area while the other will take the slightly deeper water area adjacent to its competitor. So we have here a mixture of positive feedback and adaptive advantage.

Brian Arthur illustrates the process of positive feedback with a simple model. The idea here is that balls of two different colors are chosen at a time and placed on a table. We start with one ball of each color on the table. The rule for choosing the next color is given by a function of the fraction of balls already on the table. If the fraction of balls of a given color is larger than a random variable that lies between 0 and 1, then another ball of that color is added onto the table. By adding another ball of that color its fraction is increased. As a result, for the next drawing, it is even more likely to draw another ball of the very same color . . . . Positive feedback prevails.

In our model, the balls are either blue or red. We start with one of each color on the table. The system is described by two state variables that are represented by the stocks BLUE and RED. From these stocks we can calculate the fraction of each color.

The key equations driving the model are

$$\text{ADD BLUE} = \text{IF RANDOM NUMBER} < \text{BLUE FRACTION}$$
$$\text{THEN 1 ELSE 0} \tag{1}$$

$$\text{ADD RED} = \text{IF ADD BLUE} \neq 1 \text{ THEN 1 ELSE 0} \tag{2}$$

The STELLA model is shown in Figure 6.1 and the details of the calculation are listed at the end of this section in the model equations. Before you look them up, try to model the two-color positive feedback model yourself.

The fraction of each color oscillates wildly and then settles down to a particular value (Fig. 6.2). However, this value is different on consecutive runs! Such a phenomenon is thought by Arthur to be similar to a specific technology that comes to dominate (or disappear from) the market. The process is a matter of luck at least in part.

---

[3]For an application to the competition between two firms see Ruth, M. and B. Hannon (1997) *Modeling Dynamic Economic Systems*, Springer-Verlag, New York.
[4]Grace, J. 1987. The Impact of Preemption on the Zonation of Two Typha Species Along Lakeshores, *Ecological Monographs*, Vol. 57, pp. 283–303.
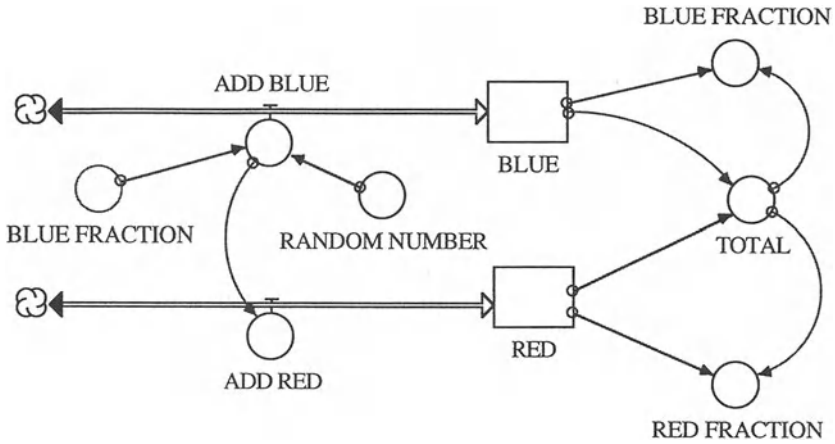
FIGURE 6.1

Set up a sensitivity run for the blue fraction: 20 runs with the initial values of the BLUE and RED always the same, at 1. Do you see a trend? Now do the whole analysis over with the initial value in both colors set at 10. Now do you see a trend? What is going on here? Note that the initial ratio is the mean value of the fixed fractions, and the number of initial balls sets the standard deviation on the distribution of the fixed fractions.
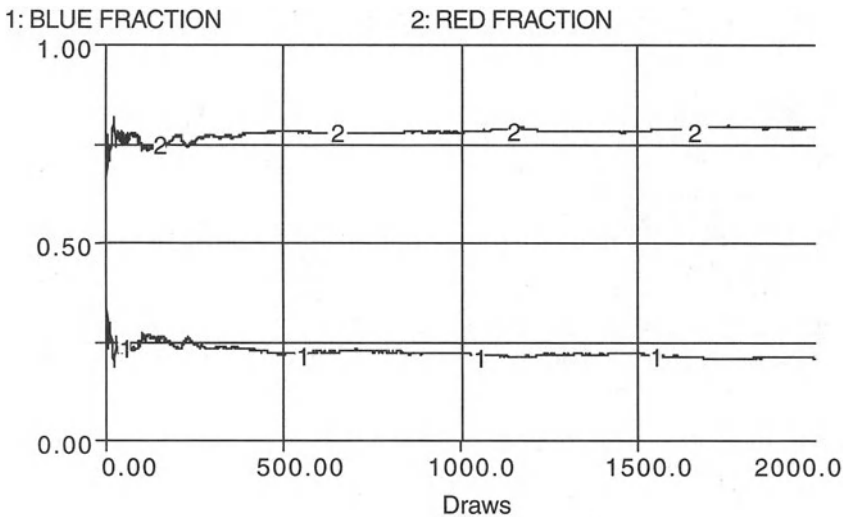

FIGURE 6.2

```
                TWO-COLOR POSITIVE FEEDBACK MODEL

BLUE(t) = BLUE(t - dt) + (ADD_BLUE) * dt
INIT BLUE = 1  {Numbers of Balls}
INFLOWS:
ADD_BLUE = IF RANDOM_NUMBER < BLUE_FRACTION THEN 1
ELSE 0
{< means positive feedback with many fixed points of
red/blue fraction.
 > means negative feedback and the fractions always go
to 0.5.
measured in Balls per Draw}

RED(t) = RED(t - dt) + (ADD_RED) * dt
INIT RED = 1   {Numbers of Balls}
INFLOWS:
ADD_RED = IF ADD_BLUE <> 1 THEN 1 ELSE 0 {The < in the
add_blue variable means: the blue fraction is the
probability that a blue ball will be drawn and if it is
greater than random, it will be drawn, if not the red
ball is drawn; measured in Balls per Draw}

BLUE_FRACTION = BLUE/TOTAL {Number of Balls per Total
Number of Balls}
RANDOM_NUMBER = RANDOM(0,1)
RED_FRACTION = RED/TOTAL {Number of Balls per Total
Number of Balls}
TOTAL = BLUE+RED {Number of Balls}
```

6.2 Positive Feedback with Fixed Points[5]

In the model of the previous section, the relationship between the BLUE FRACTION and the probability was a linear one. That is, the BLUE FRACTION was a substitute for the probability that the next choice would be blue. The higher the BLUE FRACTION, the greater the chance that a blue ball would be picked next. However, that linear choice is somewhat arbitrary. It could be any sort of realtionship. Here is a nonlinear connection:

$$ADD\ BLUE = IF\ RANDOM\ NUMBER < BLUE\ PROBABILITY$$
$$THEN\ 1\ ELSE\ 0 \qquad (3)$$

where BLUE PROBABILITY is defined by the graph in Figure 6.3.

---

[5]See Arthur, B., Y.M. Ermoliev, and Y.M. Kaniovski. 1994. Path-Dependent Processes and the Emergence of Macrostructure, in B. Arthur, *Increasing Returns and Path Dependence in the Economy*, University of Michigan Press, Ann Arbor, MI, pp. 33–48.
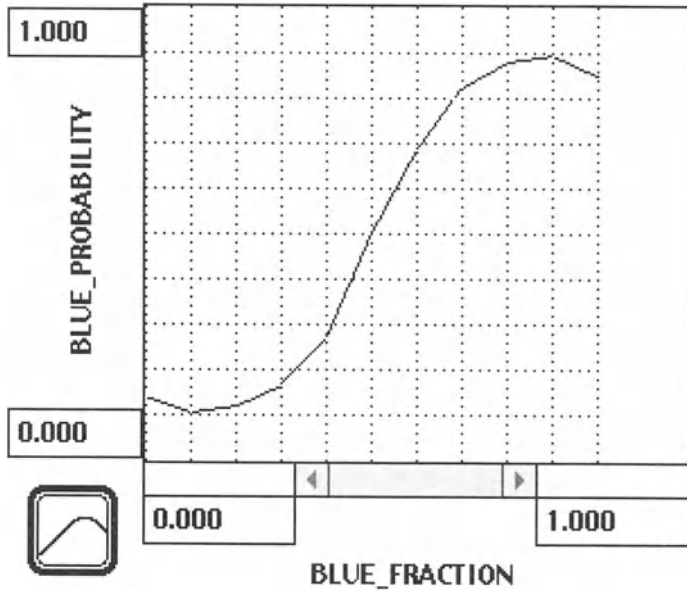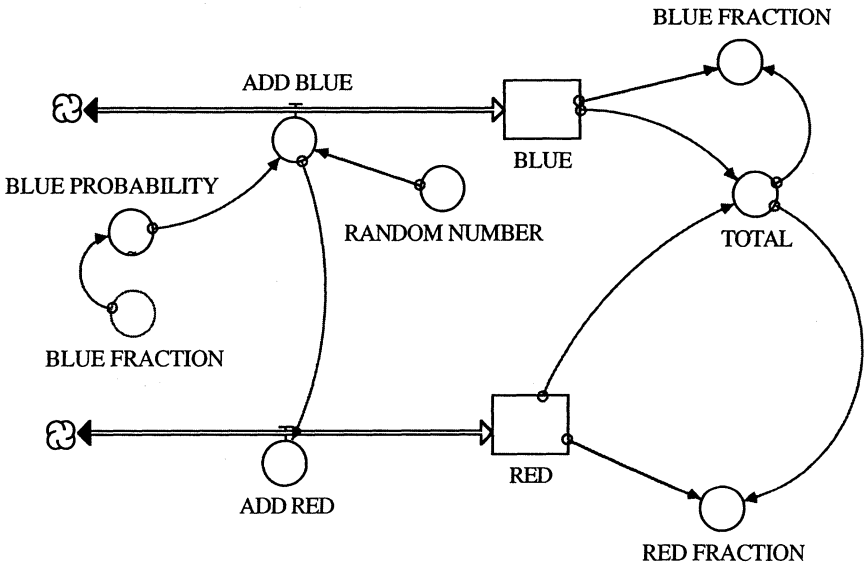
FIGURE 6.3

FIGURE 6.4

The complete model is shown in Figure 6.4. Note that the sort of connection established now produces fixed points for the long-term BLUE FRACTION rather than the random long-term behavior seen in the previous section of this chapter. These fixed points are determined by the intersection of the curved line in BLUE PROBABILITY and the 45 degree line going from 0,0 to 1,1 in Figure 6.3. There are three such intersections. When the intersecting curve comes from above the 45 degree line, a stable fixed point is set for the long-term BLUE FRACTION. There are two such fixed points in the model now. The other intersection point comes from below the 45 degree line, establishing an unstable fixed point that is never seen as a solution.

You may visualize this stable/unstable relationship as follows: When the curve is above the 45 degree line, the chances are greater than usual that a blue ball will be picked next and the tendency is to increase the BLUE FRACTION. When the curve is below, there is a less than usual chance of picking a blue ball on the next round of choice, so the BLUE FRACTION decreases.

So, the positive feedback problem becomes determinant in the long run if the BLUE PROBABILITY function is nonlinear, depending on the location of the stable fixed points. The BLUE PROBABILITY curve could of course crisscross the 45 degree line many times, creating many stable fixed points and then this model becomes increasingly indeterminant. Try different sketches of the BLUE PROBABILITY, figure out where the stable points are, and see how predictable the solutions are. The results of our model with the specification of the BLUE PROBABILITY as in Figure 6.3 is shown in Figure 6.5.
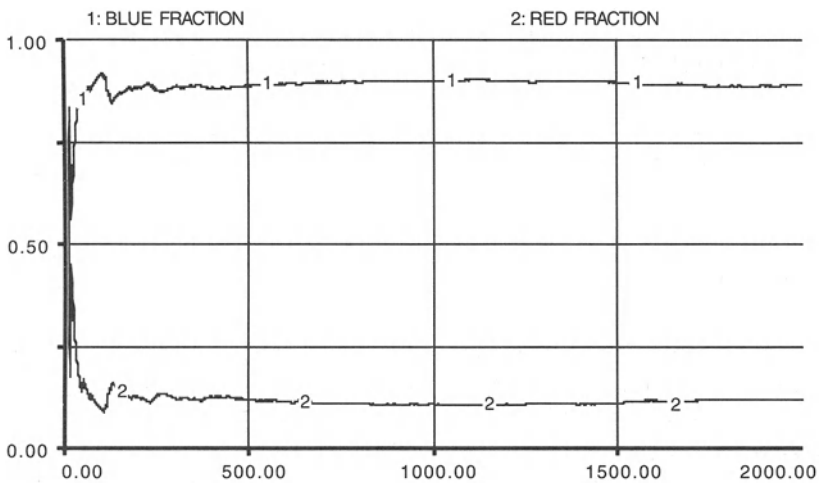

FIGURE 6.5

POSITIVE FEEDBACK WITH FIXED POINTS

```
BLUE(t) = BLUE(t - dt) + (ADD_BLUE) * dt
INIT BLUE = 1  {Numbers of Balls}
INFLOWS:
ADD_BLUE = IF RANDOM_NUMBER < BLUE_PROBABILITY THEN 1
ELSE 0
{< means positive feedback with many fixed points of
red/blue fraction.
 > means negative feedback and the fractions always go
to 0.5.
measured in Balls per Draw}

RED(t) = RED(t - dt) + (ADD_RED) * dt
INIT RED = 1  {Numbers of Balls}
INFLOWS:
ADD_RED = IF ADD_BLUE ≠ 1 THEN 1 ELSE 0 {The < in the
add_blue variable means: the blue fraction is the
probability that a blue ball will be drawn and if it is
greater than random, it will be drawn, if not the red
ball is drawn; measured in Balls per Draw}

BLUE_FRACTION = BLUE/TOTAL {Number of Balls per Total
Number of Balls}
RANDOM_NUMBER = RANDOM(0,1)
RED_FRACTION = RED/TOTAL {Number of Balls per Total
Number of Balls}
TOTAL = BLUE+RED {Number of Balls}
BLUE_PROBABILITY = GRAPH(BLUE_FRACTION)
(0.00, 0.14), (0.1, 0.105), (0.2, 0.12), (0.3, 0.165),
(0.4, 0.27), (0.5, 0.5), (0.6, 0.68), (0.7, 0.82),
(0.8, 0.88), (0.9, 0.89), (1, 0.85)
```

6.3 Elaborations

To see what negative feedback will do, reverse the inequality between the random number and the blue fraction in the model of Section 6.1. The end result is always 1/2. This is what we mean above when we say that negative feedback processes are an equilibrating force. Now return to the problem of positive feedback and add a third color. The results are similar to those found earlier. But note how much harder it is to write the correct conditions for adding the various colors. Try it yourself before checking the list of equations at the end of this section.
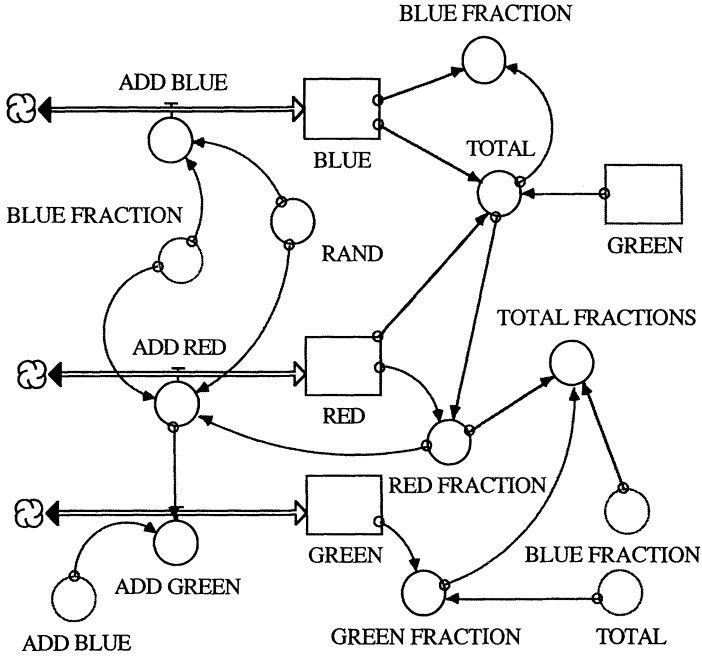
FIGURE 6.6

The model of the three-color positive feedback is shown in Figure 6.6 and the results of one model run are plotted in Figure 6.7.

Now try to combine the effects of both kinds of feedback. Can you show the results of such a combination where the relative strength of the negative feedback is four times as great as the strength of the simultaneously occurring positive feedback?
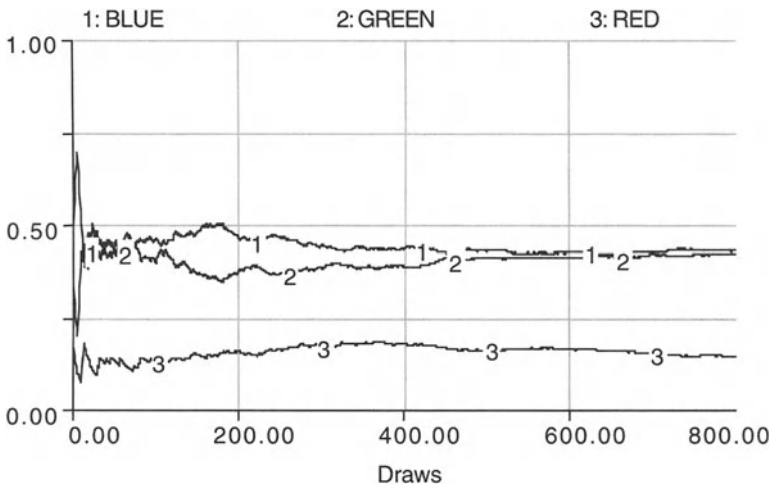

FIGURE 6.7

This little example should make clear the nature of feedback. Try to create examples of three-color positive and negative feedback. For example, try to make the effect of the feedback nonlinear, e.g., the number of balls added is a function of that ball's fraction of dominance, first proportionally and then inversely.

Now, return to the two-color model of positive feedback and find a modified IF statement that enables you to model the dynamics of the two shore-water-oriented plants that spread through root growth. As discussed above, that model should exhibit a mixture of positive feedback and adaptive advantage.

THREE-COLOR POSITIVE FEEDBACK MODEL

```
BLUE(t) = BLUE(t - dt) + (ADD_BLUE) * dt
INIT BLUE = 1   {Number of Balls}
INFLOWS:
ADD_BLUE = IF RAND < BLUE_FRACTION THEN 1 ELSE 0 {<
means positive feedback with many fixed points of
red/blue fraction. Number of Balls per Draw}


GREEN(t) = GREEN(t - dt) + (ADD_GREEN) * dt
INIT GREEN = 1 {Number of Balls}
INFLOWS:
ADD_GREEN = IF (ADD_BLUE = 0) AND (ADD_RED = 0) THEN 1
ELSE 0 {Number of Balls per Draw}


RED(t) = RED(t - dt) + (ADD_RED) * dt
INIT RED = 1 {Number of Balls}
INFLOWS:
ADD_RED = IF (RAND >= BLUE_FRACTION) AND (RAND <
(BLUE_FRACTION + RED_FRACTION)) THEN 1 ELSE 0 {Number
of Balls per Draw}


BLUE_FRACTION = BLUE/TOTAL {Number of Balls per Total
Number of Balls}
GREEN_FRACTION = GREEN/TOTAL {Number of Balls per
Total Number of Balls}
RAND = RANDOM(0,1)
RED_FRACTION = RED/TOTAL {Number of Balls per Total
Number of Balls}
TOTAL = BLUE + RED + GREEN {Number of Balls}
TOTAL_FRACTIONS =
GREEN_FRACTION+RED_FRACTION+BLUE_FRACTION {Number of
Balls per Total Number of Balls}
```

7
Derivatives and Lags

> Honour, 'Tis a derivative from me to mine, And only that I stand for.
> —William Shakspeare, *The Winter's Tale*

7.1 Introduction

STELLA is designed to incrementally add a rate variable to a stock, a process that is called integration. This is very important in finding the solutions to rate or differential equations. However, often we need to do just the opposite, to differentiate or find the slope of a curve or rate of change of a variable. The main reason that we would want to find a rate of change in STELLA is to be able to identify the maximum or minimum value of a variable. We may be generating a stream of numbers that represent the way in which some variable is changing in time, but how do we find the largest or smallest of this stream as these numbers are calculated? We find the derivative of the variable and we keep track of this value. When it goes to zero, the variable has reached a maximum or minimum value. Such an event is used as a trigger to tell the rest of the program to stop changing because the desired condition has been found. For example, we may be modeling a firm that attempts to maximize its profits. We will start the firm off with a low output per time period and calculate the corresponding profit. Then, we incrementally increase output over time, thereby increasing revenues from sale of that output but also production costs. With increasing output, revenues increase steadily but profits increase at a decreasing rate because costs increase, too. At the point at which the profit function reaches a maximum, production should be kept constant so that the firm continues to produce at the maximum profit level.

To find the derivative of a variable, we can use the derivative function provided by STELLA in the built-in list, or we can take the value of the function less its value one small time step, DT, in the past and divide this difference by DT: this period's value minus last period's value (corrected for the size of DT) is just the rate of change within one entire time interval.

Develop a model of the integral of a function of time. Integration over time means that we calculate a stock that increases incrementally with time. The flow into the stock is just that function of time. For simplicity, assume that function is

$$F(T) = (TIME)^{\wedge}2 \tag{1}$$

Take the derivative of the integral. In STELLA, that derivative is just the difference between the delayed value of the stock and the value of the stock in time period t, divided by DT. Calculate the delayed value of the integral

of F(T) with the built-in function DELAY. The DELAY built-in requires that you specify the variable you wish to delay, and how long that delay should be. In our case, the delay should be only one small time step, DT, so:

DELAY INTEGRAL F OF T = DELAY(INTEGRAL F OF T,DT)   (2)

Consequently, the derivative of the integral of F(T) is

DERIVATIVE INTEGRAL OF F(T) = (INTEGRAL F OF T

                                        − DELAY INTEGRAL F OF T)/DT   (3)

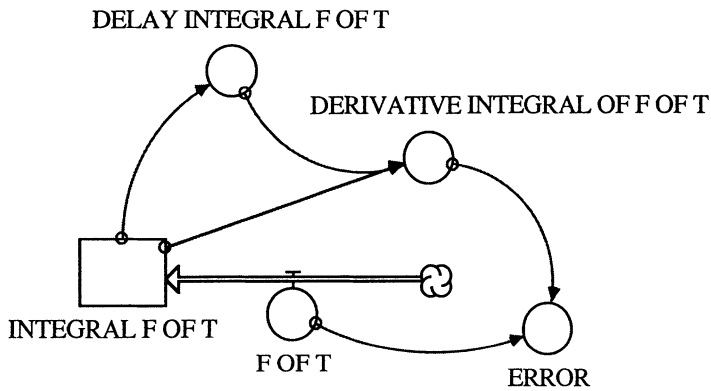Note again that we need to divide by DT to correct for the size of the time step.
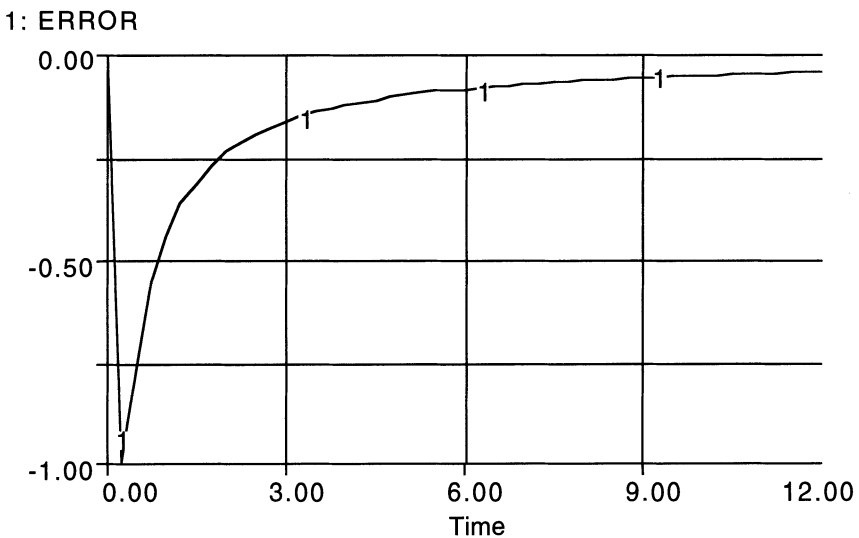

FIGURE 7.1

1: ERROR

FIGURE 7.2

The percentage error caused by using the numerical solution for the integral instead of the analytical one can be calculated by comparing the value of the derivative calculated by STELLA with the actual value of the flow (Fig. 7.1). Note how smaller DTs reduce that error. In the next section, we provide some simple illustrations and applications of derivatives and lags.

The graph in Figure 7.2 shows that, in absolute terms, the error is initially very large (minus infinity, to be precise). Subsequent calculation lead to smaller errors, expressed as a percentage of the growing state variable F of T.

INTEGRATION MODEL

```
INTEGRAL_F_OF_T(t) = INTEGRAL_F_OF_T(t - dt) +
(F_OF_T) * dt
INIT INTEGRAL_F_OF_T = 0
INFLOWS:
F_OF_T = TIME^2


DELAY_INTEGRAL_F_OF_T = DELAY(INTEGRAL_F_OF_T,DT)
DERIVATIVE_INTEGRAL_OF_F_OF_T = (INTEGRAL_F_OF_T -
DELAY_INTEGRAL_F_OF_T)/DT
ERROR = IF TIME > 0 THEN
(DERIVATIVE_INTEGRAL_OF_F_OF_T-F_OF_T)/F_OF_T
ELSE 0
```

7.2 Applications of Derivatives and Lags

7.2.1 Simple Population Model

The basic function performed by programs such as STELLA is integration, the summing of incremental changes to a state variable. The derivative is an undoing of integration, and derivatives of generated results can also be calculated with STELLA.

The derivative is useful in finding the extremes of the range of a variable. The built-in function DELAY is used to calculate the delayed, or lagged, value, which in turn is used to calculate the derivative as the difference between the actual and the delayed value. Here are two cases for the identification of extrema using the DELAY function. The first is a switch (0-1) that signals the peak growth rate in a population model. The population model is very similar to the ones developed earlier. The variable SWITTCH is calculated as the difference between the value of BIRTHS at time period t and BIRTHS at time period t–DT. If this difference is positive, we know that the present number of births exceeds that calculated for the time period one DT earlier. Thus, we did not yet encounter the maximum in BIRTHS. If the difference is smaller than zero, then a peak must have occurred and the

variable SWITTCH should signal the presence of that peak. The variable SWITTCH is set up to do exactly that. We deliberately spelled its name differently from the built-in function "SWITCH," which serves some other purpose.

In our model, the carrying capacity is set to 500, the initial population level is 10, and the birth rate is set constant at .0006. The number of births is calculated as

$$\text{BIRTH} = \text{BIRTH FRACTION} * \text{POPULATION} \\ * (\text{CARRYING CAPACITY-POPULATION}) \quad (4)$$

The switch is defined to generate a number 1 if the difference between current and delayed births is zero or less, and a number 0 as long as current births exceed the births in the previous period:

$$\text{SWITTCH} = \text{IF} ((\text{BIRTHS-DELAY(BIRTHS,DT)}) \leq 0) \\ \text{AND (TIME>1) THEN 1 ELSE 0} \quad (5)$$

Note also that the switch is only activated after the first period, or it could go off at the model outset just because there is no history to the model that would allow us to calculate the delayed value for the births.

The complete model is shown in Figure 7.3 and the results are depicted in Figure 7.4. Births rise to a peak at which the switch signals the reversal in the slope of the curve of births through time. Experiment with alternative model specifications (for example, set up a goal-seeking model for the population) and ensure that your switch accurately picks up the peak in the number of births per time period.

POPULATION MODEL

```
POPULATION(t) = POPULATION(t - dt) + (BIRTHS) * dt
INIT POPULATION = 10 {Organisms per Unit Area}
INFLOWS:
BIRTHS = BIRTH_FRACTION*POPULATION*(CARRYING_CAPACITY-
POPULATION)
{Organisms per Unit Area per Time Period}

BIRTH_FRACTION = .0006 {Organisms Born per Number of
Organisms Present per Unit Area per Time Period}
CARRYING_CAPACITY = 500 {organisms per unit area}
SWITTCH = IF ((BIRTHS-DELAY(BIRTHS,DT))<=0) AND
(TIME>1) THEN 1 ELSE 0
```

7.2.2 Two-Population Model

This problem reveals a more complicated switch than the one discussed above. Here, we have two populations with different birth rates. The model

CARRYING CAPACITY

POPULATION

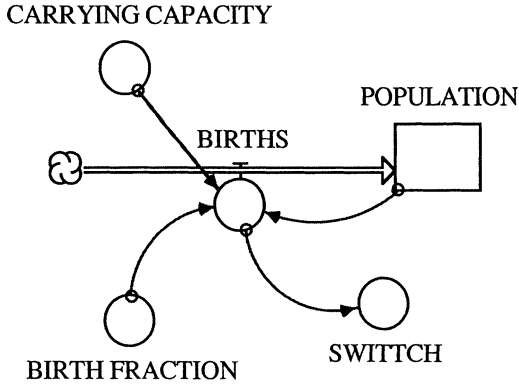BIRTHS

BIRTH FRACTION

SWITTCH

FIGURE 7.3

enables us to find numerically when the peaks for the entire population occur. There are clearly two peaks, provided we started with small initial populations.

The SUM BIRTH variable calculates the total number of births in each period, and DELTA SUM BIRTHS calculates the change in total births over the time frame t–DT:

$$\text{DELTA SUM BIRTHS} = \text{SUM BIRTHS} - \text{DELAY(SUM BIRTHS,DT,0)} \quad (6)$$

This is then used to generate a value of 1 in the SIGNAL variable every time a peak in the number of births per time period is reached or passed:

$$\text{SIGNAL} = \text{IF DELTA SUM BIRTHS} \leq 0 \text{ THEN 1 ELSE 0} \quad (7)$$

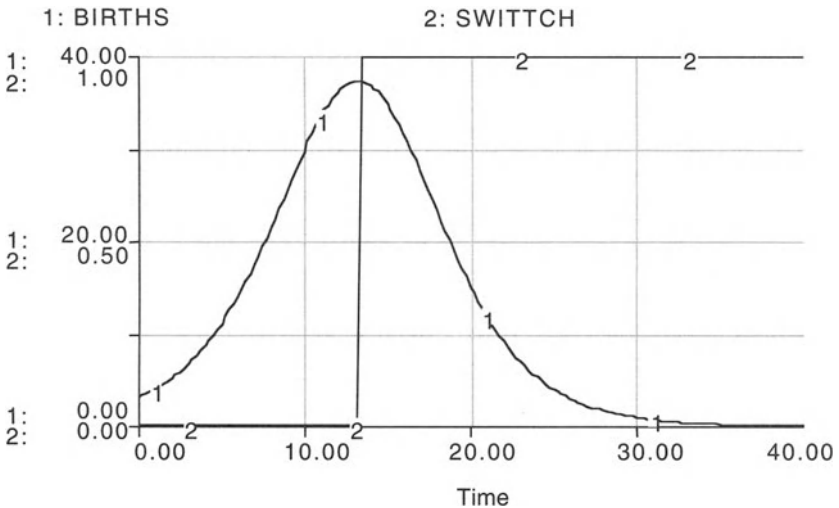1: BIRTHS   2: SWITTCH

FIGURE 7.4
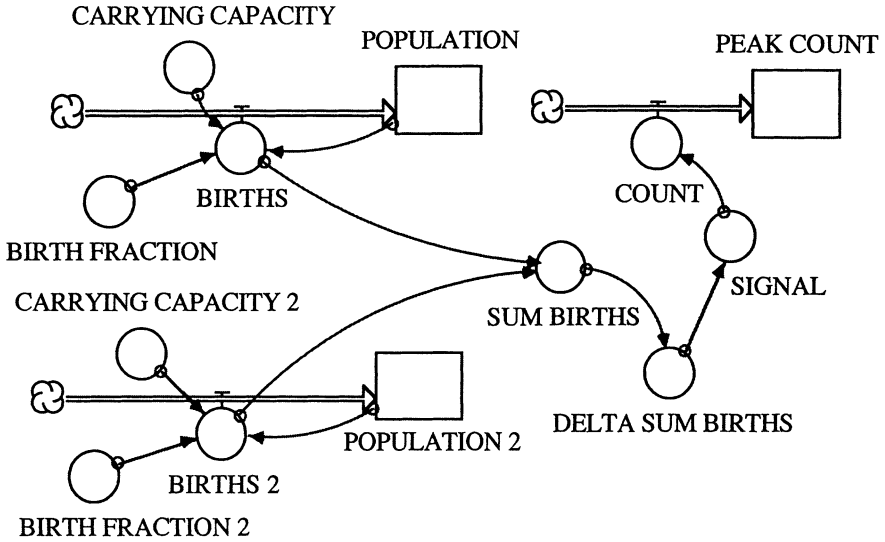
FIGURE 7.5

COUNT generates a value based on the SIGNAL that is corrected for the length of DT:

COUNT = IF SIGNAL – DELAY(SIGNAL,DT) > 0 THEN 1/DT ELSE 0   (8)

PEAK COUNT keeps track of the number of peaks that occurred (Fig. 7.5). Thus, this model can easily be expanded to accommodate multiple peaks and model runs with different time step lengths. Figure 7.6 shows for the results of the two-population case.
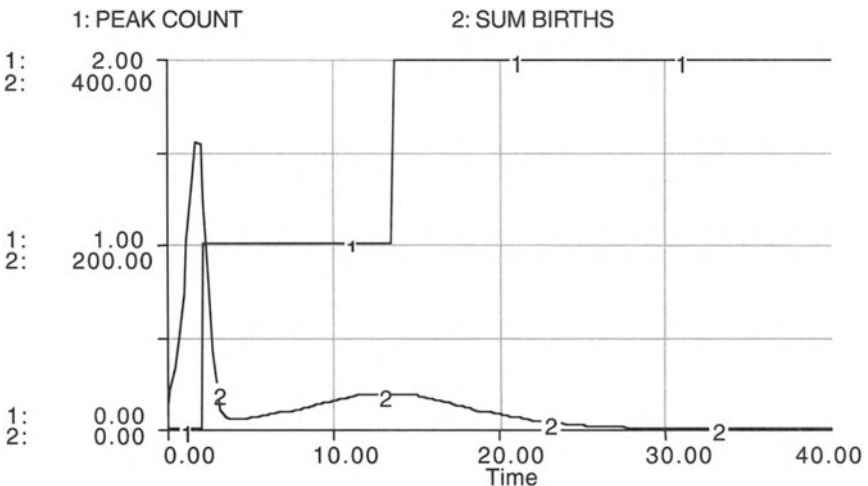

FIGURE 7.6

TWO POPULATION MODEL

```
PEAK_COUNT(t) = PEAK_COUNT(t - dt) + (COUNT) * dt
INIT PEAK_COUNT = 0 {Total Number of Peaks up to the
Current Time Period}
INFLOWS:
COUNT = IF SIGNAL - DELAY(SIGNAL,DT) > 0 THEN 1/DT
ELSE 0  {Number of Peaks per Time Period up to the
Current Time Period}


POPULATION(t) = POPULATION(t - dt) + (BIRTHS) * dt
INIT POPULATION = 10 {Organisms per Unit Area}


INFLOWS:
BIRTHS = BIRTH_FRACTION*POPULATION*(CARRYING_CAPACITY-
POPULATION)  {Organisms per Unit Area per Time Period}
POPULATION_2(t) = POPULATION_2(t - dt) + (BIRTHS_2) *
dt
INIT POPULATION_2 = 10 {Organisms per Unit Area}
INFLOWS:
BIRTHS_2 =
BIRTH_FRACTION_2*POPULATION_2*(CARRYING_CAPACITY_2-
POPULATION_2)  {Organisms per Unit Area per Time
Period}


BIRTH_FRACTION = .0006  {Organisms Born per Unit Area
per Time Period per Number of Organisms Present}
BIRTH_FRACTION_2 = .005  {Organisms Born per Unit Area
per Time Period per Number of Organisms Present}
CARRYING_CAPACITY = 500 {Organisms per Unit Area}
CARRYING_CAPACITY_2 = 500  {Organisms per Unit Area}
DELTA_SUM_BIRTHS = SUM_BIRTHS-DELAY(SUM_BIRTHS,DT,0)
{Organisms per Unit Area per Time Period}
SIGNAL = IF DELTA_SUM_BIRTHS<=0 THEN 1 ELSE 0 {Number
of Peaks}
SUM_BIRTHS = BIRTHS+BIRTHS_2  {Organisms per Unit Area
per Time Period}
```

Part 3

Chemistry Models

8

The Law of Mass Action

> No, no, no. Not love: we know better than that. Let's call it chem-
> istry . . .
>
> —George Bernard Shaw, *You Never Can Tell*

8.1 Breakdown of Nitrogen Dioxide into Nitrogen Oxide and Oxygen

The atmosphere consists of several layers that are distinguished by chemi-
cal composition, temperature, and pressure. About 99% of the volume of
clean dry air in the innermost layer, the troposphere, consists of two gases:
nitrogen (78%) and oxygen (21%). The remaining air in the troposphere has
slightly less than 1% argon and about 0.035% carbon dioxide. Air in the
troposphere also holds water vapor in amounts varying from 0.01% by vol-
ume at the frigid poles to 5% in the humid tropics.

The second layer of the atmosphere, extending from about 10 to 50 kilo-
meters above the earth's surface, is called the stratosphere. It contains small
amounts of gaseous ozone (O_3) that filters out about 99% of incoming
harmful, ultraviolet (UV) radiation. This filtering action by the thin gauze of
ozone in the stratosphere protects us from increased sunburn, skin cancer,
eye cancer, and eye cataracts. This global sun screen also protects plants
and aquatic organisms.

By filtering out high-energy UV radiation, stratospheric ozone also keeps
much of the oxygen in the troposphere from being converted into toxic
ozone. The trace amounts of ozone that do form in the troposphere as a
component of urban smog damage plants, the respiratory system of people
and other animals, and materials such as rubber. Thus, our good health and
that of many other species depends on having enough "good" ozone in the
stratosphere and as little as possible "bad" ozone in the troposphere. Un-
fortunately, human activities increase concentrations of ozone in the tropo-
sphere and decrease ozone concentrations in the stratosphere. Let us model
these two processes separately. First, we turn our attention to the workings
of the chemical processes that lead to an increase of "bad" ozone in the
troposphere. In the following section we will then model the depletion of
ozone in the stratosphere. Both models illustrate a fundamental principle of
chemical processes, the law of mass action.

Ozone (O_3) is a photochemical oxidant that is the most important com-
ponent of photochemical smog. Naturally occurring ozone may sometimes
significantly contribute to urban smog. Frequently, however, tropospheric
ozone is created by the breakdown of the anthropogenic pollutant nitrogen
dioxide (NO_2). Sunlight contributes to the formation of photochemical

smog by triggering the breakdown of NO_2. This process happens, simplified, as follows: Sunlight causes nitrogen dioxide to break down into nitrogen oxide and monatomic oxygen, O. This O-atom combines then with O_2 to form O_3:

$$NO_2 \rightarrow NO + O; \; O + O_2 \rightarrow O_3 \tag{1}$$
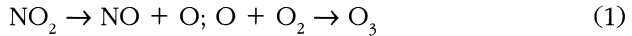
A STELLA model of this chemical reaction requires an initial stock of NO_2 that is depleted in a reaction at a rate that depends on the concentration of NO_2 and a given rate constant. The buildup of the O_2 and NO stocks occurs of course at this same rate, depending specifically on the amounts of these compounds that are available in the reactant NO_2. A second chemical process is linked to the first one. It involves the creation of ozone from the monatomic oxygen released in the NO_2 breakdown and the oxygen in the troposphere.

Let us develop a model for the first part of this reaction, the breakdown of nitrogen oxide. This problem reveals the basic idea in chemical kinetics called the law of mass action. It is an empirical rule that states that reaction speeds are proportional to the concentration(s) of the initial ingredients. For the case of two reactants at concentrations R1 and R2 that form one product at concentration P, the law of mass action is

$$P = k * R1 * R2 \tag{2}$$

with k denoting the reaction rate constant. We will use this law in the following problems in a simplified form and revisit it again in the model of an epidemic that we develop in Chapter 19.

The model of the breakdown of NO_2 is shown in Figure 8.1 for hypothetical data. It contains the law of mass action in the following forms:

$$\text{INCREASE IN O CONCENTRATION}$$
$$= \text{O PER NO2} * \text{REACTION VELOCITY} \tag{3}$$

$$\text{INCREASE IN NO CONCENTRATION}$$
$$= \text{REACTION VELOCITY} * \text{NO PER NO2} \tag{4}$$

$$\text{DECREASE IN NO2 CONCENTRATION} = \text{REACTION VELOCITY} \tag{5}$$

We begin the model with an initial concentration of 2 moles of NO_2 per cubic meter of air. The stocks of nitrogen and oxygen are normalized to zero. The reaction rate constant is arbitrarily set to 0.1.

Before you model the changes in the concentrations of NO_2, NO, and O, make a guess about the shape of the resulting curves. Run the model for alternative values of the REACTION RATE CONSTANT. Then, run the model with a modified mass action law. In order to do this, introduce an exponent for NO2 CONCENTRATION in the REACTION VELOCITY equation. Such an exponent may reflect the temperature dependence of a chemical reaction. For example, the higher the temperature, the faster the reaction takes place, the less of the original substance is left at the next time step, and the lower the increase in temperature will be for subsequent reactions.
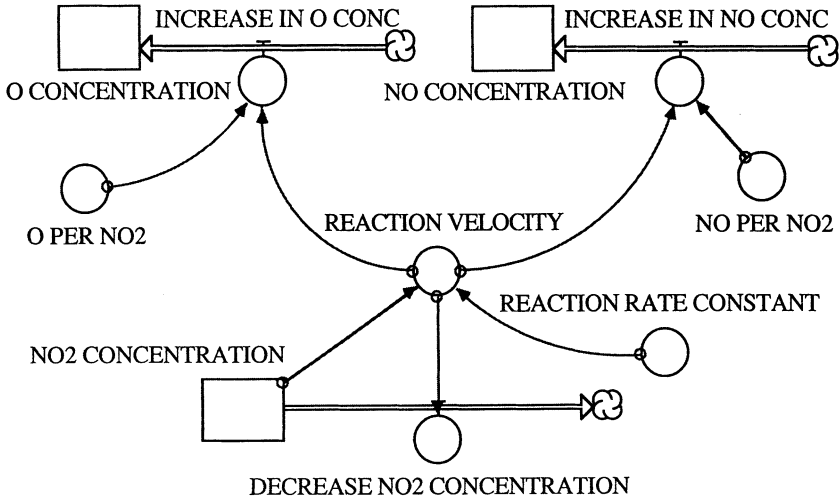
FIGURE 8.1

The model is run here with an assumed reaction rate constant that does not necessarily reflect the actual breakdown of NO_2 in the troposphere (Fig. 8.2). Try to find data in a chemistry or environmental chemistry textbook that enables you to better quantify the speed at which NO_2 breaks down under the influence of solar radiation. Then try to introduce seasonal fluctuation, reflecting the fact that NO_2 decay has its most severe implications in the summer months.

Now introduce the second part of the chemical reaction that takes place in the creation of photochemical smog. Introduce a new stock of O_2 that is large enough to provide a sufficient number of oxygen molecules to react
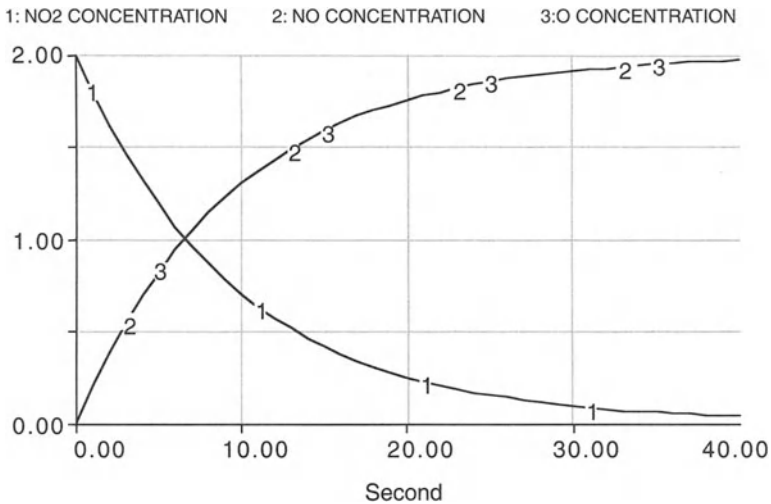
FIGURE 8.2

with all monatomic oxygen that is created in the breakdown of NO_2. Again, introduce seasonal fluctuation into the model.

A related reaction is the breakdown of N_2O_5 into NO_2 and O_2. You can fairly quickly create a model of this and similar chemical reactions, following the model above.

```
                       NO2 DECAY MODEL

NO2_CONCENTRATION(t) = NO2_CONCENTRATION(t - dt) + (-
DECREASE_NO2_CONCENTRATION) * dt
INIT NO2_CONCENTRATION = 2 {Moles per Cubic Meter}
OUTFLOWS:
DECREASE_NO2_CONCENTRATION = REACTION_VELOCITY
{Decrease in NO2 concentration; simple first-order
reaction; measured in Moles per Cubic Meter per Second}

NO_CONCENTRATION(t) = NO_CONCENTRATION(t - dt) +
(INCREASE_IN_NO_CONC) * dt
INIT NO_CONCENTRATION = 0 {Moles per Cubic Meter}
INFLOWS:
INCREASE_IN_NO_CONC = REACTION_VELOCITY*NO_PER_NO2
{Increase in NO concentration as a result of NO2 decay;
measured in Moles per Cubic Meter per Second}

O_CONCENTRATION(t) = O_CONCENTRATION(t - dt) +
(INCREASE_IN_O_CONC) * dt
INIT O_CONCENTRATION = 0 {Moles per Cubic Meter}
INFLOWS:
INCREASE_IN_O_CONC = O_PER_NO2*REACTION_VELOCITY
{Increase in O concentration as a result of NO2 decay;
measured in Moles per Cubic Meter per Second}

NO_PER_NO2 = 1 {Moles NO produced per Mole NO2
Decomposed - from stoichiometry of reaction}
O_PER_NO2 = 1 {Moles O produced per Mole NO2
Decomposed - from stoichiometry of reaction}
REACTION_RATE_CONSTANT = .1 {1/Second}
REACTION_VELOCITY =
REACTION_RATE_CONSTANT*NO2_CONCENTRATION
```

8.2 Stratospheric Ozone Depletion

As we briefly discussed above, ozone in the stratosphere protects life on earth from harmful solar radiation. Were ozone levels to fall only 10%, the

increase in ultraviolet rays would be 20%. The National Academy of Sciences has estimated that every 3% reduction in O_3 would result in 20,000 more skin cancer sufferers annually in the United States alone.

Depletion of stratospheric ozone is primarily caused by accumulation of CFCs (chlorofluorocarbons) and to some minor degree by nitrogen oxides. Major sources of CFC release are leaking air conditioners and refrigerators, evaporation of industrial solvents, and production of plastic foams. One of the properties that makes CFCs so useful in a variety of industrial applications is that they are chemically fairly inert and therefore very stable. Unfortunately, that property also influences their adverse effect on the atmosphere as they remain intact. CFCs can remain in the atmosphere for more than 100 years depending on the type. This stability allows them to reach the stratosphere, where they are broken down by intense solar radiation, and their components enter into other chemical reactions. One of these reactions results in the depletion of ozone in the stratosphere. The products of this reaction lead to a chain of reactions which, in turn, decrease ozone, following the steps in Table 8.1.

This chain reaction is catalytic, since the chlorine reactant, which initiates the reaction in Step C, is not permanently removed but appears as a product from Step D, ready to repeat the process over and over again. These chemicals stay in the atmosphere until the final rainout occurs, usually after a year or two, during which each chlorine atom destroys approximately 100,000 molecules of ozone. Ozone depletion by humans attains significance on a global scale as the ultimate consequence of the yearly emission to the atmosphere of almost one million tons of CFCs, which translates into an ozone loss 100,000 times larger.

The model shown in Figure 8.3 captures the dynamics of this autocatalytic process of ozone depletion in the presence of CFCs. The values for the reaction constants are assumed, and you may want to change them to investigate the sensitivity of the model results. Additionally, you may want to modify the mass action equation as in the example before, introducing exponents for some of the reactants. Consult an environmental chemistry textbook to support your assumptions about the mass action equation and to find realistic reaction rate constants under alternative assumptions on atmospheric temperature.

The model developed here does not explicitly trace the intermediate product of free oxygen atoms that you may want to integrate into the model together with step A of the reaction process. In contrast to the other

TABLE 8.1.

Step	Reactants		Products
A	$O + O_2$	$\rightarrow$	O_3
B	$CFCl_3$	$\rightarrow$	$CFCl_2 + Cl$
C	$Cl + O_3$	$\rightarrow$	$ClO + O_2$
D	$ClO + O$	$\rightarrow$	$Cl + O_2$

FIGURE 8.3

steps of the chain of chemical reactions, Step A is typically significantly slower. Find a reaction rate constant for the formation of ozone that would be required to offset the depletion that takes place in your model system. We also do not keep track here of the $CFCl_2$ since it does not enter any other reaction that is of interest for the model.

As you can see from Figure 8.4, the concentrations of O_3 and $CFCl_3$ decline while the concentration of Cl and ClO stabilize over the modeled time horizon. Note that these results are derived for the case in which no new $CFCl_3$ enters the atmosphere. How would the results be affected if the concentration of $CFCl_3$ were held constant or increased due to continued emissions? Make an educated guess, and then model these new assumptions and observe their consequences.

1: O3 CONC   2: CFCL3 CONC   3: CLO CONC   4: CL CONC

FIGURE 8.4

In this model we did not yet consider the fact that the chlorine atom stays in the atmosphere for 1 to 2 years and is removed steadily by rain. As a result of this deficiency in our model, the reactions continue until all ozone is depleted. To make the model more realistic, add the removal process.

STRATOSPHERIC OZONE DEPLETION

```
CFCL3_CONC(t) = CFCL3_CONC(t - dt) + (-
DECREASE_CFCL3_CONC) * dt
INIT CFCL3_CONC = 2 {Moles per Cubic Meter}
OUTFLOWS:
DECREASE_CFCL3_CONC = REACT_VELOC_1 {Decrease in NO2
concentration; simple first-order reaction; measured in
Moles per Cubic Meter per Time Period}

CLO_CONC(t) = CLO_CONC(t - dt) + (INCREASE_CLO_CONC -
DECREASE_CLO_CONC) * dt
INIT CLO_CONC = 0
INFLOWS:
INCREASE_CLO_CONC = REACT_VELOC_2
OUTFLOWS:
DECREASE_CLO_CONC = REACT_VELOC_3

CL_CONC(t) = CL_CONC(t - dt) +
(INCREASE_IN_CL_CONC_STEP_C + INCREASE_CL_CONC_STEP_D
- DECREASE_CL_CONC) * dt
```

```
INIT CL_CONC = 0 {Moles per Cubic Meter}
INFLOWS:
INCREASE_IN_CL_CONC_STEP_C =
REACT_VELOC_1*CL_PER_CFCL3 {Increase in NO
concentration as a result of NO2 decay; measured in
Moles per Cubic Meter per Time Period}
INCREASE_CL_CONC_STEP_B = CL_PER_CLO*DECEASE_CLO_CONC
OUTFLOWS:
DECREASE_CL_CONC = REACT_VELOC_2

O3_CONC(t) = O3_CONC(t - dt) + (- DECREASE_O3_CONC) *
dt
INIT O3_CONC = 100
OUTFLOWS:
DECREASE_O3_CONC = O_PER_O3_FOR_CLO*REACT_VELOC_2

CL_PER_CFCL3 = 1 {Moles NO produced per Mole NO2
Decomposed - from stoichiometry of reaction}
CL_PER_CLO = 1
O_PER_O3_FOR_CLO = 1
REACT_RATE_CONST_1 = .01 {1/Time Period}
REACT_RATE_CONST_2 = .1
REACT_RATE_CONST_3 = .05
REACT_VELOC_1 = REACT_RATE_CONST_1*CFCL3_CONC
REACT_VELOC_2 = REACT_RATE_CONST_2*CL_CONC
REACT_VELOC_3 =
```

9
Chance-Cleland Model
for Enzyme-Substrate Interaction

> Shelley called poets the unacknowledged legislators of the world, and the epithet was well chosen. The catalyst is unchanged, unabsorbed; its activity therefore not acknowledged.
>
> —Henry Read, *Politics of the Unpolitical*

Let us next develop a model of a chemical reaction in which an enzyme, E, behaves like a catalyst. There is an initial stock of this enzyme available, some of which will be temporarily tied up due to the production of an intermediate substance, C. Ultimately, the enzyme is released again in unchanged form as some product, P, is formed from the substance C. An additional substance active in this process is the substrate S, which is converted into the product P. Think of the substrate S as dextrose and the product P as fructose: the enzyme E mechanically locks onto the substrate molecule, breaks it into a new molecule, fructose, and is released again after the chemical reaction occurred.[1]

For simplicity, let us denote the reaction rate constants as K1, K2, and K3. With this notation, the basic reaction equation is:

$$E + S - K1 \rightarrow \ \leftarrow K2 - C - K3 \rightarrow E + P \tag{1}$$
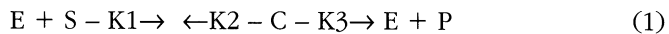
The four basic differential equations that define the rate expressions, or flows, are:

$$dS/dt = K1*S*E - K2*C \tag{2}$$

$$dC/dt = K1*S*E - (K2 + K3)*C \tag{3}$$

$$dE/dt = dC/dt \tag{4}$$

$$dP/dt = K3*C \tag{5}$$

In this model, we treat all substances that occur in the chemical reactions as concentrations. We denote the change in these concentrations as S RATE, C RATE, E RATE, and P RATE, and define them as the differential equations above. The basic assumption for the model is that the product of the concentrations is linearly proportional to the reaction rate, the "law" of mass action.

Rather than trying to simply program the differential equations, sketch out your model as a series of four stocks that are filled and emptied

---

[1]See Spain, J.D. 1982. *BASIC Microcomputer Models in Biology*, Addison-Wesley, Reading, MA.

according to the chemical equations. You should find this a less compli-
cated procedure. Then compare your model to that in Figure 9.1. Make an
educated guess of the dynamics that will result, run your model and see
whether its behavior coincides with the results in Figure 9.2.

Can you make this model with fewer stocks, say just ones for C and P? Try
to remove P at a constant rate. Assume that the enzyme is lost along with P
at a rate that is in proportion to the E concentration (e.g., 0.01*E). Figure out
how fast E and S must be added to the reaction to maintain the removal of
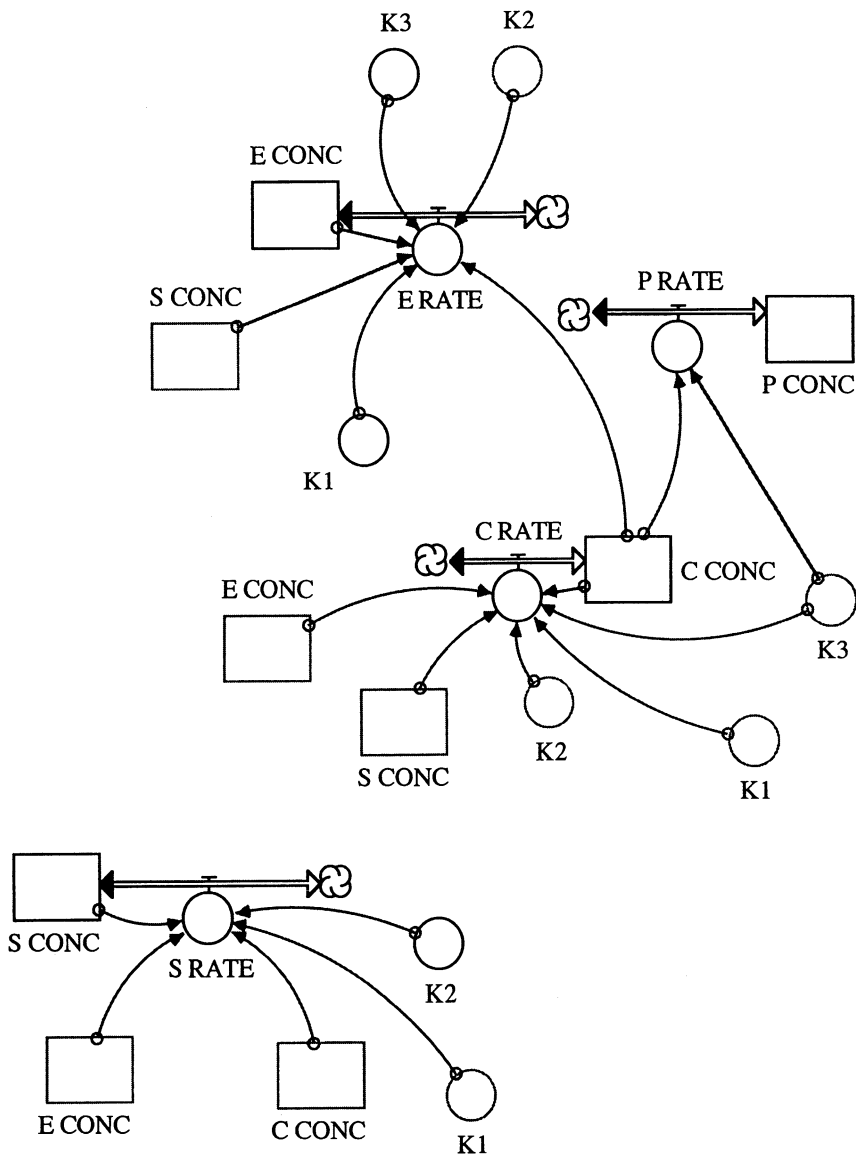
FIGURE 9.1
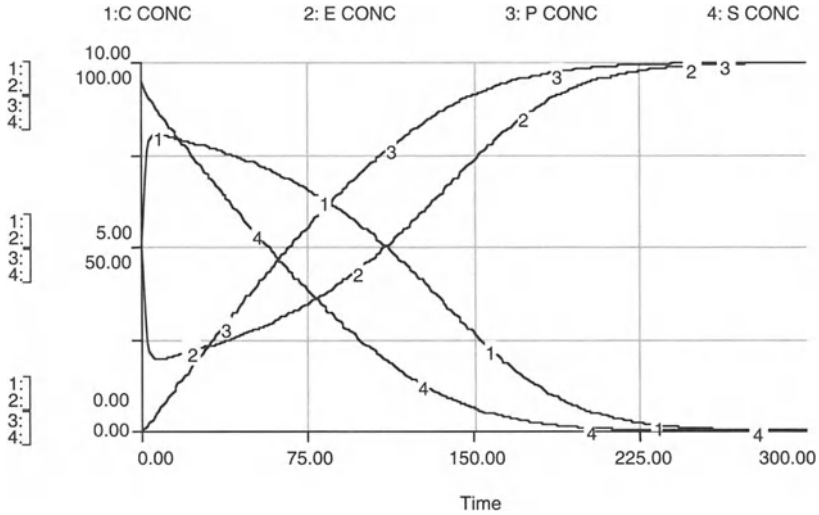
FIGURE 9.2

P and the loss of E. The idea here is to find a steady state solution. You can find the steady state substrate addition rate by adding the inputs and losses symbolically to the above differential equations and then setting the derivatives equal to 0, and solving for the steady state add rate for S.

ENZYME-SUBSTRATE REACTION

```
C_CONC(t) = C_CONC(t - dt) + (C_RATE) * dt
INIT C_CONC = 0 {Moles per Cubic Meter}
INFLOWS:
C_RATE = K1*S_CONC*E_CONC-(K2+K3)*C_CONC {Moles per
Cubic Meter per Time Period}

E_CONC(t) = E_CONC(t - dt) + (- E_RATE) * dt
INIT E_CONC = 10 {Moles per Cubic Meter}
OUTFLOWS:
E_RATE = K1*S_CONC*E_CONC-(K2+K3)*C_CONC {Moles per
Cubic Meter per Time Period}

P_CONC(t) = P_CONC(t - dt) + (P_RATE) * dt
INIT P_CONC = 0 {Moles per Cubic Meter}
INFLOWS:
P_RATE = K3*C_CONC {This is the Michaelis-Menten
enzyme model; P_Rate measured in Moles per Cubic Meter
per Time Period}
```

```
S_CONC(t) = S_CONC(t - dt) + (- S_RATE) * dt
INIT S_CONC = 100 {Moles per Cubic Meter}
OUTFLOWS:
S_RATE = K1*S_CONC*E_CONC-K2*C_CONC {Moles per Cubic
Meter per Time Period}

K1 = .005 {1/Time Period}
K2 = .005 {1/Time Period}
K3 = .1 {1/Time Period}
```

10
The Olsen Oscillator

The sightless belfry clock . . . had . . . rung, vibrate with triumph.
—Tait's Mag. XVI. 9

In this chapter we build on the law of mass action that we applied in Chapters 8 and 9 to model a chemical oscillator known as the Olsen oscillator. Chemical oscillators are unique in that they do not necessarily reach a single equilibrium state for a given set of initial conditions; rather, they seem to oscillate between two or more bounds of stability.

Chemical oscillators are generally quite complicated and, as of yet, often not fully understood. However, some general requirements to produce oscillatory behavior are known. First, the system must be nonlinear. This requires that certain reaction intermediates must appear as second-order (or higher) terms in the elementary rate expressions. A second requirement is that the system is open; that is, at least one component (oxygen, for instance) must be able to enter and/or leave the system. Thirdly, the initial concentrations must put the system far from equilibrium.

The Olsen oscillator does indeed exhibit all of these characteristics. It is a "simple" system, consisting of only four substances, which we name, for simplicity, A, B, X, and Y. The actual mechanism, however, is not nearly so simple, consisting of at least 26 different intermediates and about 28 steps. However, we abstract from many of these processes, yet capture the main features of the oscillator.

The rate expressions for the products of the reactions can be determined according to the law of mass action, just like we did for the simple kinetic models of the previous chapter. The rate constants in this problem are denoted by K1, K2, . . . , K9. The rate expressions, explaining the change in the substances A, B, X, and Y, are given as follows:

$$\text{A:} \quad dA/dt = K7*(1-A) - K3*A*B*Y \tag{1}$$

$$\text{B:} \quad dB/dt = K8 - K1*B*X - K3*A*B*Y - K9*B \tag{2}$$

$$\text{X:} \quad dX/dt = K1*B*X + 3*K3*A*B*Y + K6 - 2*K2*X^2 - K4*X \tag{3}$$

$$\text{Y:} \quad dY/dt = 2*K2*X^2 - K3*A*B*Y - K5*Y \tag{4}$$

The nonlinearity required for the oscillation is present in equations 3 and 4 by raising the concentration of X to the power 2.

The model of the Olsen oscillator is shown in Figure 10.1. The flows are specified using the differential equations 1 to 4 above. Those terms that have a positive sign are used in the FORM . . . inflows into the respective

FIGURE 10.1

stock, the ones with negative signs define the DESTROY outflows from the stocks. For example,

$$\text{FORM A} = \text{K7}^*(1 - A) \tag{5}$$

$$\text{DESTROY A} = \text{K3}^*A^*B^*Y \tag{6}$$

Let us get back to the workings of the model itself. This is a nonlinear system that yields oscillation of some periodicity. As theory and experimentation show us, we can observe the phenomenon of period doubling when we change one of the oscillating variables within certain bounds. If these oscillating variables are changed further, we will eventually observe chaotic behavior of the system. This was, in fact, observed. The original investigators of the Olsen oscillator chose to study changes in A that would oscillate between 0 and 1. K8 and K9, two variables influential in the creation of the oscillation, can be physically changed in an actual system. K8 can be changed by changing the influx of substance B into the system, and K9 can be changed by changing the concentration of some of the substances added to the system.

After setting up the model as in Figure 10.1, experiment with the time step and the integration method to get the model running. For our model run in Figure 10.2, we chose a time step of DT = 0.01 and Euler's method as the integration technique. We also set the reaction rate constants as follows:

K1 = 3.28	K6 = $1.25 \cdot 10^{-6}$
K2 = 2000	K7 = .1
K3 = 2.24	K8 = .1030125
K4 = 20	K9 = .001
K5 = 5.35	
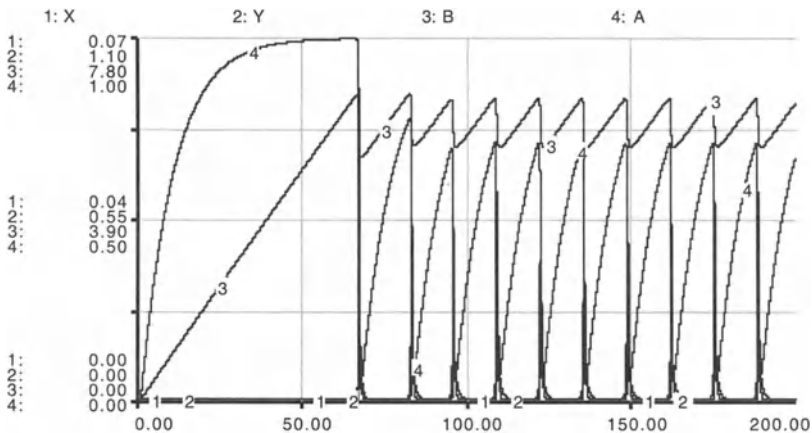
FIGURE 10.2

and our initial conditions A = $1 \cdot 10^{-6}$, B = 0, X = $1 \cdot 10^{-12}$, and Y = $1 \cdot 10^{-12}$. Change the DT and the integration method and note the significant changes in the results.

With this model, we can show that the form of the oscillations is dependent on the choice of the reaction constants (Figs. 10.2 and 10.3). In the Olsen oscillator, some of these constants depend on the concentrations of the substances being used. Let us first evaluate impacts of the choice of K9 on the oscillation.[1] For this purpose, fix the values for the other parameters and vary K9. Keep all parameter values as above and then set for subsequent runs of the model K9 = .0001; K9 = .01; and K9 = .05. It becomes readily apparent that the period of the oscillations is dependent on the value of K9, such that a larger K9 gives a longer period of oscillation. For K9 = .05, the system no longer oscillates.

Next, determine whether the form of oscillations is dependent on K1. Choose K1 = 2.8 and keep the other reaction constants the same. Choose as alternative values for K9 once K9 = .003, then K9 = .001, and finally K9 = 0. The trend here is now toward chaos. The period of oscillation doubles as we change K9 from K9 = .003 to K9 = .001. Chaos is observed at K9 = 0. We will discuss chaos in more detail in Chapter 37, but you should be al-
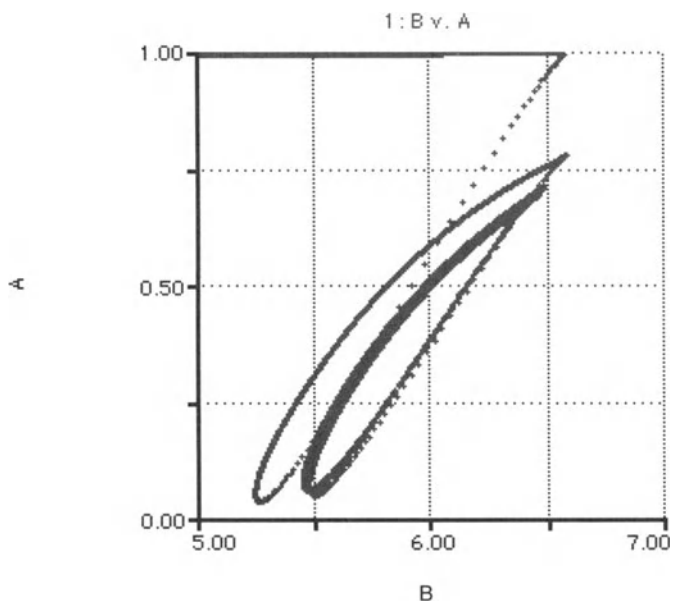
FIGURE 10.3

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[1]The parameter values chosen for the simulation runs in the following figures are listed in the equation pad at the end of this chapter.

ready aware of the possibility of "strange" model behavior as some parameter value is slightly altered.

   Finally, determine if the system is sensitive to K8. Choose the values K1 = 2.8 and K9 = .0001, then vary K8 for subsequent runs: K8 = .08; K8 = .15; and K8 = .20. Plot the corresponding graphs and note how they differ from the others. These marked differences indicate that the system is very much dependent on values assumed by the parameter K8. In fact, for K8 = .20, there is a damping effect that eventually brings the system to equilibrium. In this case there are no more oscillations!

```
                      OLSEN OSCILLATOR

A(t)  =  A(t - dt)  +  (FORM_A - DESTROY_A)  *  dt
INIT A  =  1E-6
INFLOWS:
FORM_A  =  K7*(1-A)
OUTFLOWS:
DESTROY_A  =  K3*A*B*Y


B(t)  =  B(t - dt)  +  (FORM_B - DESTROY_B)  *  dt
INIT B  =  0
INFLOWS:
FORM_B  =  K8
OUTFLOWS:
DESTROY_B  =  (K1*B*X)+(K3*A*B*Y)+(K9*B)


X(t)  =  X(t - dt)  +  (FORM_X - DESTROY_X)  *  dt
INIT X  =  1E-12
INFLOWS:
FORM_X  =  K1*B*X+3*K3*A*B*Y+K6
OUTFLOWS:
DESTROY_X  =  2*K2*X^2+K4*X


Y(t)  =  Y(t - dt)  +  (FORM_Y - DESTROY_Y)  *  dt
INIT Y  =  1E-12
INFLOWS:
FORM_Y  =  2*K2*X^2
OUTFLOWS:
DESTROY_Y  =  K3*A*B*Y+K5*Y


K1  =  3.28
K2  =  2000
K3  =  2.24
K4  =  20
```

```
K5 = 5.35
K6 = 1.25E-6
K7 = .1
K8 = .1030125
K9 = .001
```

Part 4

Genetics Models

11

Mating of Alleles

> Since the struggle for existence is chiefly the struggle for subsistence, a careful comparative account of the food of various competing species and genera at different places and seasons and at all ages of the individual . . . cannot fail to throw much light upon the details, causes and effects of the struggle.
>
> —S. Forbes, *The Lake as Microcosm*

11.1 Heterozygosity and Disease Resistance

Genetic theory is used in two basic commercial ways today: animal and plant breeding in agriculture and in the field of conservation biology. In the former, the basic strategy is to identify and eliminate undesirable, susceptible alleles by selective breeding. The susceptibility can be to a particular disease or perhaps it is manifested as a structural problem, such as weak anchor roots in corn plants.

Selection for particular alleles for agricultural purposes may decrease susceptibility to a particular disease or overcome some structural deficiency. However, such a selection process may also mean loss of some allele that was necessary to cope with some other problem. Thus, selective breeding may enhance susceptibility with regard to one problem but remove the ability to cope with another one. Conservation biologists see this path as one of walking the knife edge. Rather than increase the specialization of a plant or animal population, conservation biologists take just the opposite track. They want to maximize allele diversity in a locale of plants or herd of animals.

Let us distinguish two alleles, A and B. Suppose that the whole population was half AA and half BB. Then suppose that some disease hit this population and killed off the AA types. Half of the genetic history of the population would be gone. The same disease would not have so strongly affected the population had it also contained individuals of AB type, so-called heterozygotes. Not only does heterozygosity increase a population's ability to cope with a variety of environmental influences, such as diseases, but a population is also more genetically adaptable in the face of an environmental challenge than are the AA or BB populations. More genetic information is retained in the heterozygote population, as a result of its history of natural selection, and therefore it is less prone to adverse conditions imposed by an altered environment.

Generally, conservation biologists want to maximize the number of AB type individuals because it increases the diversity and resilience of the population.

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11.2 The Mating of Two Alleles into a Genotype: Proving the Hardy-Weinberg Law

Let us now model the process of genotype mixing, given two alleles, the fundamental experiment in genetics. In the next chapter we will then model the process of natural selection and mutation.

The process of genotype mixing can be illustrated for two alleles, A and B, which are drawn randomly from a pool of 100 A alleles and 200 B alleles. Each time step in the model is a drawing of an allele: we have 300 alleles so the total time for the mating process is 300 time steps (actually, 302 steps are needed to clear the very last allele choice into a genotype). The 300 steps represent a generation that resulted from a mating process in which the alleles were joining simultaneously. In STELLA, we could have chosen a time step of 1/300 so that t = 1, 2, 3, . . . would represent 1, 2, 3, . . . generations. The real time required for the run would not be shorter, of course. In this simple mating process, the genotype mix is set at the end of the first generation and remains fixed for all further ones.

The results of mating two alleles are explained by the Hardy-Weinberg law. This law states that the genotype frequencies are determined in a random mating process in the first generation. These genotype frequencies are for AA, p^2; for AB, 2*p*q; for BB, q^2, where p and q are the A and B allele frequencies, respectively. In our sample problem, p = 100/300 or approximately 0.3333 and q = 200/300 or approximately 0.666667. From 300 alleles we can have 150 genotypes, so the Hardy-Weinberg law tells us that we should end up with 0.33333*0.33333*150 or 16.7 AA genotypes, 2*0.3333*0.666667*150 or 66.7 AB genotypes and finally, 0.66667*0.66667*150 or 66.7 BB genotypes.

There is another way to look at the problem of genotype mixing. If the A and B alleles were of equal frequency of occurrence, the chance of an AA genotype is .25. Since in our problem, the chance of an A choice is 0.33, the AA genotype frequency is .25*.33333, or 1/12. The AB frequency is 2*.25*.66667, or 1/3. The relative frequency of AB to AA is therefore 4. We can check for the validity of this relationship in a graph generated by our model on the mating of two alleles.

To construct the model, we must set up a stock of 100 A alleles and another stock of 200 B alleles (Fig. 11.1). We use a random number generator to pick a number between 0 and 1 so that we can compare it to the frequency of the A allele in the total stock of alleles remaining to be chosen. Let's call that random number RAND.

The A frequency is just A/(A + B) and we make an initial and a running calculation of it in the variable A FREQ:

$$A \ FREQ = A/(A + B) \tag{1}$$

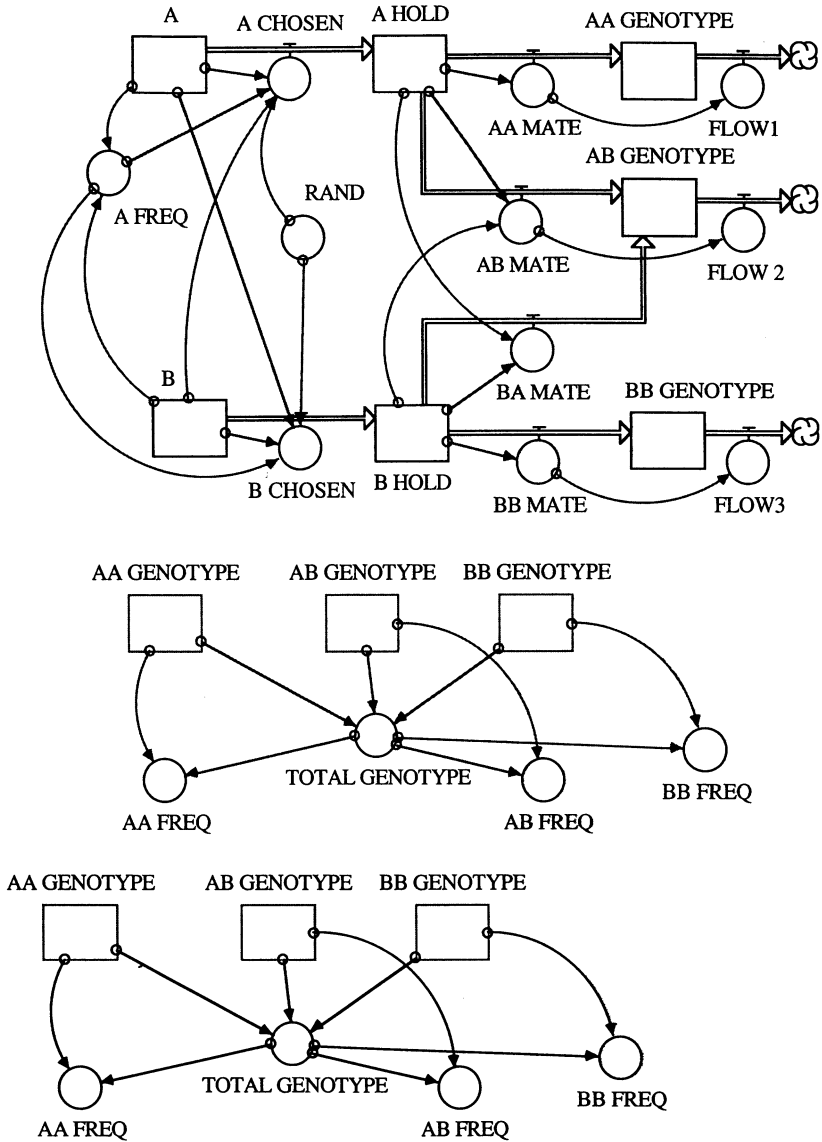If an A is randomly chosen we "store it" in a separate stock called A HOLD by adding a 1 to that stock:

FIGURE 11.1

$$\text{A CHOSEN} = \text{IF (RAND} \leq \text{A FREQ)}$$
$$\text{AND ((A} > 0) \text{ AND (B} \geq 0)) \text{ THEN 1 ELSE 0} \qquad (2)$$

If an A is chosen next, we add another 1 to the A HOLD. Now two As have been chosen sequentially, and they can be used to form an AA genotype. In this case, we wish to remove the two 1s from the A HOLD stock, and

mate them to form an AA GENOTYPE. To capture this allele mating process, the model (see Fig. 11.1) is set up so that AA MATE realizes that A HOLD has reached 2 and dumps A HOLD into the stock called AA GENOTYPE, leaving 0 in A HOLD and 2 in AA GENOTYPE. Since the number of genotypes is equal the number of pairs of alleles, we must subtract one from A GENOTYPE every time AA MATE dumps A HOLD into AA GENOTYPE. This subtraction is accomplished by FLOW1:

$$\text{FLOW 1} = \text{IF AA MATE} > 0 \text{ THEN 1 ELSE 0} \tag{3}$$

If instead of A, our second allele choice had been a B, then a one is sent to B HOLD by B CHOSEN:

$$\text{B CHOSEN} = \text{IF (RAND} > \text{A FREQ)}$$
$$\text{AND ((B} > 0) \text{ AND (A} >= 0)) \text{ THEN 1 ELSE 0} \tag{4}$$

Once we have chosen an A and a B in subsequent draws, both AB MATE and BA MATE dump a 1 into AB GENOTYPE:

$$\text{AB MATE} = \text{IF (A HOLD} = 1) \text{ AND (B HOLD} = 1) \text{ THEN 1 ELSE 0} \tag{5}$$

$$\text{BA MATE} = \text{IF (A HOLD} = 1) \text{ AND (B HOLD} = 1) \text{ THEN 1 ELSE 0} \tag{6}$$

The variable FLOW2 senses the AB MATE action and removes a 1 from AB GENOTYPE. The same steps that occur for the A HOLD process, occur for the B HOLD process.

In this way 300 alleles are converted into 150 genotypes. How does the result compare with the Hardy-Weinberg law? Run the model several times and see what you get. Compare your results with one of our runs shown in Figures 11.2 and 11.3. Could the variability of the results among your various model runs and between yours and ours be due to the small sample
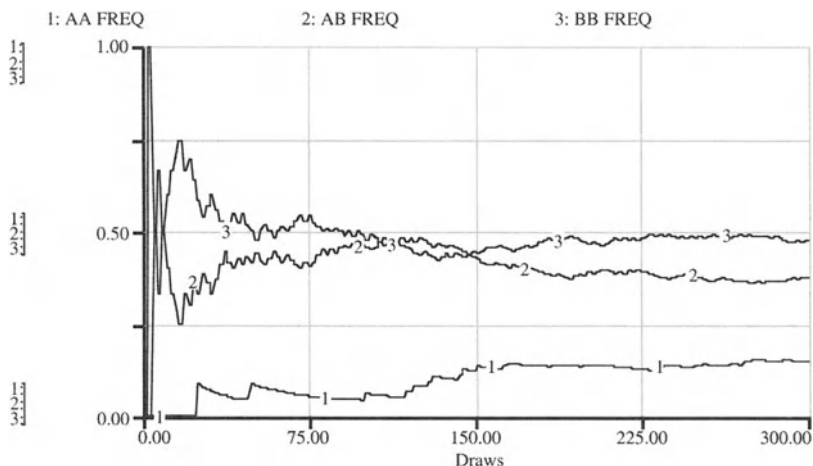
FIGURE 11.2
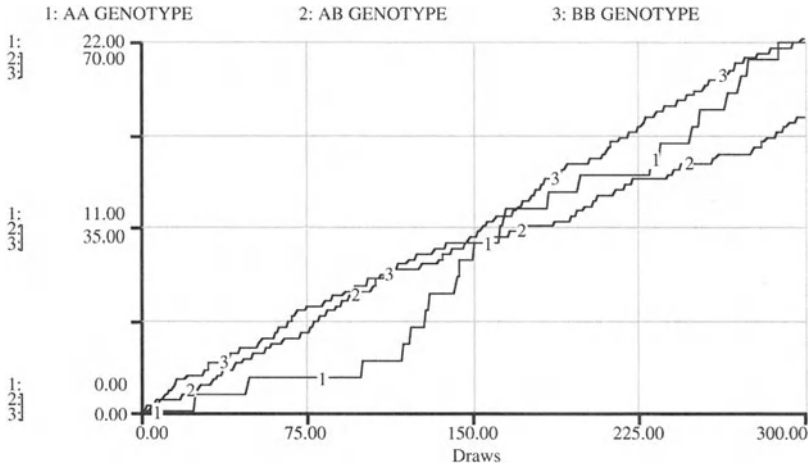
FIGURE 11.3

size? Try doubling the sample size and keep the initial frequency the same. You must also double the number of time steps (plus 2) to allow all the mating to take place. Try tripling the initial number of alleles. You should find that the larger the number of initial alleles, the closer we come to re-peating the Hardy-Weinberg prediction. Set up a model that does not have a fixed supply of the alleles but calls them from an infinite source and ob-serve the long-term convergence on the Hardy-Weinberg predicted result.

What can we learn from this experiment? First we have shown how the Hardy-Weinberg law is derived and we have learned a little more dynamic programming. We have also learned that an analytic result can be useful in cutting down on programming time and effort. As we shall see in ensuing problems where many generations are required to reach equilibrium due to differential survival rates, mutation, etc., the computer time grows to intol-erable levels and begins to obscure the point of the exercise. So this dy-namic programming business can be usefully complimented by analytic re-sults, but not before we understand the underlying principles.

PROVING THE HARDY-WEINBERG LAW

```
A(t) = A(t - dt) + (- A_CHOSEN) * dt
INIT A = 100 {Initial A number of alleles.}
OUTFLOWS:
A_CHOSEN = IF (RAND <= A_FREQ) AND ((A > 0) AND (B >=
0)) THEN 1 ELSE 0   {Process of randomly chosing A based
on its intial frequency. Other conditions prevent
negative A. }
```

```
AA_GENOTYPE(t) = AA_GENOTYPE(t - dt) + (AA_MATE -
FLOW1) * dt
INIT AA_GENOTYPE = 0 {This counter receives 2 alleles
for each genotype and therefore a 1 must be subtracted
each time the 2 alleles are added. This is done by
flow1.}
INFLOWS:
AA_MATE = IF A_HOLD = 2 THEN 2 ELSE 0 {This mating
step occurs when second A is chosen. It empties A_hold
and sends 2 to the AA genotype counter.}
OUTFLOWS:
FLOW1 = IF AA_MATE > 0 THEN 1 ELSE 0

AB_GENOTYPE(t) = AB_GENOTYPE(t - dt) + (AB_MATE +
BA_MATE - FLOW_2) * dt
INIT AB_GENOTYPE = 0 {This counter receives 2 alleles
for each genotype and therefore a 1 must be subtracted
each time the 2 alleles are added. This is done by
flow2.}
INFLOWS:
AB_MATE = IF (A_HOLD = 1) AND (B_HOLD = 1) THEN 1 ELSE
0 {This mating step occurs when a single A is stored in
A_Hold and a B is chosen. It empties A_hold and B_Hold
& sends 2 to the AB genotype counter.}
BA_MATE = IF (A_HOLD = 1) AND (B_HOLD = 1) THEN 1 ELSE
0 {This mating step occurs when a single B is stored in
B_Hold and an A is chosen. It empties A_hold and B_Hold
& sends 2 to the AB genotype counter.}
OUTFLOWS:
FLOW_2 = IF AB_MATE > 0 THEN 1 ELSE 0

A_HOLD(t) = A_HOLD(t - dt) + (A_CHOSEN - AA_MATE -
AB_MATE) * dt
INIT A_HOLD = 0 {an intermediate variable = to the
number of A chosen. One generation = A + B time steps.
If next choice is A, then AA genotype is formed; if B,
the AB is formed}
INFLOWS:
A_CHOSEN = IF (RAND <= A_FREQ) AND ((A > 0) AND (B >=
0)) THEN 1 ELSE 0   {Process of randomly chosing A based
on its intial frequency. Other conditions prevent
negative A. }
OUTFLOWS:
```

```
AA_MATE = IF A_HOLD = 2 THEN 2 ELSE 0 {This mating
step occurs when second A is chosen. It empties A_hold
and sends 2 to the AA genotype counter.}
AB_MATE = IF (A_HOLD = 1) AND (B_HOLD = 1) THEN 1 ELSE
0 {This mating step occurs when a single A is stored in
A_Hold and a B is chosen. It empties A_hold and B_Hold
& sends 2 to the AB genotype counter.}

B(t) = B(t - dt) + (- B_CHOSEN) * dt
INIT B = 200 {Initial number of B alleles.}
OUTFLOWS:
B_CHOSEN = IF (RAND > A_FREQ) AND ((B > 0) AND (A >=
0)) THEN 1 ELSE 0 {Process of randomly chosing B based
on its initial frequency. }

BB_GENOTYPE(t) = BB_GENOTYPE(t - dt) + (BB_MATE -
FLOW3) * dt
INIT BB_GENOTYPE = 0 {This counter receives 2 alleles
for each genotype and therefore a 1 must be subtracted
each time the 2 alleles are added. This is done by
flow3.}
INFLOWS:
BB_MATE = IF B_HOLD = 2 THEN 2 ELSE 0 {This mating
step occurs when second B is chosen. It empties B_hold
and sends 2 to the BB genotype counter.}
OUTFLOWS:
FLOW3 = IF BB_MATE > 0 THEN 1 ELSE 0

B_HOLD(t) = B_HOLD(t - dt) + (B_CHOSEN - BB_MATE -
BA_MATE) * dt
INIT B_HOLD = 0
INFLOWS:
B_CHOSEN = IF (RAND > A_FREQ) AND ((B > 0) AND (A >=
0)) THEN 1 ELSE 0 {Process of randomly chosing B based
on its initial frequency. }
OUTFLOWS:
BB_MATE = IF B_HOLD = 2 THEN 2 ELSE 0 {This mating
step occurs when second B is chosen. It empties B_hold
and sends 2 to the BB genotype counter.}
BA_MATE = IF (A_HOLD = 1) AND (B_HOLD = 1) THEN 1 ELSE
0 {This mating step occurs when a single B is stored in
B_Hold and an A is chosen. It empties A_hold and B_Hold
& sends 2 to the AB genotype counter.}
```

```
AA_FREQ = IF AA_GENOTYPE > 0 THEN
AA_GENOTYPE/TOTAL_GENOTYPE ELSE 0
AB_FREQ = IF AB_GENOTYPE > 0 THEN
AB_GENOTYPE/TOTAL_GENOTYPE ELSE 0
A_FREQ = A/(A + B) {Tracks the randomly changing A
allele frequency. This tells us the probability that an
A allele will be chosen by the random drawing from the
A,B allele pool.}
BB_FREQ = IF BB_GENOTYPE > 0 THEN
BB_GENOTYPE/TOTAL_GENOTYPE ELSE 0
RAND = RANDOM(0,1) {A random number genrator which
gives a number between 0 and 1.0. This number allows us
to randomly choose from the A, B allele pool.}
TOTAL_GENOTYPE = AA_GENOTYPE+AB_GENOTYPE+BB_GENOTYPE
```

12
Natural Selection and Mutation

> Each organic being is striving to increase in a geometrical ratio. . . .
> each at some period of its life, during some season of the year, during each
> generation or at intervals has to struggle for life and to suffer great destruc-
> tion . . . .The vigorous, the healthy, and the happy survive and multiply.
>   —Charles Darwin, *On the Origin of the Species*
>   *by Means of Natural Selection*

The problem of this chapter is an extension of the genetics model of the
previous chapter. Given the frequency of appearance of two alleles, A and
B, the previous model randomly selected and combined these alleles into
three possible genotypes: AA, AB, and BB, whose frequency of appearance
is predicted by the Hardy-Weinberg law. In this chapter, we incorporate the
law directly and then allow the three genotypes to break up into alleles
again, after the genotypes each are prone to specific fertility and survival
rates. Each allele also has a tendency to mutate into the other type before
recombining into the new genotypes. The survival and fertility rates com-
bine to form a fitness measure and these two effects consort to give the
new genotype frequencies.

  The model is made up of three parts, each connected through the use of
"ghosts." One part of the model calculates the addition of alleles dependent
on the survival rates of the genotypes and their fertilities (Fig. 12.1). To cal-
culate the "generation" of a new A allele, we multiply the number of alleles
in a genotype by that genotype's fertility and survival rates:

$$\text{NEW A} = 2 * \text{AA GENOTYPE} * \text{AA FERT} * \text{AA SURVIVE RATE}$$
$$+ \text{AB GENOTYPE} * \text{AB FERT} * \text{AB SURVIVE RATE} \quad (1)$$

$$\text{NEW B} = 2 * \text{BB GENOTYPE} * \text{BB FERT} * \text{BB SURVIVE RATE}$$
$$+ \text{AB GENOTYPE} * \text{AB FERT} * \text{AB SURVIVE RATE} \quad (2)$$

  This part of the model (see Fig. 12.1) also captures mutation, shown as a
random process. For simplicity of exposition, the model is set to allow for
only one direction of mutation (A to B) on any model run. To revert the di-
rection, simply remove the multiplication by zero in the first line of the
MUTATION equation and instead multiply the second line by zero.

$$\text{MUTATION} = .3 * \text{B ALLELES} * \text{RANDOM}(0,1) * 0$$
$$+ .03 * \text{A ALLELES} * \text{RANDOM}(0,1) \quad (3)$$

  The second part of the model (Fig. 12.2) captures the mating process,
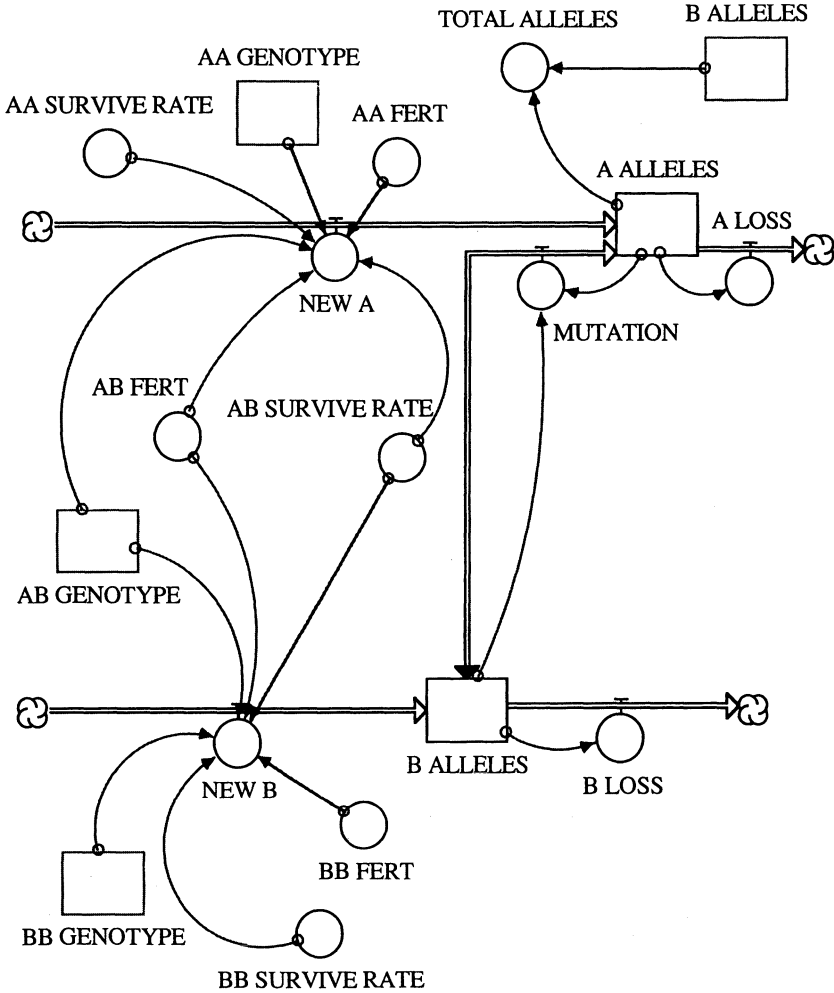based on the Hardy-Weinberg law. For example, the rate at which AA

FIGURE 12.1

genotypes are formed is specified by the relative frequency of A alleles in the population:

$$AA \ RATE = TOTAL \ ALLELES/2 * A \ FREQ^2 \qquad (4)$$

The third part of the model (Fig. 12.3) simply calculates the frequencies of each genotype. These frequencies enter the Hardy-Weinberg law used in the second part of the model.

This model needs to be run at a DT less than 1 in order to update stocks frequently enough. If stocks are not updated frequently enough, some of them may become 0 at times, leading to a division by 0 in the calculation of the genotype frequencies. What does a DT < 1 imply for the interpretation of the model's behavior over time with regard to "generations"?
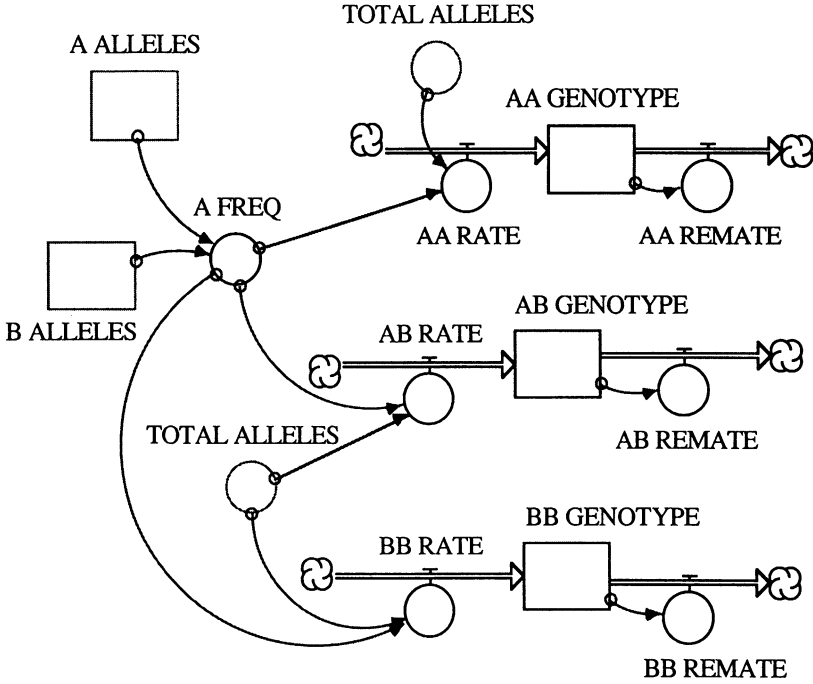
FIGURE 12.2

The graph in Figure 12.4 shows a randomly varying A frequency that stays near to the original frequency of 0.33333. Open the mutation circle and set both mutation rates to 0. Run the model again to see that A is dominant with the current six fertility and survival rate settings. Thus, the mutation from A to B effectively destroys the dominance of the A allele. Run the model with various settings to see the effect of different rates. You should

FIGURE 12.3

1: A FREQ   2: AA FREQ   3: AB FREQ   4: BB FREQ
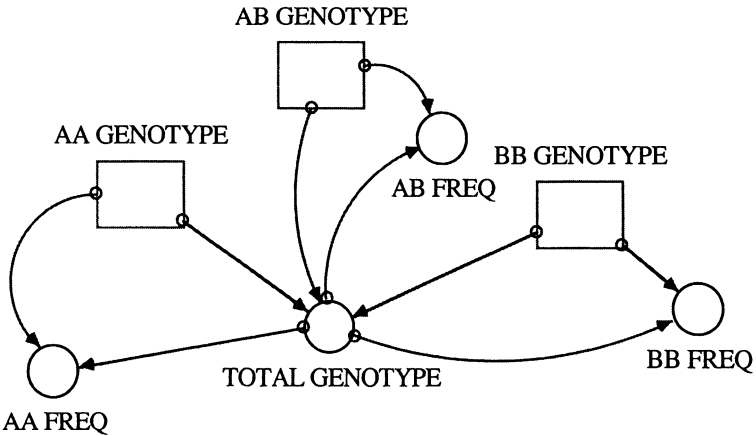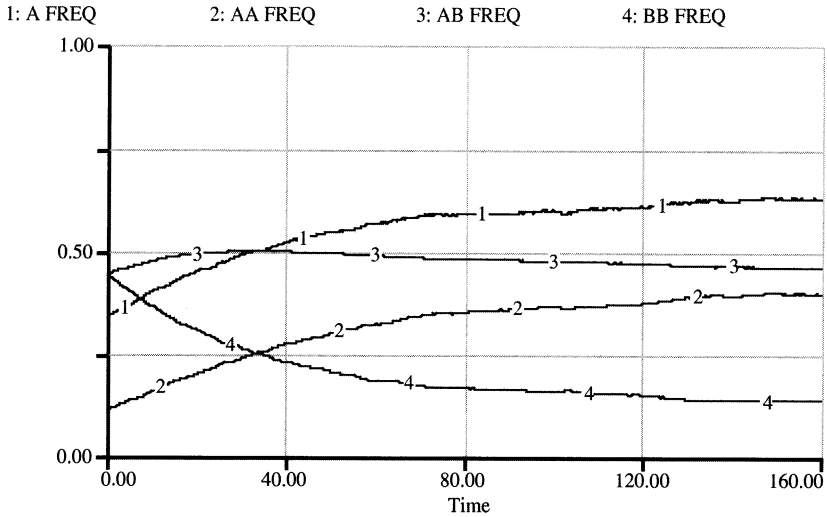
FIGURE 12.4

find that the fertility and survival rate can be thought of as a single fitness parameter, and that the relative sizes of these parameters is all that matters in the determination of surviving genotype frequencies. Try to keep the genotype growth rate from being exponential. Can you rig a connection between the survival rate or fertility rate and the genotype frequency to bring the number of genotypes to an eventual steady state?

There is a heresy afoot in evolutionary biology. Its adherents claim that at least in part, mutation is not random.[1] The mutation rates of bacteria in some experiments seem to speed up and produce a greater fraction of mutants that are most suited to the new environmental stress (e.g., a rise in ambient temperature makes the current genotype distribution less than optimal). Can you arrange such a model?

Another interesting question concerns the issue of human-caused selection versus conservation. This question arises, for example, out of the work of David Hopcraft of the Hopcraft Wildlife Ranch, on the Athi River in Kenya. Hopcraft has given up his ancestor's ways of raising English and Scottish cattle on the hot, dry African planes. He instead is able to make greater income by harvesting the "extra" males from the animal herds on his ranch, after the dominant male has been established in the breeding groups. His observations of such herds showed that the "natural" assem-

---

[1] Hall, B.G. 1990. Spontaneous point mutations that occur more often when advantageous than when neutral, *Genetics*, Vol. 126, pp. 5–16. Cairns, J., J. Overbaugh, and S. Miller. 1988. The origin of mutants, *Nature*, Vol. 335, pp. 142–145. Mittler, J.E. and R.E. Lenski. 1990. New data on excisions of Mu from E. Coli MCS2 cast doubt on directed mutation hypothesis, *Nature*, Vol. 344, pp. 173–175.

blage contained many nonbreeding males at any given time. These animals were just consuming grass that could have gone to greater populations of producing females. He harvests from this large, youngish male population. In one sense, he is cleverly farming the nutrient base and the Kenyan sunlight.

Closer observation shows that the "extra" males are actually part of the breeding population. The dominant male may account for, say, half of the offspring in any given year, the rest coming with the help of the "extra" males. Also, it is not necessarily the case that a dominant male this year will also be one in the next mating season if the extra males are still around.

We tend to think of the dominant male as doing all the breeding since that process leads to healthier offspring. But it would also lead to fewer "ABs" in the natural population, and thus the natural population would be at a greater than necessary risk of demise. Thus, the theories of dominance and territoriality at their extreme are significantly flawed with regard to maintaining genetic diversity in a population.

In real populations, the complete defense of a harem by the male is apparently so difficult and the drive to breed on the part of the "extra" males is so great, that the harem defenses are significantly breached, thereby increasing genetic diversity. As a result, the genetic contribution of the nondominant males may increase the stability of the overall population.

So, nature seems to be creating a tension of two opposing forces in the breeding populations of wild animals at least. One force is seen in the selection by females of strong, large, symmetrically-built males, a force surely producing size, speed, and general health that allows the herd to minimize loss to predators. The counterforce, the allowance of nondominant male breeding produces diversity, stability of population size, and resistance to unpredictable challenge, from disease for example. So is the Hopcraft method really that crafty? It seems to be much better than allowing imported cattle to stumble around inappropriate terrain, but is such selective removal of the extra males leaving the future herd unstable in the face of genetically based challenge? To answer these questions, you can mimic the Hopcraft situation and its possible consequences and cures with a STELLA model based on insight from the previous chapters. In your model you may find that it is advantageous to distinguish the population of animals by age classes. We will show in the following three chapters some use of age cohort models.

NATURAL SELECTION AND MUTATION

```
AA_GENOTYPE(t) = AA_GENOTYPE(t - dt) + (AA_RATE -
AA_REMATE) * dt
INIT AA_GENOTYPE = 0
INFLOWS:
AA_RATE = TOTAL_ALLELES/2*A_FREQ^2   {Using the Hardy-
Weinberg result to obtain the genotype frequencies.}
```

```
OUTFLOWS:
AA_REMATE = AA_GENOTYPE {This flow represents the
conversion of the genotype to alleles, representing the
generation change.}

AB_GENOTYPE(t) = AB_GENOTYPE(t - dt) + (AB_RATE -
AB_REMATE) * dt
INIT AB_GENOTYPE = 0
INFLOWS:
AB_RATE = 2*TOTAL_ALLELES/2*A_FREQ*(1 - A_FREQ) {Using
the Hardy-Weinberg result to obtain the genotype
frequencies.}
OUTFLOWS:
AB_REMATE = AB_GENOTYPE {This flow represents the
conversion of the genotype to alleles, representing the
generation change.}

A_ALLELES(t) = A_ALLELES(t - dt) + (NEW_A + MUTATION -
A_LOSS) * dt
INIT A_ALLELES = 100
INFLOWS:
NEW_A = 2*AA_GENOTYPE*AA_FERT*AA_SURVIVE_RATE
+AB_GENOTYPE*AB_FERT*AB_SURVIVE_RATE {This is the way
that genotypes break up into alleles again, after a
generation of time has passed.}
MUTATION = .3*B_ALLELES*RANDOM(0,1)*0 +
.03*A_ALLELES*RANDOM(0,1)
OUTFLOWS:
A_LOSS = A_ALLELES {This flow represents the converson
of the alleles into genetypes.}

BB_GENOTYPE(t) = BB_GENOTYPE(t - dt) + (BB_RATE -
BB_REMATE) * dt
INIT BB_GENOTYPE = 0
INFLOWS:
BB_RATE = TOTAL_ALLELES/2*(1 - A_FREQ)^2 {Using the
Hardy-Weinberg result to obtain the genotype
frequencies.}
OUTFLOWS:
BB_REMATE = BB_GENOTYPE {This flow represents the
conversion of the genotype to alleles, representing the
generation change.}
```

```
B_ALLELES(t) = B_ALLELES(t - dt) + (NEW_B - B_LOSS -
MUTATION) * dt
INIT B_ALLELES = 200
INFLOWS:
NEW_B = 2*BB_GENOTYPE*BB_FERT*BB_SURVIVE_RATE +
AB_GENOTYPE*AB_FERT*AB_SURVIVE_RATE {This is the way
that genotypes break up into alleles again, after a
generation of time has passed.}
OUTFLOWS:
B_LOSS = B_ALLELES {This flow represents the converson
of the alleles into genetypes.}
MUTATION = .3*B_ALLELES*RANDOM(0,1)*0 +
.03*A_ALLELES*RANDOM(0,1)

AA_FERT = 2 {Fraction per time step of one.}
AA_FREQ = IF TIME > 0 THEN AA_GENOTYPE/TOTAL_GENOTYPE
ELSE 0
AA_SURVIVE_RATE = .9 {Fraction per time step of one.}
AB_FERT = 2 {Fraction per time step of one.}
AB_FREQ = IF TIME > 0 THEN AB_GENOTYPE/TOTAL_GENOTYPE
ELSE 0
AB_SURVIVE_RATE = 1 {Fraction per time step of one.}
A_FREQ = A_ALLELES/(A_ALLELES+ B_ALLELES)
BB_FERT = 2 {Fraction per time step of one.}
BB_FREQ = IF TIME > 0 THEN BB_GENOTYPE/TOTAL_GENOTYPE
ELSE 0
BB_SURVIVE_RATE = .9
TOTAL_ALLELES = A_ALLELES + B_ALLELES
TOTAL_GENOTYPE = AA_GENOTYPE+AB_GENOTYPE+BB_GENOTYPE
```

Part 5

Ecological Models

13
Artificial Worms

> Any intelligent fool can make things bigger, more complex, and more
> violent. It takes a touch of genius—and a lot of courage—to move in the
> opposite direction.
>
> —Albert Einstein, *Ideas and Opinions*

Natural selection works in the presence of chance or randomness. As random mutations take place, the form or function of organisms changes. With changes in the "appearance" of individual organisms may come an enhanced ability to survive and to pass on the respective traits to subsequent generations. However, because mutations continue to take place, and because the environment within which natural selection occurs is not constant, it is not automatic that the fitness of offspring increases from one generation to the next.

The following model illustrates the workings of natural selection and randomness for a population of six worms of different lengths. Each individual worm can, in principle, mate with any one of the others. The decision of who mates with whom is randomly made. Of the three randomly chosen pairs of mating worms, only the longer one of the two will survive to bear exactly two offspring. The offspring, in turn, are either shorter or longer (randomly decided) than their parent. Their actual length is determined by a coin toss. If the coin shows head, one of the offspring will be exactly 1 unit longer than the parent, and the other one will be exactly 2 units longer than the parent. In the case that the coin shows tail, the two offspring are, respectively, 1 and 2 units shorter than the parent. The moment the offspring are generated, the parent dies.[1] As a result, a total of six new worms are present whose lengths may differ from those of their six parents. Will the lengths of the new generations of worms be similar to those of their parents, or over generations, will worms get longer or shorter?

Because of the randomness that determines the choice of mating partners and the randomness that determines the length of the offspring we do not know in advance how long each of the next generation worms will be. What we do know, however, is that of the pairs that mate, only the longer one survives. So, a mechanism is built in to the dynamics of this artificial, evolutionary process that favors longer worms, and we would therefore expect that as time passes, later-born worms are on average longer than their predecessors.

In order to model the evolution of our worms, note that we need to solve three interrelated problems. First, we need to randomly pair the worms.

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[1]This is the worm growth model briefly mentioned in Cohen, J. and I. Stewart. 1994. *The Collapse of Chaos*, Penguin Books, New York, pp. 105–106.

Then we need to decide who in each pair is longer, and "kill" the shorter ones off. Third, we need to toss a coin to determine the length of the offspring and add these new worms to our system while we remove the information about their parents, i.e., let the remaining three parents die.

The first of these three modeling problems can be solved by assuming that we cast six dice such that at the end all six dice show a different number. Then we always pair the same dice—the first die with the second, the third with the fourth, and the fifth with the sixth. For example, if the dice give the numbers 2, 4, 1, 6, 3, 5, then we take this to mean that we should mate WORM 2 with WORM 4, WORM 1 with WORM 6, and WORM 3 with WORM 5. To make things easier, we may actually fix one die—say the first one—and cast only the other five die. That little trick still leaves the process perfectly random, because any time five die are cast such that their numbers are different from each other, the number for the sixth is automatically determined anyway. We may as well start with that one. Here, we arbitrarily set it to 1.

Once all dice have been cast such that they each show a different number, their sum is 21 (1+2+3+4+5+6=21). At that time, the worms are officially paired and can begin their fight to the death (with the winner reproducing). Then the system is reset and the stocks, which contain the results of the round of casting the dice, are emptied, and the dice are cast anew.

The dice of this model are represented as stocks named STOCK 1, STOCK 2, etc., whose contents are held constant until all dice show random numbers between 2 and 6, and all are different from each other. The exception is the first die, STOCK 1, whose value is fixed at 1.

Take, for example, the second die. The converter DIE 2 in Figure 13.1 generates a random number between 1 and 7:

$$DIE\ 2 = INT(RANDOM(1,7)) \qquad (1)$$

By only taking the integer of the random number we generate a string of numbers between 1 and 6. The odds that exactly a 7 is generated are minuscule. Next, we need to make sure that we only accept a number that has not already been cast by another die—we want to avoid mating one worm with itself. We do this, for example for DIE 2, with the following conditional statement that states that as long as STOCK 2 is zero and as long as DIE 2 shows a different number from those recorded in the other stocks, we can accept that die's number. If not, we continue to cast that die until its number differs from any of the others, and we then add it to the stock.

RAND 2 = IF (STOCK 2 = 0 AND DIE 2 ≠ STOCK 1 AND DIE 2 ≠ STOCK 3
        AND DIE 2 ≠ STOCK 4 AND DIE 2 ≠ STOCK 5
        AND DIE 2 ≠ STOCK 6
        AND DIE 2 ≠ DIE 6 AND DIE 2 ≠ DIE 5 AND DIE 2 ≠ DIE 4
        AND DIE 2 ≠ DIE 3)
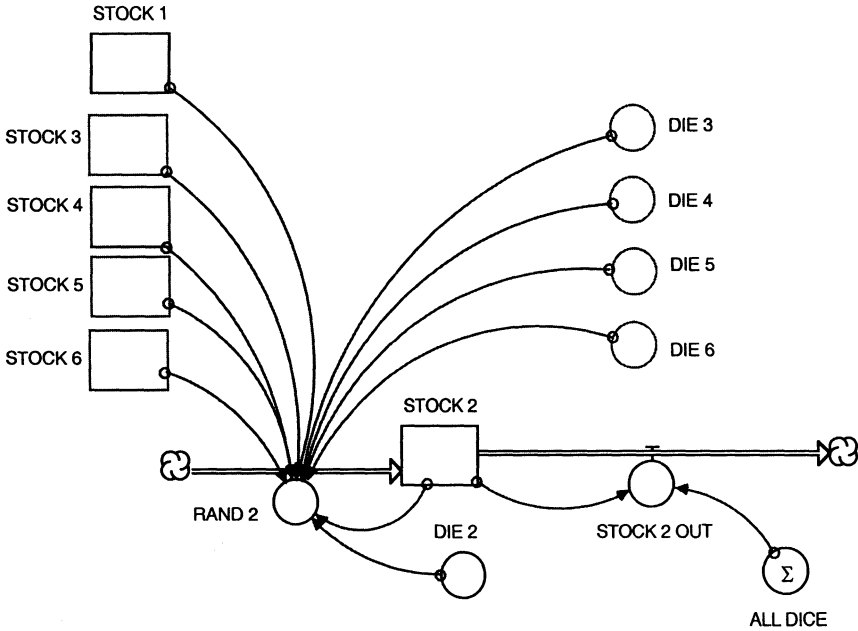   THEN DIE 2 ELSE 0   (2)

FIGURE 13.1

Once all dice have been cast such that they each show a different number, their sum is 21. We calculate that sum in the ALL DICE converter, using STELLA's "Summer" option. Once all dice show a different number all worms are randomly, yet unambiguously, paired with each other to begin their fight to the death. The system is reset by emptying STOCK 2, STOCK 3 etc., and the dice are cast anew.

STOCK 2 OUT = IF ALL DICE = 21 THEN STOCK 2 ELSE 0   (3)

So far, we have only concerned ourselves with the means by which we can randomly pair up the six worms. But what about their lengths? To keep track of worm length, we generate six new state variables, and we assign them some arbitrary initial values for the lengths of the worms. For simplicity, we set here the initial values of WORM 1 = 1, WORM 2 = 2, WORM 3 = 3, and so on.

Note in Figure 13.2 how we set up the model structure such that we always update the same pair of stocks for the worms. Who these worms are, however, is determined by casting the dice—it is not as fixed as the graphical model representation may suggest. For example, WORM 3 and WORM 4 are always those worms that were identified as the first and second worms in the second pair of worms that is formed for the mating process, but from one round of casting dice to the next, these are different worms whose lengths may differ from those of their parents.
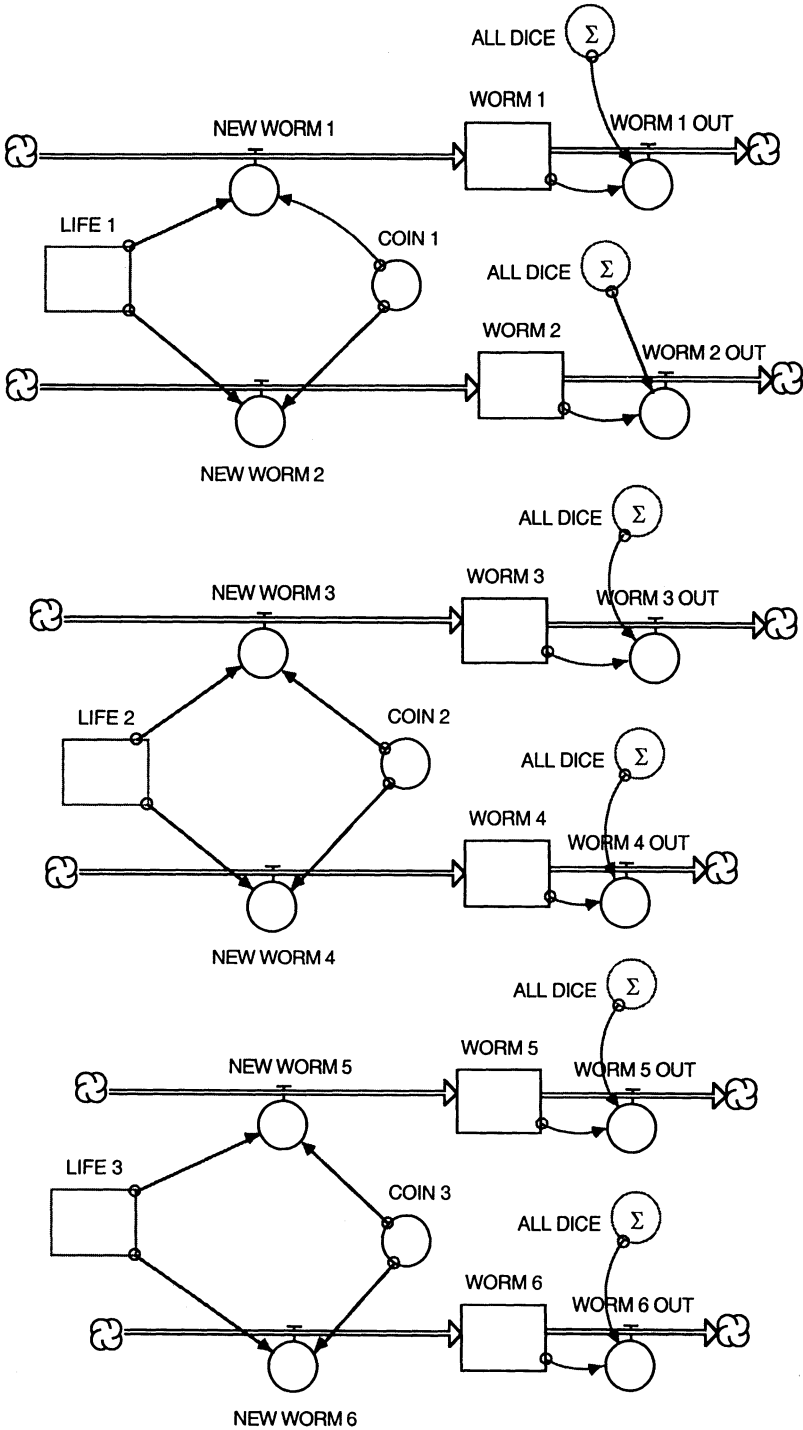
FIGURE 13.2

Outflows from the WORM 1, WORM 2, etc. stocks occur once all dice are cast, i.e., when ALL DICE = 21. Those worms are now ready to fight for their lives and then generate offspring. For example, the outflow from the WORM 1 stock is

$$\text{WORM 1 OUT} = \text{IF ALL DICE} = 21 \text{ THEN WORM 1 ELSE 0} \qquad (4)$$

The inflows are determined on the basis of a coin toss and who of the two worms in a pair survived the fight for life. We will discuss the inflows into WORM 1, WORM 2, etc. below, after we deal with the processes by which we determine the winners of the fights and update the respective other stocks in our model.

Take, for example, the second pair of worms—the pair that was determined by tossing the third and fourth dice (Fig. 13.3). The stock LIFE 2 retains information about the length of the winner of a fight. That stock is initially set to zero, as are LIFE 1 and LIFE 3—the winners of the other two fights.

The inflows into the stocks LIFE 1, LIFE 2, and LIFE 3 are called FIGHT 1, FIGHT 2, and FIGHT 3, and they contain the guts of this model. Here, we check which of the worms got matched up in the dice-casting part of this model, and then compare their lengths. For example, if the third die showed the number 2 and the fourth die a number 3, then we know that the second and third worms were paired up, and we will compute the maximum of the lengths of the two. But if the third die showed a number 2 and the second a number 4, then WORM 2 and WORM 4 were paired up and their lengths need to be compared. We repeat this process until we exhaust all combinatorial possibilities. If none of the possibilities arose, that is a sign that the process of pairing the worms by casting the dice has not yet been completed. In this case, the value of FLIGHT 2 is set to zero and the stock LIFE 2 remains unchanged.
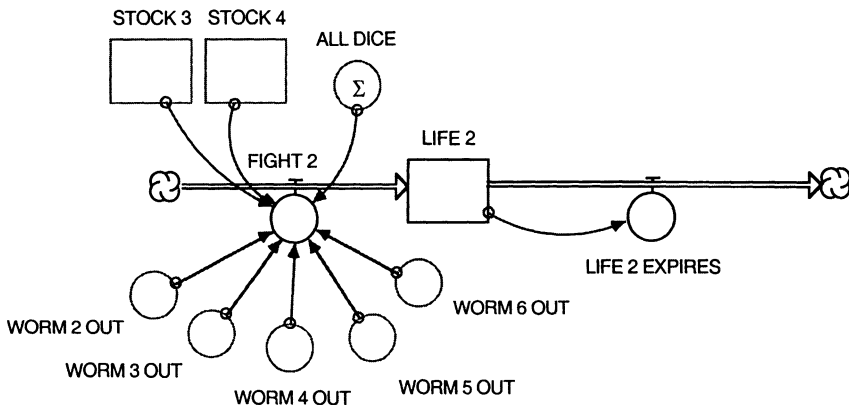
FIGURE 13.3

FIGHT 2 = IF (STOCK 3=2 AND STOCK 4=3 AND ALL DICE=21)
            THEN MAX(WORM 2 OUT,WORM 3 OUT)
            ELSE IF (STOCK 3=2 AND STOCK 4=4 AND ALL DICE=21)
            THEN MAX(WORM 2 OUT,WORM 4_OUT)
            ELSE IF (STOCK 3=2 AND STOCK 4=5 AND ALL DICE=21)
            THEN MAX(WORM 2 OUT,WORM 5 OUT)
            ELSE IF (STOCK 3=2 AND STOCK 4=6 AND ALL DICE=21)
            THEN MAX(WORM 2 OUT,WORM 6 OUT)
            ELSE IF (STOCK 3=3 AND STOCK 4=2 AND ALL DICE=21)
            THEN MAX(WORM 3 OUT,WORM 2 OUT)
            ELSE IF (STOCK 3=3 AND STOCK 4=4 AND ALL DICE=21)
            THEN MAX(WORM 3 OUT,WORM 4 OUT)
            ELSE IF (STOCK 3=3 AND STOCK 4=5 AND ALL DICE=21)
            THEN MAX(WORM 3 OUT,WORM 5 OUT)
            ELSE IF (STOCK 3=3 AND STOCK 4=6 AND ALL DICE=21)
            THEN MAX(WORM 3 OUT,WORM 6 OUT)
            ELSE IF (STOCK 3=4 AND STOCK 4=2 AND ALL DICE=21)
            THEN MAX(WORM 4 OUT,WORM 2 OUT)
            ELSE IF (STOCK 3=4 AND STOCK 4=3 AND ALL DICE=21)
            THEN MAX(WORM 4 OUT,WORM 3 OUT)
            ELSE IF (STOCK 3=4 AND STOCK 4=5 AND ALL DICE=21)
            THEN MAX(WORM 4 OUT,WORM 5 OUT)
            ELSE IF (STOCK 3=4 AND STOCK 4=6 AND ALL DICE=21)
            THEN MAX(WORM 4 OUT,WORM 6 OUT)
            ELSE IF (STOCK 3=5 AND STOCK 4=2 AND ALL DICE=21)
            THEN MAX(WORM 5 OUT,WORM 2 OUT)
            ELSE IF (STOCK 3=5 AND STOCK 4=3 AND ALL DICE=21)
            THEN MAX(WORM 5 OUT,WORM 3 OUT)
            ELSE IF (STOCK 3=5 AND STOCK 4=4 AND ALL DICE=21)
            THEN MAX(WORM 5 OUT,WORM 4 OUT)
            ELSE IF (STOCK 3=5 AND STOCK 4=6 AND ALL DICE=21)
            THEN MAX(WORM 5 OUT,WORM 6 OUT)
            ELSE IF (STOCK 3=6 AND STOCK 4=2 AND ALL DICE=21)
            THEN MAX(WORM 6 OUT,WORM 2 OUT)
            ELSE IF (STOCK 3=6 AND STOCK 4=3 AND ALL DICE=21)
            THEN MAX(WORM 6 OUT,WORM 3 OUT)
            ELSE IF (STOCK 3=6 AND STOCK 4=4 AND ALL DICE=21)
            THEN MAX(WORM 6 OUT,WORM 4 OUT)
            ELSE IF (STOCK 3=6 AND STOCK 4=5 AND ALL DICE=21)
            THEN MAX(WORM 6 OUT,WORM 5 OUT)
            ELSE 0   (5)

The FIGHT 3 flow is analogous to the one for FIGHT 2, but both of these differ slightly from the FIGHT 1 flow (Fig. 13.4). To calculate FIGHT 1 we do not need a ghost of STOCK 1, because we always know the value of that stock—earlier we set it to 1.
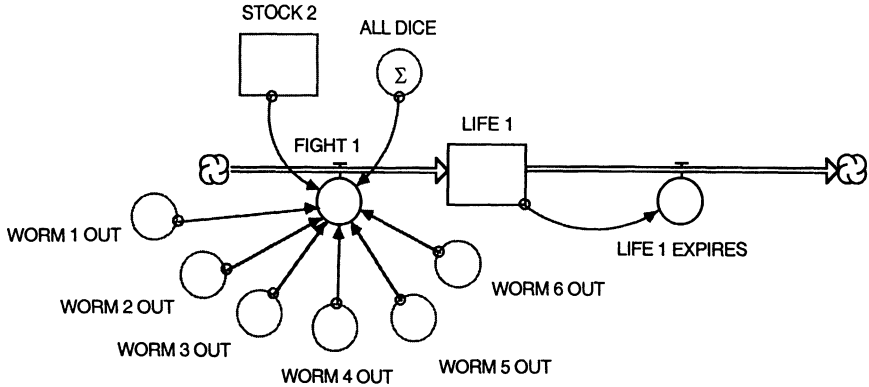
FIGURE 13.4

Once we calculated the length of the winner of the fights in each pair of worms, that worm will die, too. We model the winner's death with the outflows from the respective LIFE stocks, such as

$$\text{LIFE 2 EXPIRES} = \text{LIFE 2} \tag{6}$$

The LIFE 1, LIFE 2, and LIFE 3 stocks are used to calculate the lengths of the next generation of worms. For each pair of offspring we flip a coin, as we have done in previous chapters:

$$\text{COIN 1} = \text{RANDOM}(0,1) \tag{7}$$

$$\text{COIN 2} = \text{RANDOM}(0,1) \tag{8}$$

$$\text{COIN 3} = \text{RANDOM}(0,1) \tag{9}$$

Then we compute the length of the new worms based on the outcome of the coin toss. We interpret numbers below .5 as tail and in that case make the length of one new worm 1 unit less than that of the winner of the fight, and we make the other offspring 2 units shorter. For numbers above .5 we make the offspring 1 and 2 units longer, respectively. For example,

NEW WORM 1 = IF (LIFE 1>0 AND COIN 1<.5) THEN LIFE 1–1
             ELSE IF (LIFE 1>0 AND COIN 1>.5) THEN LIFE 1+1
             ELSE 0   (10)

NEW WORM 2 = IF (LIFE 1>0 AND COIN 1<.5) THEN LIFE 1–2
             ELSE IF (LIFE 1>0 AND COIN 1>.5) THEN LIFE 1+2
             ELSE 0   (11)

Now our model is finished. We have modules that we use to generate random pairs of worms, to compare their lengths and let the longer one win a fight, and to generate offspring whose lengths depend on the length of the winner in each fight. Figure 13.5 shows in the form of a bar chart the
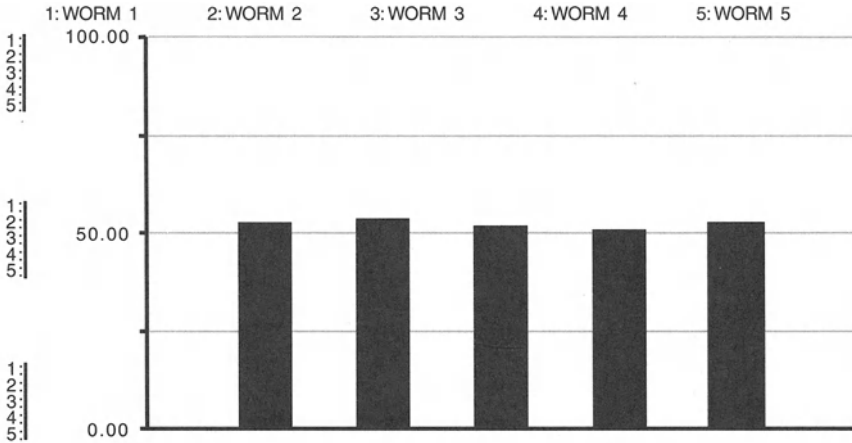
FIGURE 13.5

result of the evolution of our worms after running the model for 1000 pe-riods. To generate a bar chart, simply double-click on an open graph pad and select "Bar." Only a maximum of five variables can be plotted.

Note that the length of the model run should not be interpreted in terms of the number of generations of worms. At times, the process of casting dice leads immediately to numbers that are different from each other for all six dice. At that moment, a new generation of worms can be formed, and only in this case is one time period equal to the length of one worm gener-ation. If the dice are cast and not all dice show different numbers, we need to continue to cast them until they do. This may take several periods in the model, and as a consequence, the new generation of worms will be formed after more than one period in the model.

Of course in the world of laboratory and field experiments, the most likely picture we would have is a seasonal or breeding cycle study of the length of worms. In actual measurement of the worms, we would get just one of the pictures in the series that composes Figure 13.5. We would cat-egorize the lengths in the field most likely in terms of mean length and standard deviation. Our model allows one to actually compute these values continuously and thus provides a way to validate the model. In the absence of such data, we have a model here that provides a dynamic example of natural selection dynamics.

ARTIFICIAL WORMS

```
LIFE_1(t) = LIFE_1(t - dt) + (FIGHT_1 -
LIFE_1_EXPIRES) * dt
INIT LIFE_1 = 0
```

```
INFLOWS:
FIGHT_1 = IF (STOCK_2 = 2 AND ALL_DICE= 21) THEN
MAX(WORM_1_OUT,WORM_2_OUT)
ELSE IF (STOCK_2 = 3 AND ALL_DICE= 21) THEN
MAX(WORM_1_OUT,WORM_3_OUT)
ELSE IF (STOCK_2 = 4 AND ALL_DICE= 21) THEN
MAX(WORM_1_OUT,WORM_4_OUT)
ELSE IF (STOCK_2 = 5 AND ALL_DICE= 21) THEN
MAX(WORM_1_OUT,WORM_5_OUT)
ELSE IF (STOCK_2 = 6 AND ALL_DICE= 21) THEN
MAX(WORM_1_OUT,WORM_6_OUT)
ELSE 0
OUTFLOWS:
LIFE_1_EXPIRES = LIFE_1

LIFE_2(t) = LIFE_2(t - dt) + (FIGHT_2 -
LIFE_2_EXPIRES) * dt
INIT LIFE_2 = 0
INFLOWS:
FIGHT_2 = IF (STOCK_3=2 AND STOCK_4=3 AND
ALL_DICE=21) THEN MAX(WORM_2_OUT,WORM_3_OUT)
ELSE IF (STOCK_3=2 AND STOCK_4=4 AND ALL_DICE=21) THEN
MAX(WORM_2_OUT,WORM_4_OUT)
ELSE IF (STOCK_3=2 AND STOCK_4=5 AND ALL_DICE=21) THEN
MAX(WORM_2_OUT,WORM_5_OUT)
ELSE IF (STOCK_3=2 AND STOCK_4=6 AND ALL_DICE=21) THEN
MAX(WORM_2_OUT,WORM_6_OUT)
ELSE IF (STOCK_3=3 AND STOCK_4=2 AND ALL_DICE=21) THEN
MAX(WORM_3_OUT,WORM_2_OUT)
ELSE IF (STOCK_3=3 AND STOCK_4=4 AND ALL_DICE=21) THEN
MAX(WORM_3_OUT,WORM_4_OUT)
ELSE IF (STOCK_3=3 AND STOCK_4=5 AND ALL_DICE=21) THEN
MAX(WORM_3_OUT,WORM_5_OUT)
ELSE IF (STOCK_3=3 AND STOCK_4=6 AND ALL_DICE=21) THEN
MAX(WORM_3_OUT,WORM_6_OUT)
ELSE IF (STOCK_3=4 AND STOCK_4=2 AND ALL_DICE=21) THEN
MAX(WORM_4_OUT,WORM_2_OUT)
ELSE IF (STOCK_3=4 AND STOCK_4=3 AND ALL_DICE=21) THEN
MAX(WORM_4_OUT,WORM_3_OUT)
ELSE IF (STOCK_3=4 AND STOCK_4=5 AND ALL_DICE=21) THEN
MAX(WORM_4_OUT,WORM_5_OUT)
ELSE IF (STOCK_3=4 AND STOCK_4=6 AND ALL_DICE=21) THEN
MAX(WORM_4_OUT,WORM_6_OUT)
```

```
ELSE IF (STOCK_3=5 AND STOCK_4=2 AND ALL_DICE=21) THEN
MAX(WORM_5_OUT,WORM_2_OUT)
ELSE IF (STOCK_3=5 AND STOCK_4=3 AND ALL_DICE=21) THEN
MAX(WORM_5_OUT,WORM_3_OUT)
ELSE IF (STOCK_3=5 AND STOCK_4=4 AND ALL_DICE=21) THEN
MAX(WORM_5_OUT,WORM_4_OUT)
ELSE IF (STOCK_3=5 AND STOCK_4=6 AND ALL_DICE=21) THEN
MAX(WORM_5_OUT,WORM_6_OUT)
ELSE IF (STOCK_3=6 AND STOCK_4=2 AND ALL_DICE=21) THEN
MAX(WORM_6_OUT,WORM_2_OUT)
ELSE IF (STOCK_3=6 AND STOCK_4=3 AND ALL_DICE=21) THEN
MAX(WORM_6_OUT,WORM_3_OUT)
ELSE IF (STOCK_3=6 AND STOCK_4=4 AND ALL_DICE=21) THEN
MAX(WORM_6_OUT,WORM_4_OUT)
ELSE IF (STOCK_3=6 AND STOCK_4=5 AND ALL_DICE=21) THEN
MAX(WORM_6_OUT,WORM_5_OUT)
ELSE 0
OUTFLOWS:
LIFE_2_EXPIRES = LIFE_2

LIFE_3(t) = LIFE_3(t - dt) + (FIGHT_3 -
LIFE_3_EXPIRES) * dt
INIT LIFE_3 = 0
INFLOWS:
FIGHT_3 = IF (STOCK_5=2 AND STOCK_6=3 AND
ALL_DICE=21) THEN MAX(WORM_2_OUT,WORM_3_OUT)
ELSE IF (STOCK_5=2 AND STOCK_6=4 AND ALL_DICE=21) THEN
MAX(WORM_2_OUT,WORM_4_OUT)
ELSE IF (STOCK_5=2 AND STOCK_6=5 AND ALL_DICE=21) THEN
MAX(WORM_2_OUT,WORM_5_OUT)
ELSE IF (STOCK_5=2 AND STOCK_6=6 AND ALL_DICE=21) THEN
MAX(WORM_2_OUT,WORM_6_OUT)
ELSE IF (STOCK_5=3 AND STOCK_6=2 AND ALL_DICE=21) THEN
MAX(WORM_3_OUT,WORM_2_OUT)
ELSE IF (STOCK_5=3 AND STOCK_6=4 AND ALL_DICE=21) THEN
MAX(WORM_3_OUT,WORM_4_OUT)
ELSE IF (STOCK_5=3 AND STOCK_6=5 AND ALL_DICE=21) THEN
MAX(WORM_3_OUT,WORM_5_OUT)
ELSE IF (STOCK_5=3 AND STOCK_6=6 AND ALL_DICE=21) THEN
MAX(WORM_3_OUT,WORM_6_OUT)
ELSE IF (STOCK_5=4 AND STOCK_6=2 AND ALL_DICE=21) THEN
MAX(WORM_4_OUT,WORM_2_OUT)
```

```
ELSE IF (STOCK_5=4 AND STOCK_6=3 AND ALL_DICE=21) THEN
MAX(WORM_4_OUT,WORM_3_OUT)
ELSE IF (STOCK_5=4 AND STOCK_6=5 AND ALL_DICE=21) THEN
MAX(WORM_4_OUT,WORM_5_OUT)
ELSE IF (STOCK_5=4 AND STOCK_6=6 AND ALL_DICE=21) THEN
MAX(WORM_4_OUT,WORM_6_OUT)
ELSE IF (STOCK_5=5 AND STOCK_6=2 AND ALL_DICE=21) THEN
MAX(WORM_5_OUT,WORM_2_OUT)
ELSE IF (STOCK_5=5 AND STOCK_6=3 AND ALL_DICE=21) THEN
MAX(WORM_5_OUT,WORM_3_OUT)
ELSE IF (STOCK_5=5 AND STOCK_6=4 AND ALL_DICE=21) THEN
MAX(WORM_5_OUT,WORM_4_OUT)
ELSE IF (STOCK_5=5 AND STOCK_6=6 AND ALL_DICE=21) THEN
MAX(WORM_5_OUT,WORM_6_OUT)
ELSE IF (STOCK_5=6 AND STOCK_6=2 AND ALL_DICE=21) THEN
MAX(WORM_6_OUT,WORM_2_OUT)
ELSE IF (STOCK_5=6 AND STOCK_6=3 AND ALL_DICE=21) THEN
MAX(WORM_6_OUT,WORM_3_OUT)
ELSE IF (STOCK_5=6 AND STOCK_6=4 AND ALL_DICE=21) THEN
MAX(WORM_6_OUT,WORM_4_OUT)
ELSE IF (STOCK_5=6 AND STOCK_6=5 AND ALL_DICE=21) THEN
MAX(WORM_6_OUT,WORM_5_OUT)
ELSE 0
OUTFLOWS:
LIFE_3_EXPIRES = LIFE_3

STOCK_1(t) = STOCK_1(t - dt)
INIT STOCK_1 = 1

STOCK_2(t) = STOCK_2(t - dt) + (RAND_2 - STOCK_2_OUT)
* dt
INIT STOCK_2 = 0
INFLOWS:
RAND_2 = IF (STOCK_2 = 0 AND DIE_2 <> STOCK_1 AND
DIE_2<>STOCK_3 AND DIE_2<>STOCK_4 AND DIE_2<>STOCK_5
AND DIE_2<>STOCK_6 AND DIE_2<>DIE_6 AND DIE_2<>DIE_5
AND DIE_2<>DIE_4 AND DIE_2<>DIE_3 ) THEN DIE_2 ELSE 0
OUTFLOWS:
STOCK_2_OUT = IF ALL_DICE = 21 THEN STOCK_2 ELSE 0

STOCK_3(t) = STOCK_3(t - dt) + (RAND_3 - STOCK_3_OUT)
* dt
INIT STOCK_3 = 0
```

```
INFLOWS:
RAND_3 = IF (STOCK_3 = 0 AND DIE_3 <> STOCK_1 AND
DIE_3 <>STOCK_2 AND DIE_3<>STOCK_4 AND DIE_3<>STOCK_5
AND DIE_3<>STOCK_6 AND DIE_3<>DIE_6 AND DIE_3<>DIE_5
AND DIE_3<>DIE_4 AND DIE_3<>DIE_2 ) THEN DIE_3 ELSE 0
OUTFLOWS:
STOCK_3_OUT = IF ALL_DICE = 21 THEN STOCK_3 ELSE 0


STOCK_4(t) = STOCK_4(t - dt) + (RAND_4 - STOCK_4_OUT)
* dt
INIT STOCK_4 = 0
INFLOWS:
RAND_4 = IF (STOCK_4 = 0 AND DIE_4 <> STOCK_1 AND
DIE_4 <>STOCK_2 AND DIE_4<>STOCK_3 AND DIE_4<>STOCK_5
AND DIE_4<>STOCK_6 AND DIE_4<>DIE_6 AND DIE_4<>DIE_5
AND DIE_4<>DIE_3 AND DIE_4<>DIE_2) THEN DIE_4 ELSE 0
OUTFLOWS:
STOCK_4_OUT = IF ALL_DICE = 21 THEN STOCK_4 ELSE 0


STOCK_5(t) = STOCK_5(t - dt) + (RAND_5 - STOCK_5_OUT)
* dt
INIT STOCK_5 = 0
INFLOWS:
RAND_5 = IF (STOCK_5 = 0 AND DIE_5 <> STOCK_1 AND
DIE_5 <>STOCK_2 AND DIE_5<>STOCK_3 AND DIE_5<>STOCK_4
AND DIE_5<>STOCK_6 AND DIE_5<>DIE_6 AND DIE_5<>DIE_4
AND DIE_5<>DIE_3 AND DIE_5<>DIE_2) THEN DIE_5 ELSE 0
OUTFLOWS:
STOCK_5_OUT = IF ALL_DICE = 21 THEN STOCK_5 ELSE 0


STOCK_6(t) = STOCK_6(t - dt) + (RAND_6 - STOCK_6_OUT)
* dt
INIT STOCK_6 = 0
INFLOWS:
RAND_6 = IF (STOCK_6 = 0 AND DIE_6 <> STOCK_1 AND
DIE_6 <>STOCK_2 AND DIE_6<>STOCK_3 AND DIE_6<>STOCK_4
AND DIE_6<>STOCK_5 AND DIE_6<>DIE_5 AND DIE_6<>DIE_4
AND DIE_6<>DIE_3 AND DIE_6<>DIE_2) THEN DIE_6 ELSE 0
OUTFLOWS:
STOCK_6_OUT = IF ALL_DICE = 21 THEN STOCK_6 ELSE 0


WORM_1(t) = WORM_1(t - dt) + (NEW_WORM_1 - WORM_1_OUT)
* dt
```

```
INIT WORM_1 = 1
INFLOWS:
NEW_WORM_1 = IF (LIFE_1>0 AND COIN_1<.5) THEN LIFE_1-1
ELSE IF (LIFE_1>0 AND COIN_1>.5) THEN LIFE_1+1
ELSE 0
OUTFLOWS:
WORM_1_OUT = IF ALL_DICE = 21 THEN WORM_1 ELSE 0


WORM_2(t) = WORM_2(t - dt) + (NEW_WORM_2 - WORM_2_OUT)
* dt
INIT WORM_2 = 2
INFLOWS:
NEW_WORM_2 = IF (LIFE_1>0 AND COIN_1<.5) THEN LIFE_1-2
ELSE IF (LIFE_1>0 AND COIN_1>.5) THEN LIFE_1+2
ELSE 0
OUTFLOWS:
WORM_2_OUT = IF ALL_DICE = 21 THEN WORM_2 ELSE 0


WORM_3(t) = WORM_3(t - dt) + (NEW_WORM_3 - WORM_3_OUT)
* dt
INIT WORM_3 = 3
INFLOWS:
NEW_WORM_3 = IF (LIFE_2>0 AND COIN_2<.5) THEN LIFE_2-1
ELSE IF (LIFE_2>0 AND COIN_2>.5) THEN LIFE_2+1
ELSE 0
OUTFLOWS:
WORM_3_OUT = IF ALL_DICE = 21 THEN WORM_3 ELSE 0


WORM_4(t) = WORM_4(t - dt) + (NEW_WORM_4 - WORM_4_OUT)
* dt
INIT WORM_4 = 4
INFLOWS:
NEW_WORM_4 = IF (LIFE_2>0 AND COIN_2<.5) THEN LIFE_2-2
ELSE IF (LIFE_2>0 AND COIN_2>.5) THEN LIFE_2+2
ELSE 0
OUTFLOWS:
WORM_4_OUT = IF ALL_DICE = 21 THEN WORM_4 ELSE 0


WORM_5(t) = WORM_5(t - dt) + (NEW_WORM_5 - WORM_5_OUT)
* dt
INIT WORM_5 = 5
INFLOWS:
NEW_WORM_5 = IF (LIFE_3>0 AND COIN_3<.5) THEN LIFE_3-1
```

```
ELSE IF (LIFE_3>0 AND COIN_3>.5) THEN LIFE_3+1
ELSE 0
OUTFLOWS:
WORM_5_OUT = IF ALL_DICE = 21 THEN WORM_5 ELSE 0


WORM_6(t) = WORM_6(t - dt) + (NEW_WORM_6 - WORM_6_OUT)
* dt
INIT WORM_6 = 6
INFLOWS:
NEW_WORM_6 = IF (LIFE_3>0 AND COIN_3<.5) THEN LIFE_3-2
ELSE IF (LIFE_3>0 AND COIN_3>.5) THEN LIFE_3+2
ELSE 0
OUTFLOWS:
WORM_6_OUT = IF ALL_DICE = 21 THEN WORM_6 ELSE 0


ALL_DICE = STOCK_1 + STOCK_2 + STOCK_3 + STOCK_4 +
STOCK_5 + STOCK_6
COIN_1 = RANDOM(0,1)
COIN_2 = RANDOM(0,1)
COIN_3 = RANDOM(0,1)
DIE_2 = INT(RANDOM(1,7))
DIE_3 = INT(RANDOM(1,7))
DIE_4 = INT(RANDOM(1,7))
DIE_5 = INT(RANDOM(1,7))
DIE_6 = INT(RANDOM(1,7))
```

14
Robin Population

> The spray of the willow trembles with the wing of the redbreast
> —Edward Lytton, *Lucretia*

Table 14.1 shows a life table for the American robin.[1] The life table captures the population changes by age class, or cohort.

How can we model the population dynamics of the American robin, accounting for different survival rates at different age classes? In order to distinguish the different cohorts, we must split up the stock of population into one for each age group. We used such an approach before in Chapter 4. Next, we need to know for each of these stocks its inflows and outflows. The age-specific survival rate times each respective stock provides the growth in the population for the next age level. The die-off at each age is the difference between the population at that age and those surviving to become a year older. The collective natality is the sum of each population times its age-specific natality rate.

When we model the cohorts of the robin population, each age group will "move" through the model much like the pollutant in our earlier model on river toxins moved downstream. Similar to that model and analogous to the age-cohort model developed in Chapter 4, we need an inflow of newborns into the model. The introduction of the first set of newborns occurs in the second period of the model as it should: the newborns listed in the life table above are the offsprings of the adults already present. These newborns mature in the first year and have their own offspring in the following year:

TABLE 14.1

Age at start of interval,	x	Nx	Dx	Sx	Mx
	0	1400	903	.355	0
	1	497	268	.461	1.4
	2	229	130	.433	2.1
	3	99	63	.364	1.6
	4	36	26	.277	1.4
	5	10	4	.600	1.1
	6	6	5	.163	1.0

Nx = the number of survivors at age x,
Dx = the number of deaths x interval x,
Sx = the proportion surviving from x to x+1, the survival rate,
Mx = the natality rate for the interval x to x+1.

[1]Spain, J.D. 1982. *BASIC Microcomputer Models in Biology*, Addison-Wesley, Reading, MA, p. 110.

NATALITY = ONE YEAR OLDS * 1.4 + TWO YEAR OLD * 2.1
                + THREE YEAR OLDS * 1.6 + FOUR YEAR OLDS * 1.4
                + FIVE YEAR OLDS * 1.1 + SIX YEAR OLDS * 1.0   (1)

and

$$\text{NEWBORN RATE} = \text{NATALITY} \qquad (2)$$

FIGURE 14.1

1: ONE YEAR OLDS   3: THREE YEAR OLDS   5: FIVE YEAR OLDS
2: TWO YEAR OLDS   4: FOUR YEAR OLDS
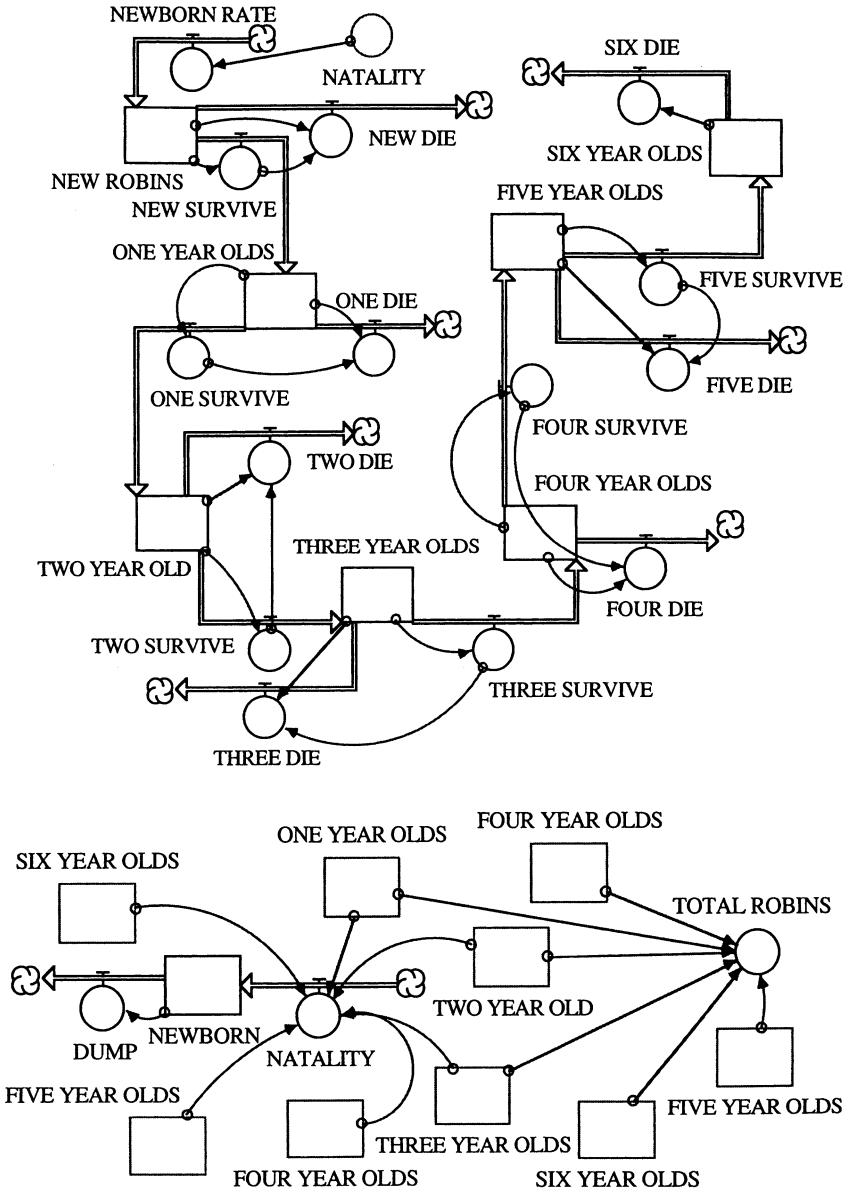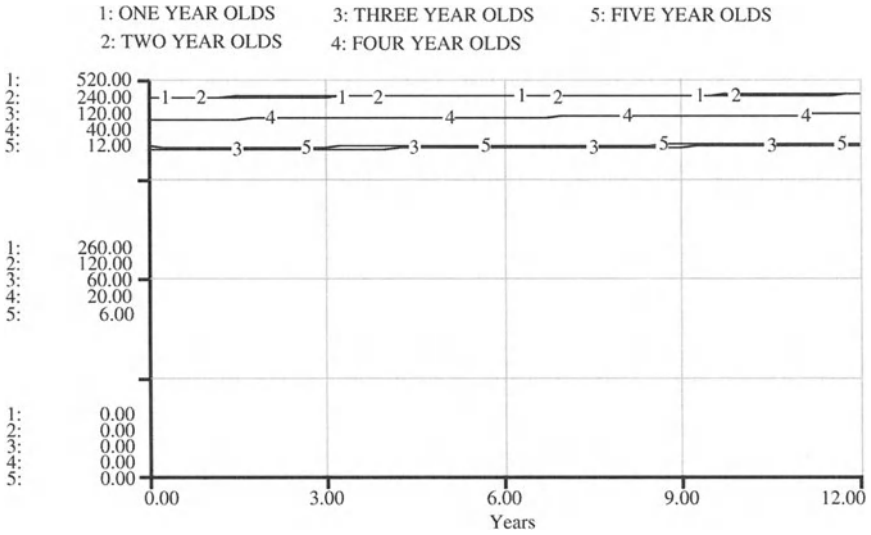
FIGURE 14.2

The complete model is shown in Figure 14.1. It captures the dynamics that underlie Table 14.1. However, the life table only covers the first six years, yet robins may well exceed that age. How should we model the age group of six and more years? In our model we rigged the die-off of the six and older group to give a nearly constant 6+-year-old group population.

Note from the results in Figure 14.2 and Figure 14.3 that all the populations are essentially stable. This is as it should be. Run the model at a

1: SIX YEAR OLDS   2: TOTAL ROBINS
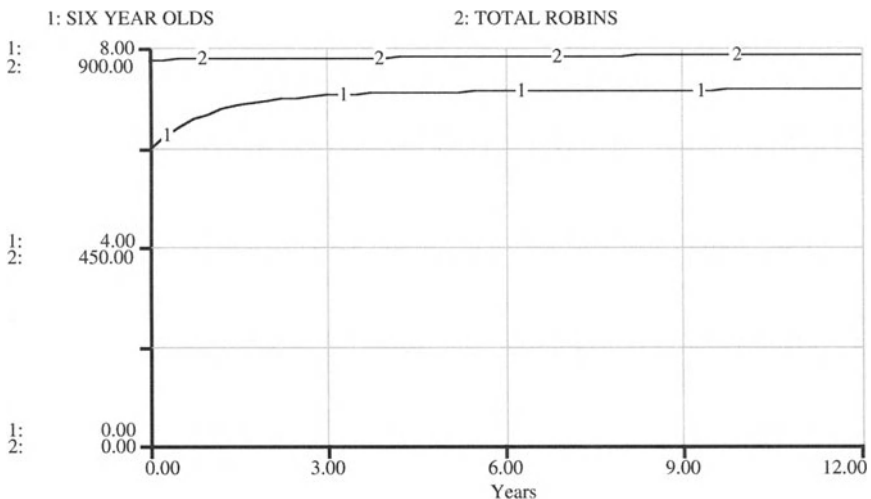
FIGURE 14.3

shorter time step to see if you can improve the accuracy. Try to imagine what would happen if you could change the behavior of the robin by increasing or decreasing its newborn survival rate. Will the model come to a new equilibrium? Do compensating offsets have to be made somewhere (where?) in the model?

ROBIN POPULATION

```
FIVE_YEAR_OLDS(t) = FIVE_YEAR_OLDS(t - dt) +
(FOUR_SURVIVE - FIVE_SURVIVE - FIVE_DIE) * dt
INIT FIVE_YEAR_OLDS = 10   {Initial number of 5 year
olds.}
INFLOWS:
FOUR_SURVIVE = FOUR_YEAR_OLDS*.277
OUTFLOWS:
FIVE_SURVIVE = .6*FIVE_YEAR_OLDS
FIVE_DIE = FIVE_YEAR_OLDS-FIVE_SURVIVE

FOUR_YEAR_OLDS(t) = FOUR_YEAR_OLDS(t - dt) +
(THREE_SURVIVE - FOUR_SURVIVE - FOUR_DIE) * dt
INIT FOUR_YEAR_OLDS = 36
INFLOWS:
THREE_SURVIVE = .364*THREE_YEAR_OLDS
OUTFLOWS:
FOUR_SURVIVE = FOUR_YEAR_OLDS*.277
FOUR_DIE = FOUR_YEAR_OLDS-FOUR_SURVIVE

NEWBORN(t) = NEWBORN(t - dt) + (NATALITY - DUMP) * dt
INIT NEWBORN = 0
INFLOWS:
NATALITY = ONE_YEAR_OLDS*1.4 + TWO_YEAR_OLD*2.1
+THREE_YEAR_OLDS*1.6 + FOUR_YEAR_OLDS*1.4 +
FIVE_YEAR_OLDS*1.1 + SIX_YEAR_OLDS*1.0
OUTFLOWS:
DUMP = NEWBORN

NEW_ROBINS(t) = NEW_ROBINS(t - dt) + (NEWBORN_RATE -
NEW_SURVIVE - NEW_DIE) * dt
INIT NEW_ROBINS = 1400
INFLOWS:
NEWBORN_RATE = NATALITY
OUTFLOWS:
NEW_SURVIVE = .355*NEW_ROBINS
NEW_DIE = NEW_ROBINS-NEW_SURVIVE
```

```
ONE_YEAR_OLDS(t) = ONE_YEAR_OLDS(t - dt) +
(NEW_SURVIVE - ONE_SURVIVE - ONE_DIE) * dt
INIT ONE_YEAR_OLDS = 497
INFLOWS:
NEW_SURVIVE = .355*NEW_ROBINS
OUTFLOWS:
ONE_SURVIVE = .461*ONE_YEAR_OLDS
ONE_DIE = ONE_YEAR_OLDS-ONE_SURVIVE

SIX_YEAR_OLDS(t) = SIX_YEAR_OLDS(t - dt) +
(FIVE_SURVIVE - SIX_DIE) * dt
INIT SIX_YEAR_OLDS = 6   {Initial number of six years
old and older robins.}
INFLOWS:
FIVE_SURVIVE = .6*FIVE_YEAR_OLDS
OUTFLOWS:
SIX_DIE = SIX_YEAR_OLDS*(1-.16)

THREE_YEAR_OLDS(t) = THREE_YEAR_OLDS(t - dt) +
(TWO_SURVIVE - THREE_SURVIVE - THREE_DIE) * dt
INIT THREE_YEAR_OLDS = 99
INFLOWS:
TWO_SURVIVE = .433*TWO_YEAR_OLD
OUTFLOWS:
THREE_SURVIVE = .364*THREE_YEAR_OLDS
THREE_DIE = THREE_YEAR_OLDS -THREE_SURVIVE

TWO_YEAR_OLD(t) = TWO_YEAR_OLD(t - dt) + (ONE_SURVIVE
- TWO_SURVIVE - TWO_DIE) * dt
INIT TWO_YEAR_OLD = 229
INFLOWS:
ONE_SURVIVE = .461*ONE_YEAR_OLDS
OUTFLOWS:
TWO_SURVIVE = .433*TWO_YEAR_OLD
TWO_DIE = TWO_YEAR_OLD-TWO_SURVIVE

TOTAL_ROBINS =
ONE_YEAR_OLDS+TWO_YEAR_OLD+THREE_YEAR_OLDS+FOUR_YEAR_O
LDS+FIVE_YEAR_OLDS+SIX_YEAR_OLDS
```

15

Two-Stage Insect Model

> We may define insects to be little animals without red blood, bones or
> cartilages, furnished with a trunk or else a mouth, opening lengthwise,
> with eyes which they are incapable of covering, and with lungs which
> have their openings in the sides.
> —Thomas Goldsmith, *Natural History*

One of the most advanced areas of dynamic modeling at the organism level
is found in entomology. Insects are an economic animal; they cause billions
of dollars of damage to food supplies around the world every year. We try
to control their population levels, having long ago realized they multiply
and evolve too fast for elimination.

To better understand the dynamics of insect populations, we model the
life cycle of an insect, simplified into two stages, egg and adult. Typically,
the data used in understanding insect population dynamics come from lab-
oratory experiments in which one watches each egg and notes that at the
end of the maturation period only a fraction of the original eggs have sur-
vived. Eggs are only pronounced dead after they fail to mature. This obser-
vation gives us two numbers, the experimental survival fraction, EXP SURV
FRAC, and the experimental maturation time, EXP MATURE TIME. The ex-
perimental maturation time is the mean time to hatch all the hatched eggs.
How can we use such data to parameterize a model, when the model time
step is dramatically different from the maturation time?

We develop a new concept: the model survival fraction, MOD SURV
FRAC. In ecological experiments, the instantaneous survival rate cannot be
measured. But the survival rate can be measured over some real time pe-
riod, the maturation time, by counting the number of eggs surviving to mat-
uration. A problem arises when we wish to model the system at a shorter
time step than the real one. We need to model at these shorter times be-
cause the characteristic time of the system may be shorter than the shortest
feasible measurement time of some particular part of the system. So we
have the experimental time step (the maturation time) and the model time
step (DT), and we must devise a conversion from experiment to model.

That conversion is based on the assumption that the survival fraction is a
declining exponential. First, we simply assume that

$$\text{EXP SURV FRAC}(t) = N(t)/N(t - DT) = \text{EXP}(-m^*t)$$
$$= \text{the dimensionless survival fraction} \qquad (1)$$

N is the population level. Later, when we attempt to verify the experimental
data with our model, we will shut off the birth and hatch rates and observe

the (necessarily exponential) decline in egg population due to death. Then we will realize that the exponential form assumption was necessary. The resulting instantaneous survival fraction is designated by some constant m. Using the experimental data to solve this equation for –m gives:

$$-m = \ln(\text{EXP SURV FRAC(T)})/T \qquad (2)$$

The *model* survival fraction (based on our choice of the time step DT), via the same exponential assumption, is

$$\text{MOD SURV FRAC (DT)} = \text{EXP}(-m*DT) \qquad (3)$$

and when the expression for –m is substituted

$$\text{MOD SURV FRAC(DT)} = \text{EXP}(\ln(\text{EXP SURV FRAC(T)}*DT/T) \qquad (4)$$

which is the basic equation for the model survival fraction. We now have the instantaneous survival fraction, and those surviving will mature or hatch at the modified maturation rate

$$\text{EGGS/T*MOD SURV FRAC(DT)} \qquad (5)$$

that is, the survivors mature at the experimentally found maturation rate, EGGS/T. Remember, eggs don't have to hatch or die. They may simply wait. But survivors do hatch.

To use this fraction, let us look first at the ADULT DEATH rate and the model survival fraction, MOD SURV FRAC divided by DT. This step converts the experimental survival fraction, EXP SURV FRAC, to a daily model survival rate (when DT = 1.0 it is equal to a one-day step). From this we can readily calculate the death rate as 1 – MOD SURV FRAC, divided by DT. A similar calculation has is done to yield the egg death rate—adjusting for the fact that the experimental period is 5 days, not 1 day (as in the adult death rate case).

The model (Fig. 15.1) is run for initial stocks of adults and eggs equal to 0 and 50, respectively, EXP ADULT SURV = 0.8, EXP MATURE TIME = 0.2, EXP SURV FRAC = 0.7, and ADULT SURV TIME = 1. The results are shown in Figure 15.2.

Next, confirm that the model does actually duplicate the experiment. Shut off the birth and hatch processes and put 100 eggs in stock. Observe that 70% of the eggs survive after 5 days, no matter what DT you choose for the model.

Suppose we are uncertain about the exact egg experimental fraction. We may suspect that using literature data is not good enough, and think that this number is within ±10%. Need we do an experiment to find this number if the total number of adults in 24 days is within ±10%. Insert a larval stage into this model with a larval survival fraction of 0.8 in 3 days' maturation time. Why doesn't the stock of adults in this model grow as is did in the first version?
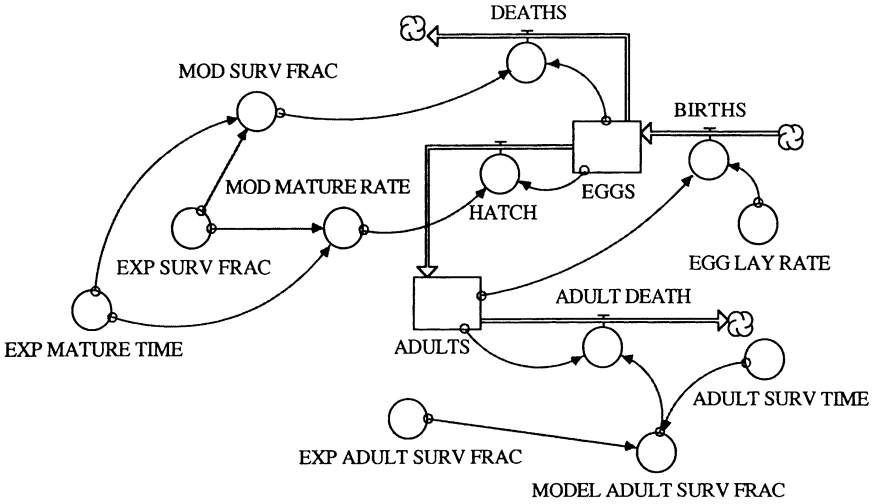
FIGURE 15.1
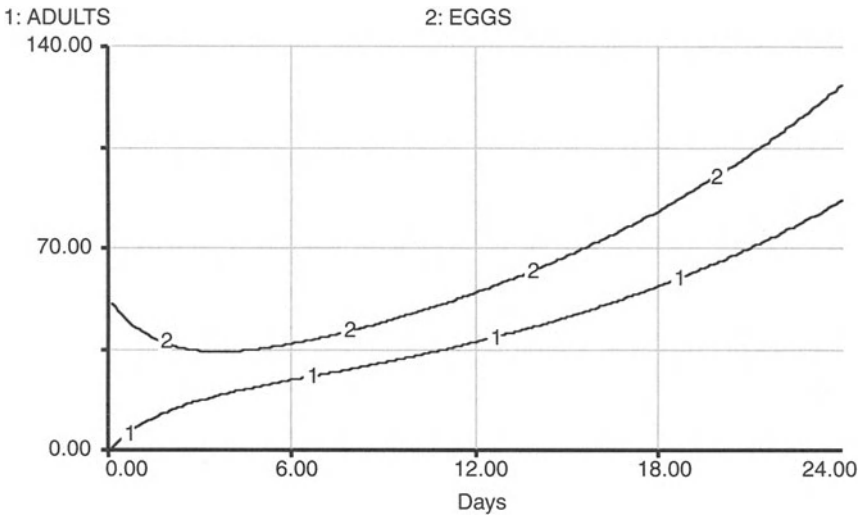
FIGURE 15.2

```
            TWO STAGE INSECT POPULATION MODEL

ADULTS(t) = ADULTS(t - dt) + (HATCH - ADULT_DEATH) * dt
INIT ADULTS = 0
INFLOWS:
HATCH = MOD_MATURE_RATE*EGGS {Some of the eggs neither
die nor mature.}
```

```
OUTFLOWS:
ADULT_DEATH = ADULTS*MODEL_ADULT_SURV_FRAC
EGGS(t) = EGGS(t - dt) + (BIRTHS - DEATHS - HATCH) *
dt
INIT EGGS = 50

INFLOWS:
BIRTHS = EGG_LAY_RATE*ADULTS
OUTFLOWS:
DEATHS = MOD_SURV_FRAC*EGGS
HATCH = MOD_MATURE_RATE*EGGS {Some of the eggs neither
die nor mature.}
ADULT_SURV_TIME = 1
DOCUMENT: One day = T = 1 = experimental period for
which adult mortality is measured.

EGG_LAY_RATE = .5 {Experimental laying rate. EGGS PER
ADULT PER DAY.}
EXP_ADULT_SURV_FRAC = .8 {Experimental daily adult
survival fraction per stage, dimensionless.}
EXP_MATURE_TIME = .2 {Experimental maturation rate,
1/DAY, i.e., 20 eggs per 100 eggs mature each day, as
noted in the experiment. In other words, a surviving
egg matures on the average in five days under the
experimental conditions. T = 5 days.}
EXP_SURV_FRAC = .7   {Experimental egg survival
fraction, dimensionless, per stage. Stage = 1/F1,
i.e., 70 eggs per 100 eggs survive each 1/F1 days, as
noted in the experiment.}
MODEL_ADULT_SURV_FRAC = (1-
EXP(LOGN(EXP_ADULT_SURV_FRAC)*DT/ADULT_SURV_TIME))/DT
{Adult mortality rate, 1/day.  Instantaneous survival
fraction + instantaneous mortality fraction = 1.}
MOD_MATURE_RATE =
EXP_MATURE_TIME*EXP(LOGN(EXP_SURV_FRAC)*DT*EXP_MATURE_T
IME)   {Model maturation rate for survivors, 1/DAY.
Hatch rate = instantaneous survival fraction*model
maturation rate.}
MOD_SURV_FRAC = (1 -
EXP(LOGN(EXP_SURV_FRAC)*EXP_MATURE_TIME*DT))/DT   {Egg
mortality rate, 1/DAY. Instantaneous survival fraction
+ instantaneous mortality fraction = 1.}
```

16
The Zebra Mussel[1]

> If we confine the concept of weeds to species adapted to human distur-
> bance, then man is by definition the first and primary weed under
> whoes influence all other weeds have evolved.
> —Jack R. Harlan, *Crops and Man*

The zebra mussel (*Dreissena polymorpha*) is a small bivalve mollusk native
to Europe. The mussel was first observed in North American lakes only re-
cently. Initial colonization may have occurred in 1986, probably from larvae
discharged in ballast water. The zebra mussel is a potentially serious pest.
In high densities, it presents major problems for both human-made struc-
tures and for the ecology of infested bodies of water. One of the most
harmful impacts is the colonization of intake cribs and pipes serving water
treatment plants, power generating stations, and industries. Attracted to
swift-moving water carrying large amounts of nutrients, mussels quickly
colonize and block these intake pipes. As an efficient feeder, the zebra
mussel is capable of removing large amounts of seston from the water. A
benthic (bottom-dwelling) organism, the zebra mussel effectively removes
nutrients from the water column and deposits them on the bottom of the
lake, river, or estuary in which it lives. In addition to diverting primary pro-
ductivity from the plankton to the benthos, zebra mussels may also cover
substrates used by other organisms and foul sedentary benthic organisms.

There is concern about the long-term effects of shifting large amounts of
organic matter from the pelagic to benthic zones. Effects of these dramatic
changes in water clarity and energy distribution on invertebrate, aquatic
plant, and fish communities have yet to be determined. Therefore, let us
develop a model to investigate the potential impact that the introduction of
zebra mussels will have on a small lake in Northern America and forecast
the potential growth of the zebra mussel in some small lake. The questions
to be answered by the model are: If zebra mussels are introduced into the
lake, how will the population grow over time? What are the critical param-
eters that influence the size of the population? How significant an impact
will the equilibrium mussel population have on the lake's ecosystem?

In order to enhance the understanding of the model we split it up into
three parts, or modules, that are interrelated and together comprise the
critical dynamic aspects of the questions being addressed. Partitioning a
model into individual, easily comprehensible parts is always helpful in
making the model structure transparent. In the model of the zebra mussel,

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the growth module captures the growth processes of the zebra mussel population. The sustainability module determines the long-term, sustainable level of the population by incorporating some key physical parameters. The filtration module yields a relative indication of the impact of the population on the lake by computing the frequency with which the total population filters the entire water volume of the lake. An explanation of the structure of each of these modules and the data and assumptions that were used to construct them follows.

The growth module in Figure 16.1 is made up of four population cohorts (one juvenile and three adult). The average life span of a zebra mussel appears to vary among populations in different areas. We chose a life span of 4 years, which seemed most appropriate for the analysis. Zebra mussel populations typically reproduce once a year (usually sometime during the summer) according to a mass synchronous spawning behavior. Therefore, each cohort represents one generation of mussels. The juvenile cohort represents all those mussels that have successfully attached to some suitable
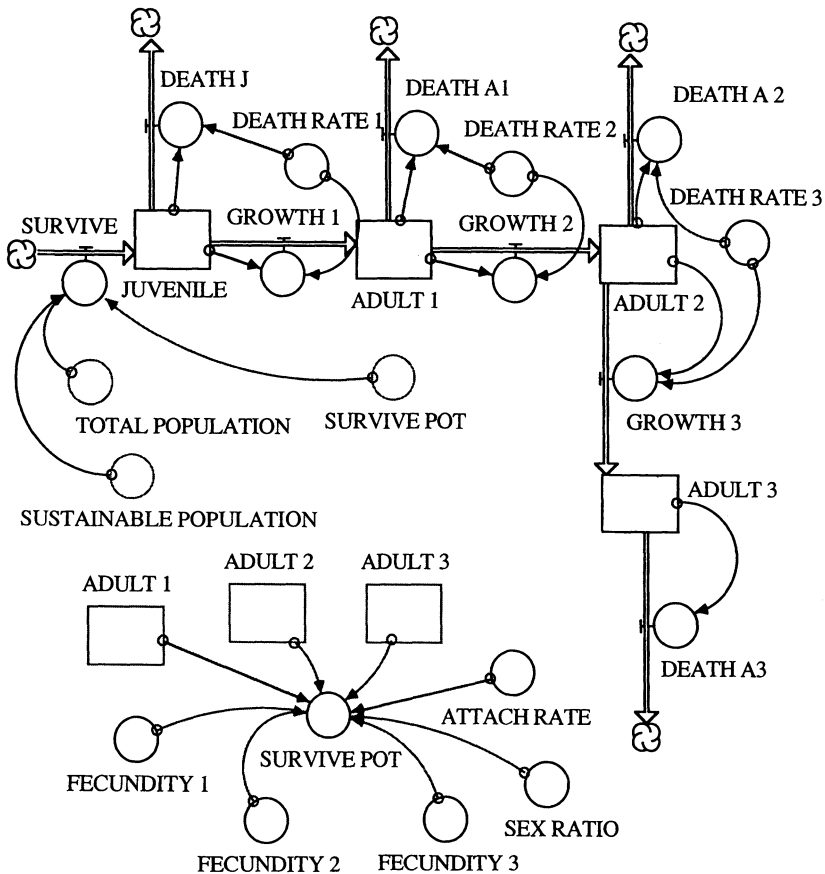
FIGURE 16.1

substrate and have grown to a certain average size by the end of the first year. These individuals are not yet sexually mature. The three adult cohorts are the populations of each generation that have reached sexual maturity and survived to the end of the second, third, and fourth years.

The number of individual mussels that enter the juvenile cohort in a certain year is a function of the number of adult female zebra mussels and their fecundities. Although fecundities are high (30,000 to 40,000 larvae per female), larval mortality rates are fairly high as well. The survival rate (.008) represents the percentage of larvae that are successful in attaching to some suitable substrate and grow to a certain size by the end of the year.

The survival potential is then the number of mussels that reach the juvenile cohort under normal circumstances (no limiting conditions). However, the actual number of mussels that survive the first year depends on whether or not the overall population is close to or above the sustainable population of the lake. The sustainable population is determined in the sustainability module described in detail below. So, the actual number that survive is calculated as follows: If the total population is less than the sustainable population in any year, then SURVIVE equals the smaller of the survival potential, SURVIVE POT, and the difference between SUSTAINABLE POPULATION and TOTAL POPULATION. If the total population is greater than or equal to the sustainable population, then SURVIVE equals SURVIVE POT times a factor that is less than 1 and decreases exponentially as total population gets larger. The result is that the more the total population exceeds the sustainable population, the smaller the number of juveniles that survive during that year.

At first glance, we would expect that the sustainable population will not be exceeded. However, due to the way in which the populations move between cohorts, the total population does in fact overshoot the sustainable level in certain years. Adult populations die at certain rates between years leaving the remainder, which successfully grows to a new average size and enters the next cohort. At the end of the fourth year it is assumed that the entire generation dies out. The sustainability module of the model is shown in Figure 16.2.

The sustainability module determines the equilibrium population level that is expected to be reached over the long term. The key parameter that determines this level is the availability of suitable substrate material upon which the mussels can attach. Zebra mussels require hard substrates and cannot live in muddy conditions. Other potentially limiting factors, such as extremes in water temperature, calcium deficiency, and extremes in nutrient availability, are not considered here.

Available hard substrate is calculated by multiplying the total area of the lake bottom by the percentage of the lake bottom that is hard. This percentage can only be estimated and its initial value is given. To determine the sustainable population at any point in time, the average density and distribution of the population must also be known. Observed density figures
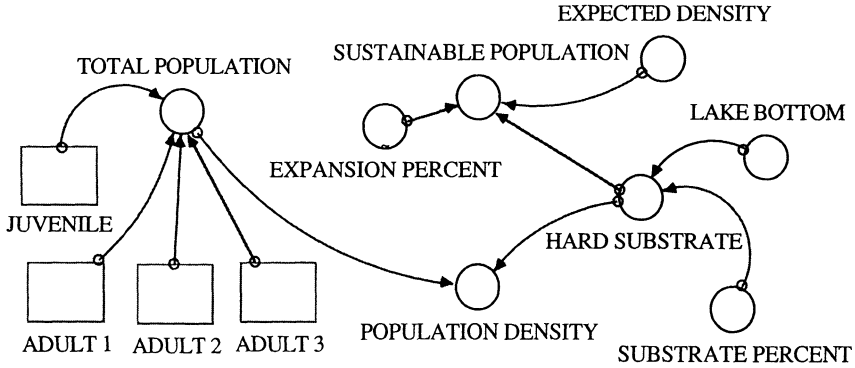
FIGURE 16.2

vary widely in the literature. Therefore, we have chosen a range over which we run our model. Our expected density figure is taken to be that density that is sustainable over the long term.

The expansion percent parameter shows the percentage of the lake that has been colonized over time and is represented in graphical form. We are assuming that infestation occurs at a specific point (as from an infested bait bucket dumped overboard) and spreads from there. Zebra mussel larvae are distributed via lake currents as well as boat traffic. We assume that in the first month 20% of the lake is infested with the mussel. The expansion percentage increases drastically during the first few months and reaches a maximum of 100% by the end of the first year. The percentage expansion remains at that level for the relevant future. For simplicity, we assume the values shown in Figure 16.3.

Try experimenting with the expansion rate in alternative runs of the model. For example, assume that the expansion percentage is reduced to 20% at the beginning of each year, and proceeds to increase over the course of the year again toward 100%. This processes is then repeated each year. Toward that end, the EXPANSION PERCENT can be modeled with the built-in function "MOD." MOD(TIME,12)+1 converts simulation time into months, starting at 1. After 12 months of simulation time, MOD(TIME,12)+1 will reset itself to 1.

In the filtration module of Figure 16.4, LAKE TURNOVER is calculated. This serves as a relative indication of how great an impact the population will have on the current lake ecosystem. LAKE TURNOVER is the number of times per day that the total mussel population filters the total volume of water in the lake. The filtration rate of an individual mussel is a function of its shell length and named here FILTER1, FILTER2, . . . , for the respective age classes 1, 2, ... An initial shell length, LENGTH1, LENGTH2, . . . , is assumed for the juvenile population and shell length grows each year as a function of the previous year's length.
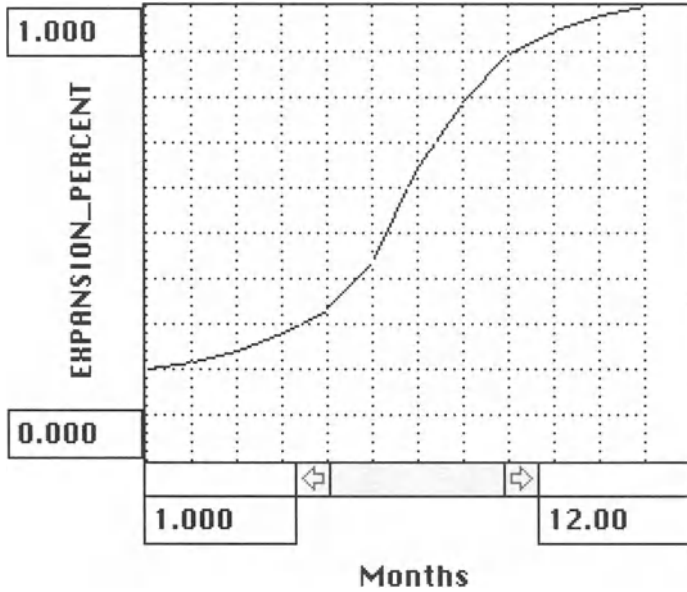
FIGURE 16.3

The amount of water filtered by each cohort is calculated by multiplying the individual filtration rates times the population of each cohort. Daily lake turnover is the sum of the daily filtration volumes of each cohort divided by the total lake volume. In addition, the number of days per complete lake turnover is calculated by dividing the daily lake turnover into one complete turn.
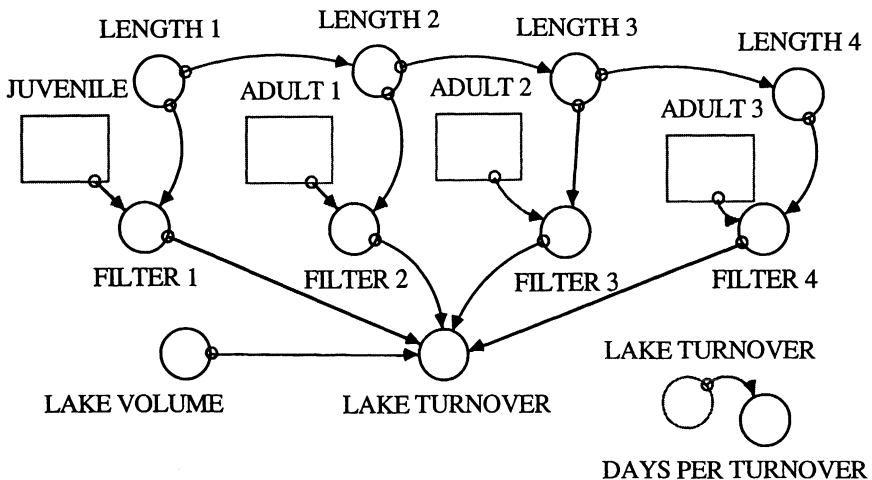
FIGURE 16.4

Now the model is complete and can be used to develop a base case model of zebra mussel growth in our lake. The base case model is based on average values for model parameters as they are published in the literature.[2] You find these data in the model equations at the end of the chapter.

The results of the base case model can be compared with alternative scenarios to determine the impact and sensitivity of various parameters on the growth patterns. Such a procedure is important if not all data are known or given, or if there is uncertainty surrounding parameter estimates.

For a sensitivity analysis, vary the value for SUBSTRATE PERCENT between 50 and 90% and the death rates for the first three age classes within the intervals listed in Table 16.1. Change only one parameter value at a time.

The base case growth, density, and lake turnover pathways are shown in Figure 16.5. We have assumed that the initial invading population consists of 100 juveniles. In the early years, population grows exponentially and reaches a total population peak of $8.08*10^8$ mussels in year 13. The population then oscillates with decreasing extremes toward a steady state population of $7.79*10^8$. Population density follows the same pattern as total population growth. Density peaks at 834 mussels/per square meter and stabilizes at 772 mussels per square meter.

Similarly, lake turnover (Fig. 16.6) closely follows the population pattern since lake turnover is directly related to the number of zebra mussels in the

TABLE 16.1

	Minimum	Maximum
DEATH RATE 1	0.06	0.12
DEATH RATE 2	0.09	0.15
DEATH RATE 3	0.18	0.18

[2]See, for example: Bij de Vaate, A. 1991. Distribution and Aspects of Population Dynamics of the Zebra Mussel, *Dreissena polymorpha* (Pallas, 1771), in the Lake Ijsselmeer Area (The Netherlands). *Oecologia* Vol. 86, pp. 40–50; Griffiths, R.W., W.P. Kovalak, and D.W. Schloesser. 1989. The Zebra Mussel, Dreissena polymorpha, in North America: Impact on Raw Water Users, in *Proceedings: EPRI Service Water System Reliability Improvement Seminar*. Electric Power Research Institute, Palo Alto, CA, pp. 11–27; Griffiths, R.W., D.W. Schloesser, J.H. Leach, and W.P. Kovalak. 1991. Distribution and Dispersal of the Zebra Mussel (*Dreissena polymorpha*) in the Great Lakes Region, *Canadian Journal of Fisheries and Aquatic Science*, Vol. 48, pp. 1381–1388; Haag, W.R. and D.W. Garton. 1992. Synchronous Spawning in a Recently Established Population of the Zebra Mussel, *Dreissena polymorpha*, in Western Lake Erie, USA., *Hydrobiologia*, Vol. 234, pp. 103–110; Kryger, J. and H.U. Riisgard. 1988. Filtration Rate Capacities in 6 Species of European Freshwater Bivalves, *Oecologia*, Vol. 77, pp. 34–38; Mackie, G. 1991. Biology of the Exotic Zebra Mussel, *Dreissena polymorpha*, in Relation to Native Bivalves and Its Potential Impact in Lake St. Clair, Hydrobiologia, Vol. 219, pp. 251–268; Strayer, D.L. 1991. Projected Distribution of the Zebra Mussel, *Dreissena polymorpha*, in North America, *Canadian Journal of Fisheries and Aquatic Science*, Vol. 48, pp. 1389–1395.
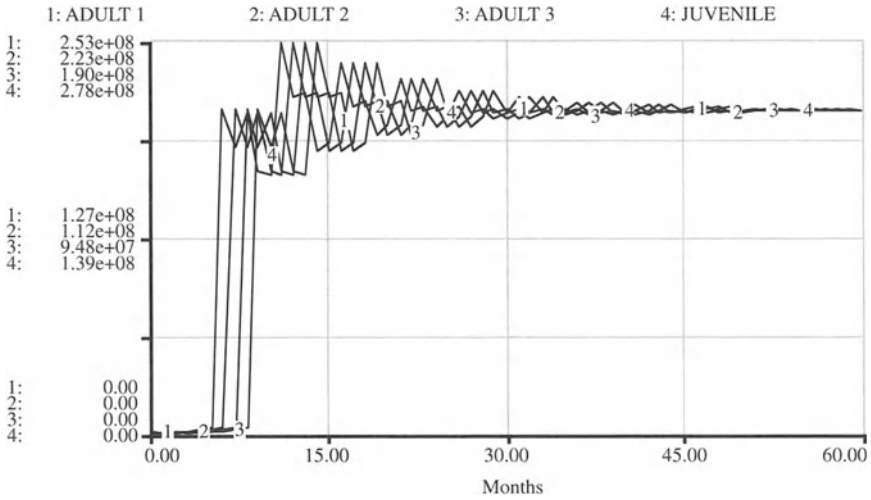
FIGURE 16.5

lake, but it is slightly lagged. Lake turnover peaks in year 14 at 0.37 (total lake turnover per day) and stabilizes at 0.33. This variable is more intuitive if presented as the number of days required for the mussel population to filter the entire volume of the lake. In the peak of the base case, year 14, the mussel population filters the volume of the lake in 2.69 days. This rate changes as population drops and stabilizes at 2.99 days/turnover.
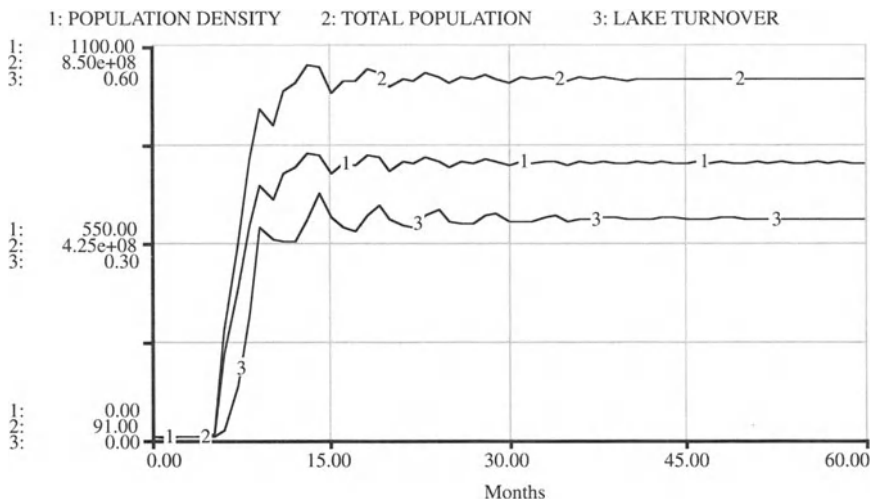
FIGURE 16.6

ZEBRA MUSSEL MODEL

ADULT_1(t) = ADULT_1(t - dt) + (GROWTH_1 - GROWTH_2 -
DEATH_A1) * dt
INIT ADULT_1 = 0 {Number of Individuals}
INFLOWS:
GROWTH_1 = (1-DEATH_RATE_1)*JUVENILE {Individuals per
Month}
OUTFLOWS:
GROWTH_2 = (1-DEATH_RATE_2)*ADULT_1 {Individuals per
Month}
DEATH_A1 = DEATH_RATE_2*ADULT_1 {Individuals per
Month}

ADULT_2(t) = ADULT_2(t - dt) + (GROWTH_2 - GROWTH_3 -
DEATH_A_2) * dt
INIT ADULT_2 = 0 {Number of Individuals}
INFLOWS:
GROWTH_2 = (1-DEATH_RATE_2)*ADULT_1 {Individuals per
Month}
OUTFLOWS:
GROWTH_3 = (1-DEATH_RATE_3)*ADULT_2 {Individuals per
Month}
DEATH_A_2 = DEATH_RATE_3*ADULT_2 {Individuals per
Month}

ADULT_3(t) = ADULT_3(t - dt) + (GROWTH_3 - DEATH_A3) *
dt
INIT ADULT_3 = 0 {Number of Individuals}
INFLOWS:
GROWTH_3 = (1-DEATH_RATE_3)*ADULT_2 {Individuals per
Month}
OUTFLOWS:
DEATH_A3 = 1.0*ADULT_3 {Individuals per Month}

JUVENILE(t) = JUVENILE(t - dt) + (SURVIVE - GROWTH_1 -
DEATH_J) * dt
INIT JUVENILE = 100 {Number of Individuals}
INFLOWS:
SURVIVE = IF (TOTAL_POPULATION<SUSTAINABLE_POPULATION)
THEN MIN(SURVIVE_POT,SUSTAINABLE_POPULATION-
TOTAL_POPULATION)
ELSE ((1-(TOTAL_POPULATION-
SUSTAINABLE_POPULATION)/TOTAL_POPULATION)^2)*SURVIVE_
POT {Number of Individuals per Month}

```
OUTFLOWS:
GROWTH_1 = (1-DEATH_RATE_1)*JUVENILE {Individuals per
Month}
DEATH_J = DEATH_RATE_1*JUVENILE {Individuals per
Month}
ATTACH_RATE = .8*.01 {20% larval mortality in veliger
stage, 99% mortality post-veliger; Bij de Vaate 1991
p.46; Percent of Larvae Which Successfully Attach to
Suitable Substrate and Survive at Least to the End of
the Year}

DAYS_PER_TURNOVER = 1/LAKE_TURNOVER {number of days
for complete lake turnover}
DEATH_RATE_1 = .09
{Bij de Vaate, 1991,p.10}
DEATH_RATE_2 = .12
DEATH_RATE_3 = .15
EXPECTED_DENSITY = 1000
{sq m-2; Strayer; Ch 43}
FECUNDITY_1 = 30000 {Mackie, 1991; Larvae per Female}
FECUNDITY_2 = 35000 {Larvae per Female}
FECUNDITY_3 = 40000 {Mackie, 1991; Larvae per Female}
FILTER_1 = JUVENILE*(6.82*(1.54E-5*LENGTH_1^2.42)^.88)
{L/hr; Kryger and Rilsgard, 1988}
FILTER_2 = ADULT_1*(6.82*(1.54E-5*LENGTH_2^2.42)^.88)
{L/hr; Kryger and Rilsgard, 1988}
FILTER_3 = ADULT_2*(6.82*(1.54E-5*LENGTH_3^2.42)^.88)
{L/hr; Kryger and Rilsgard, 1988}
FILTER_4 = ADULT_3*(6.82*(1.54E-5*LENGTH_4^2.42)^.88)
{L/hr; Kryger and Rilsgard, 1988}
HARD_SUBSTRATE = LAKE_BOTTOM*SUBSTRATE_PERCENT
{sq. m}
LAKE_BOTTOM = (1.3*1E+6+0.142*1E+6)
LAKE_TURNOVER =
(FILTER_1+FILTER_2+FILTER_3+FILTER_4)*24/LAKE_VOLUME
{total lake turnover per day}
LAKE_VOLUME = 8524472*1000 {liters(cubic meters *
1000L/cu.M); BLA}
LENGTH_1 = 5 {mm}
LENGTH_2 = LENGTH_1+(.006*(LENGTH_1^2)-
.56*LENGTH_1+12.1)
{Bij de Vatte, 1991; mm}
LENGTH_3 = LENGTH_2+(.006*(LENGTH_2^2)-
.56*LENGTH_2+12.1) {mm}
```

```
LENGTH_4 = LENGTH_3+(.006*(LENGTH_3^2)-
.56*LENGTH_3+12.1) {mm}
POPULATION_DENSITY = TOTAL_POPULATION/HARD_SUBSTRATE
SEX_RATIO = .6
{Mackie, 1991,p.255}

SUBSTRATE_PERCENT = .70
SURVIVE_POT =
ATTACH_RATE*(SEX_RATIO*(FECUNDITY_1*ADULT_1+FECUNDITY_
2*ADULT_2+FECUNDITY_3*ADULT_3)) {Number of
Individuals}
SUSTAINABLE_POPULATION =
EXPANSION_PERCENT*HARD_SUBSTRATE*EXPECTED_DENSITY
{Number of Individuals}
TOTAL_POPULATION = JUVENILE+ADULT_1+ADULT_2+ADULT_3
{Number of Individuals}
EXPANSION_PERCENT = GRAPH(TIME)
(1.00, 0.2), (2.00, 0.215), (3.00, 0.24), (4.00, 0.28),
(5.00, 0.33), (6.00, 0.435), (7.00, 0.64), (8.00,
0.78), (9.00, 0.89), (10.0, 0.945), (11.0, 0.98),
(12.0, 1.00)
```

17

Single Cell Forest

> The real trouble with our world is not that it is an unreasonable world,
> nor even that it is a reasonable one. The commonest kind of trouble is
> that it is nearly reasonable, but not quite. Life is not an illogicality; yet it
> is a trap for logicians. It looks just a little more mathematical and regu-
> lar than it is; its exactitude is obvious, but its inexactitude is hidden; its
> wildness lies in wait.
>
> —G. K. Chesterton, *Orthodoxy*

In the ecological models of the previous chapters much attention is given to the dynamics of a single population. In contrast, in this chapter we model the succession of trees in a forest ecosystem. In the next we deal with predator-prey interactions.

When we look at a forest we see that different areas of forest are at different stages in the succession of dominant species: Areas that recently underwent a disturbance are characterized by fast-growing species that prosper in good-light conditions. However, their growth generates shade and provides the environment for another species to grow and ultimately take over.

Areas at different successional stages are called patches. A collection of patches forms a landscape. While the population distribution for a specific patch may change dramatically over time, the population distribution for the landscape tends to remain fairly constant. This observation gives rise to many interesting questions. One may ask: What are the most important factors that influence patch size? How does the species composition within a patch change? How does the shape and location of a patch change? Once these questions have been answered, perturbations to this system can be investigated. The effect of block cutting and single species replanting may be modeled or the effect of forest fire suppression.

The factors that influence patch size and longevity can be difficult to model. In order to try to model a patch, as the first step in this overall process, we defined a unit cell. The unit cell is defined as the maximum forest area that is required for a tree in one cell to evenly distribute its seeds to all of the surrounding cells. A patch may contain as little as one cell, or as many cells as are in the landscape.

For this model, the unit cell area was set at 100 (length units^2). The model considers only the succession in one cell. Once the factors that influence succession in this model have been determined, a large grid of cells can be set up and linked together using STELLA's array features discussed in Chapter 4.1.2. Closely follow the discussion below and then try to develop a multicell forest model yourself.

For simplicity, we assume that the forest of our model is composed of only three species of trees, beech, spruce, and redwood. The critical factors influencing tree growth and population were determined to be shade tolerance, maturation time, seed production and dispersal rate, and seed germination probability. Maturation time is defined as a constant for each species. For beech, it is 10 years, for spruce it is 15 years, and for redwood it is 25 years. Seed production and dispersal rate was also defined as a constant. For beech, the seed rate is 5000 seeds per tree per year, for spruce it is 3000, for redwood it is 1000.

Shade tolerance is a characteristic of the species of tree. Beech are modeled as the least shade-tolerant, so that their seed germination probability and death rates are most affected by the amount of sunlight received. Redwood are modeled as the most shade-tolerant.

The seed germination probability is a base percentage of the number of seeds, modified by the amount of rain and sunlight received in the year. For beech trees, the base number of seeds that will germinate is 0.004% of the total number of beech seeds. In a rainy year (RAINFALL $\geq$ 100) this is increased by 0.0009% while in a drought year (RAINFALL $\leq$ 40) it is only increased by 0.0003%. Likewise, if the seeds receive a lot of sunlight (ENERGY REACHING GROUND $\geq$ 900,000) the number of seeds that germinate is increased by 0.0006%, and if the seeds do not get much sunlight (ENERGY REACHING GROUND $\leq$ 50,000) the number is increased by only 0.00003%.

The module of tree growth is shown in Figure 17.1 for the example of beech. The structure of the modules for spruce and redwood is in essence
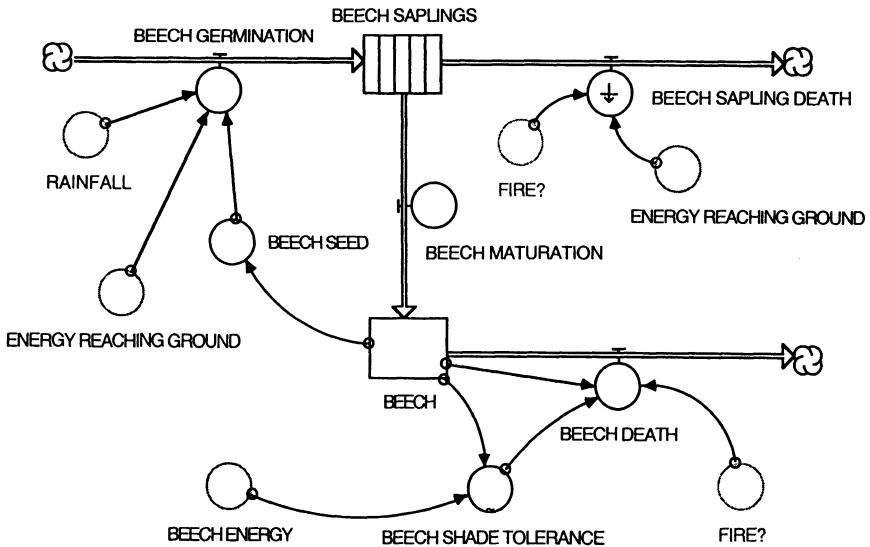
FIGURE 17.1

the same. Note that we have collapsed the age cohorts of beech into two stocks. The mature beech are in the reservoir titled BEECH, while 0- to 10-year-old saplings are in the BEECH SAPLINGS stock. The latter stock is not our traditional reservoir, but a conveyor. The conveyor functions like a conveyor belt. The entities that enter first move along the conveyor and remain on it for a specified transit time. In our case, a transit time of 10 years implies that saplings "move" every year from one slot of the conveyor belt to the next until they leave the conveyor at the end of year 10.

To create a conveyor in STELLA, select the stock symbol and double-click on it. Select Conveyor among the choices listed at the top of the dialog window. The appearance of the stock symbol changes and you are asked to specify a transit time. That would be 10 years in the case of beech, 15 in the case of spruce, and 25 in the case of redwood. The saplings that reach an age of 10 years move on to become mature beech. A second outflow from the BEECH SAPLING conveyor is used to specify "leakage" from the conveyor. These are saplings that die according to their natural mortality rate or an elevated rate due to too little or too much solar energy reaching the ground. Note that when you draw the first outflow from the conveyor (in our case the BEECH MATURATION flow pipe), that outflow is automatically defined by the logic of the conveyor—no question mark appears in the flow symbol asking you to specify the right-hand side of an equation. In contrast, for the second (and any subsequent) outflow you are asked to specify a leakage fraction—a percentage rate by which the values in each slot of the conveyor are reduced in each time period.

In our model, shade tolerance and seed germination probability depend on the amount of energy that the trees receive. This, in turn, depends on how large the tree is (its crown size), and how many of each kind of trees live in a cell. The solar energy flux is set at 1,000,000 units. The size of a beech tree crown is defined as 2 (area units). Therefore, a cell can comfortably contain up to 50 beech trees. If the number of beech trees exceeds 50, then they are in competition for the sunlight, and the energy available per tree is reduced. In addition, as the number of trees increases, the amount of sunlight (energy) that reaches the ground (either directly or through the leaves of the trees) decreases. This means that the seeds and seedlings have less energy available to them and they will not germinate or grow as quickly. The crown size for spruce is defined as 5 and for redwood it is 20.

The module for the calculations of the canopy size and energy availability is shown in Figure 17.2. Tree biomass by species is modeled as in Figure 17.3 with BIOMASS CONTRIBUTION reflecting the impact that different amounts of biomass may have on the occurrence of forest fires. Forest fires are discussed below.

All units here are rather arbitrary. What matters is their relative order of magnitude. The purpose here is not to develop an exactly quantified model but rather to illuminate the dynamics behind the succession of the various tree species in our artificial forest. Of course, with some extra effort you can
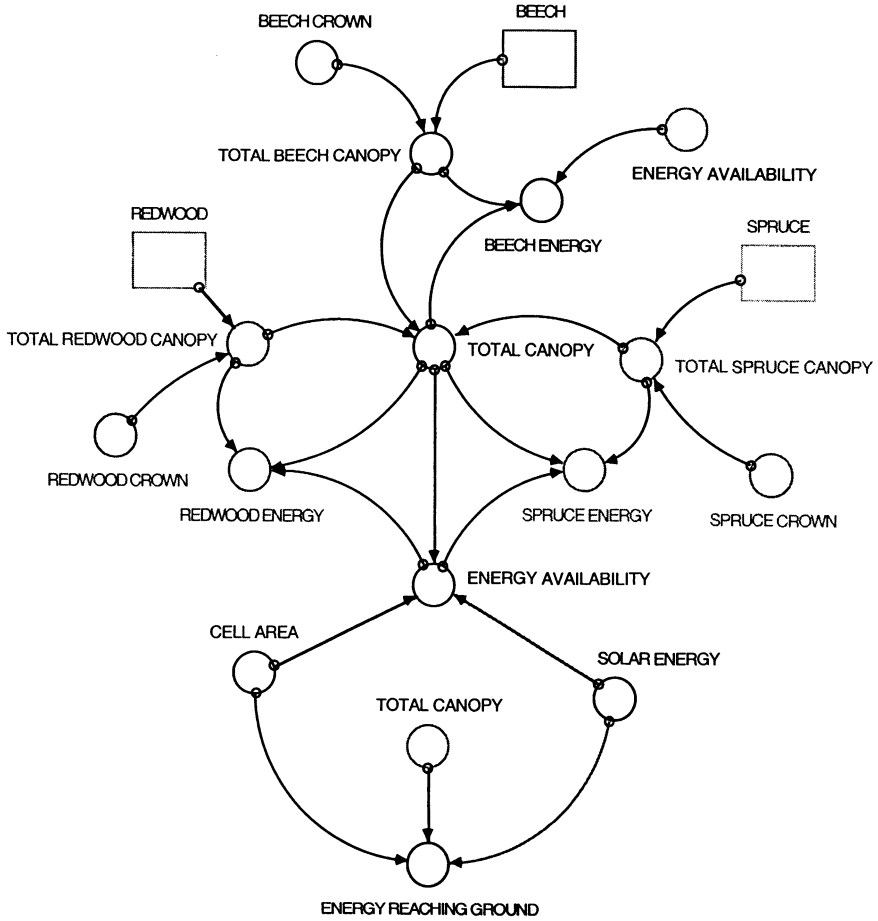
FIGURE 17.2

find meaningful data to quantify the model and play out its dynamics. The quantitative results will be different, but the qualitative insights that this model generates should not be substantially changed. We have chosen here to make our model descriptive rather than a predictive. It provides us with some interesting insight into forest dynamics and can serve as a guide for data collection and analysis.

To investigate the dynamics of our forest system, we introduce a like-lihood for forest fires to occur. In the absence of human interventions, forest fires are caused exclusively by lightning strikes. In this model, the frequency of lightning strikes depends on the amount of annual rainfall. If rainfall is high in a given year, the likelihood for storms increases, and thus a higher number of lightning strikes can be caused. Whether or not a light-ning strike causes a fire depends on several factors. The first of these is the
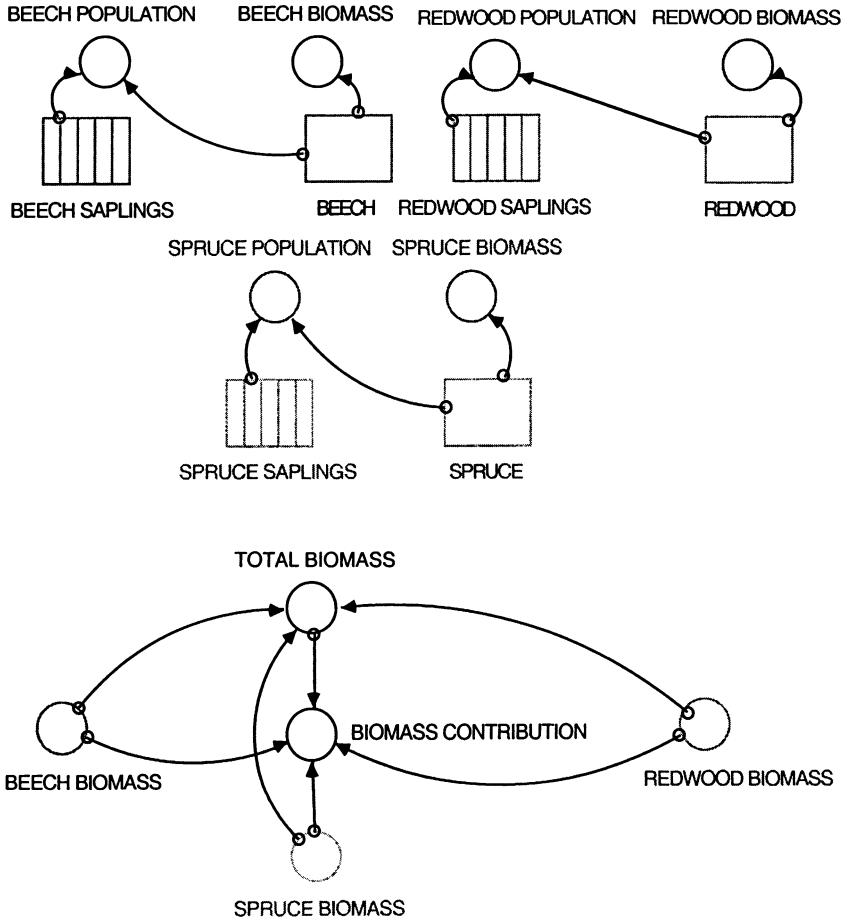
FIGURE 17.3

amount of rainfall for the year. In a wet year, lightning strikes are less likely to cause a fire than during drought. Other factors that can contribute are the number of years since the last fire. If it has been a long time since a fire, there is a large amount of forest floor debris, that when it becomes dry can very easily touch off a fire. The module to calculate forest fire occurrence is shown in Figure 17.4.

Run the model first without the possibility of any fires. To do this change the equation for lightning from

$$\text{LIGHTNING} = \text{IF RAINFALL} > 90 \text{ THEN } 1 \text{ ELSE } 0 \qquad (1)$$

to

$$\text{LIGHTNING} = \text{IF RAINFALL} > 90 \text{ THEN } 0 \text{ ELSE } 0 \qquad (2)$$
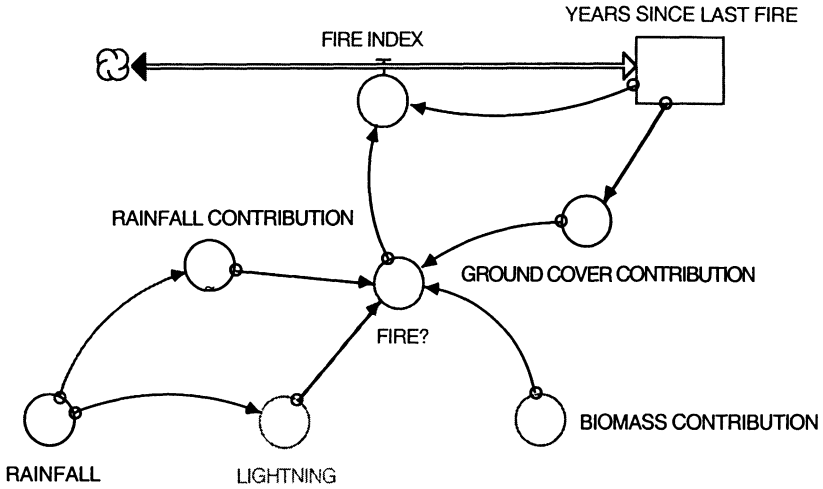
FIGURE 17.4

The resulting population curve for each of the species is shown in Figure 17.5. Beech trees start out with a large population and maintain their dominance for a bit more than 125 years. Then the redwood population becomes the dominant one. The spruce population experiences a peak in population after around 100 years, but then it decreases, and it never becomes dominant.

These results of Figure 17.5 match to some degree the succession of species in a forest patch in which natural and human-induced disturbances are
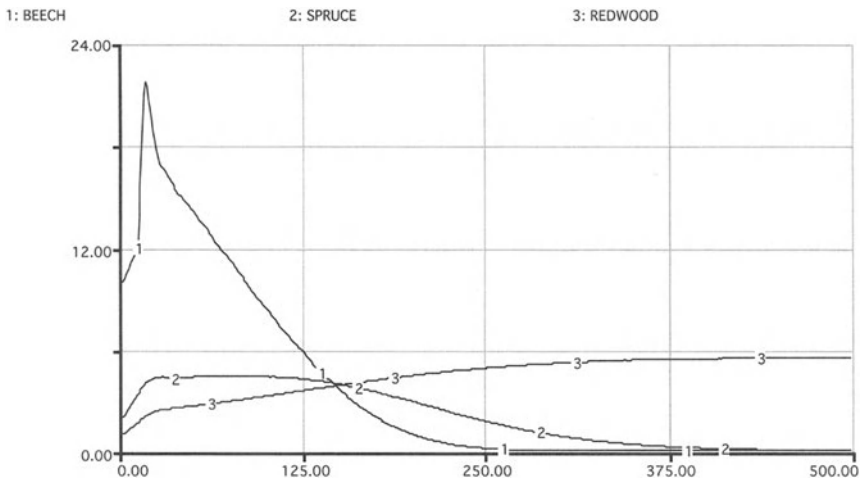
FIGURE 17.5
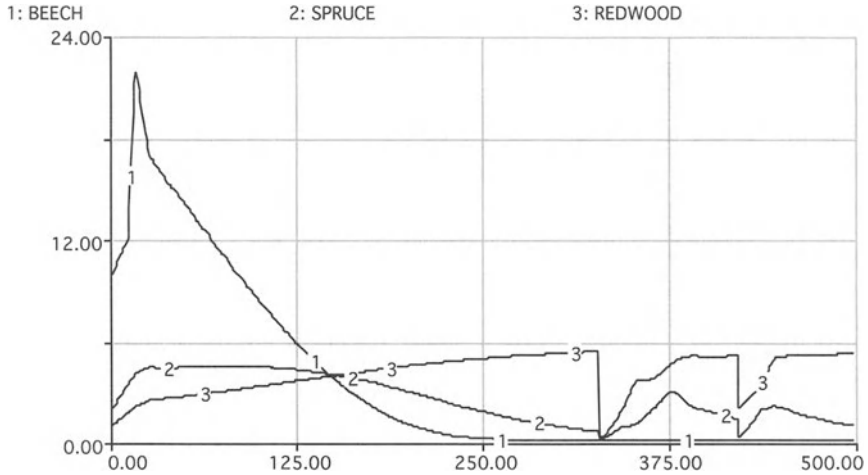
FIGURE 17.6

minimized. We can use this model to investigate the impacts of forest fires on species composition. As you can see from Figure 17.6, in which we allowed for forest fires to occur, no species ever has time to become dominant after each round of fires. The tree populations greatly decrease, and then they would increase to some degree, but the fires occurred too frequently to allow any species to continuously dominate a cell.

Run the model several times without making any changes to it. Since there are several random elements (outbreak of fires and death rates of trees), each run will show a completely different result. As we indicated above, the parameters used here are simply for illustrative purposes, and the results will also vary for different parameter choices. For example, change the fire probability and observe the results. Alternatively, change the rainfall pattern. Then return to our suggestion above and develop a multicell forest model using STELLA's array features. In that model, you may want to allow any cell to contribute seeds of each species for germination on a neighboring cell. Set up the conditions such that beech generate more seeds than spruce, and spruce more than redwood. Additionally, consider different ways in which a forest fire may spread from cell to cell. How do these spatial features influence the population dynamics in your forest?

SINGLE CELL FOREST

BEECH MODULE
```
BEECH(t) = BEECH(t - dt) + (BEECH_MATURATION -
BEECH_DEATH) * dt
INIT BEECH = 10 {Initial Beech trees in Box}
```

```
INFLOWS:
BEECH_MATURATION = CONVEYOR OUTFLOW
OUTFLOWS:
BEECH_DEATH = IF FIRE?=1 THEN (.95 +
.05*RANDOM(0,1))*BEECH ELSE BEECH_SHADE_TOLERANCE*BEECH
{Death fraction of beech trees}

BEECH_SAPLINGS(t) = BEECH_SAPLINGS(t - dt) +
(BEECH_GERMINATION - BEECH_MATURATION -
BEECH_SAPLING_DEATH) * dt
INIT BEECH_SAPLINGS = 5 {Initial beech saplings in
cell}
        TRANSIT TIME = 10
        INFLOW LIMIT = ∞
        CAPACITY = ∞
INFLOWS:
BEECH_GERMINATION = (.00004 +
(ENERGY_REACHING_GROUND/100000000000) +
(RAINFALL/22000000)) * BEECH_SEED {The number of beech
seeds that germinate per year}
OUTFLOWS:
BEECH_MATURATION = CONVEYOR OUTFLOW
BEECH_SAPLING_DEATH = LEAKAGE OUTFLOW
        LEAKAGE FRACTION = IF FIRE?=1 THEN 1 ELSE IF
ENERGY_REACHING_GROUND < 100000 THEN (1.2-
(ENERGY_REACHING_GROUND/100000)) ELSE IF
(ENERGY_REACHING_GROUND<200000 AND
ENERGY_REACHING_GROUND≥100000) THEN (0.1-
((ENERGY_REACHING_GROUND-200000)/1000000)) ELSE 0.1
        NO-LEAK ZONE = 0
BEECH_SEED = 5000*BEECH {Number of beech seeds
produced each year}
BEECH_SHADE_TOLERANCE = GRAPH(BEECH_ENERGY/BEECH)
(0.00, 1.00), (2000, 0.9), (4000, 0.805), (6000,
0.705), (8000, 0.6), (10000, 0.5), (12000, 0.395),
(14000, 0.295), (16000, 0.195), (18000, 0.105), (20000,
0.025)
```

BIOMASS MODULE
```
BEECH_BIOMASS = 2*BEECH
BEECH_POPULATION = BEECH+BEECH_SAPLINGS {Total number
of beech trees in cell}
BIOMASS_CONTRIBUTION = IF (TOTAL_BIOMASS >= 90 AND
(BEECH_BIOMASS/TOTAL_BIOMASS >= 0.75 OR
SPRUCE_BIOMASS/TOTAL_BIOMASS >= 0.75 OR
```

REDWOOD_BIOMASS/TOTAL_BIOMASS >= 0.75)) THEN 0.05 ELSE
IF (TOTAL_BIOMASS >= 90 AND
(BEECH_BIOMASS/TOTAL_BIOMASS >= 0.9 OR
SPRUCE_BIOMASS/TOTAL_BIOMASS >= 0.9 OR
REDWOOD_BIOMASS/TOTAL_BIOMASS >= 0.9)) THEN 0.15 ELSE
0
REDWOOD_BIOMASS = 10*REDWOOD
REDWOOD_POPULATION = REDWOOD+REDWOOD_SAPLINGS {Total
number of redwood trees in cell}
SPRUCE_BIOMASS = 5*SPRUCE
SPRUCE_POPULATION = SPRUCE+SPRUCE_SAPLINGS {Total
number of spruce trees in box}
TOTAL_BIOMASS =
BEECH_BIOMASS+SPRUCE_BIOMASS+REDWOOD_BIOMASS

ENERGY MODULE
BEECH_CROWN = 2 {Average amount of shade that one
Beech tree casts. Area/tree}
BEECH_ENERGY =
ENERGY_AVAILABILITY*TOTAL_BEECH_CANOPY/TOTAL_CANOPY
{Amount of energy received by beech trees}
CELL_AREA = 100 {Cell area available for tree growth}
ENERGY_AVAILABILITY = IF (TOTAL_CANOPY > CELL_AREA)
THEN SOLAR_ENERGY*CELL_AREA/TOTAL_CANOPY ELSE
SOLAR_ENERGY
ENERGY_REACHING_GROUND = IF (TOTAL_CANOPY < CELL_AREA)
THEN SOLAR_ENERGY*(1-(0.95*TOTAL_CANOPY/CELL_AREA))
ELSE SOLAR_ENERGY*.05
REDWOOD_CROWN = 20 {Average amount of shade that one
Redwood tree casts. Area/tree}
REDWOOD_ENERGY =
ENERGY_AVAILABILITY*TOTAL_REDWOOD_CANOPY/TOTAL_CANOPY
{Amount of energy received by redwood trees}
SPRUCE_CROWN = 5 {Average amount of shade that one
Spruce tree casts. Area/tree}
SPRUCE_ENERGY =
ENERGY_AVAILABILITY*TOTAL_SPRUCE_CANOPY/TOTAL_CANOPY
{Amount of energy received by spruce trees}
TOTAL_BEECH_CANOPY = BEECH*BEECH_CROWN {Total area
covered by beech trees}
TOTAL_CANOPY =
TOTAL_BEECH_CANOPY+TOTAL_REDWOOD_CANOPY+TOTAL_SPRUCE_
CANOPY

{Total area covered by all trees}
TOTAL_REDWOOD_CANOPY = REDWOOD*REDWOOD_CROWN {Total
area covered by redwood trees}
TOTAL_SPRUCE_CANOPY = SPRUCE*SPRUCE_CROWN {Total area
covered by spruce trees}

FIRE MODULE
YEARS_SINCE_LAST_FIRE(t) = YEARS_SINCE_LAST_FIRE(t -
dt) + (FIRE_INDEX) * dt
INIT YEARS_SINCE_LAST_FIRE = 0
INFLOWS:
FIRE_INDEX = IF FIRE? = 1 THEN (-
1*YEARS_SINCE_LAST_FIRE) ELSE 1
{Increments Years Since Last Fire by 1 if there is no
fire, otherwise it resets YSLF to 0}

FIRE? = IF (LIGHTNING > 0) AND (RANDOM(0,1) <
BIOMASS_CONTRIBUTION+RAINFALL_CONTRIBUTION+GROUND_
COVER_CONTRIBUTION) THEN 1 ELSE 0
GROUND_COVER_CONTRIBUTION = IF
YEARS_SINCE_LAST_FIRE>20 THEN 0.2 ELSE 0
RAINFALL_CONTRIBUTION = GRAPH(RAINFALL)
(0.00, 0.67), (10.0, 0.5), (20.0, 0.35), (30.0, 0.25),
(40.0, 0.17), (50.0, 0.1), (60.0, 0.035), (70.0, 0.00),
(80.0, -0.05), (90.0, -0.1), (100, -0.19), (110, -
0.28), (120, -0.39), (130, -0.5)

REDWOOD MODULE
REDWOOD(t) = REDWOOD(t - dt) + (REDWOOD_MATURATION -
REDWOOD_DEATH) * dt
INIT REDWOOD = 1 {Initial Redwood trees in cell}
INFLOWS:
REDWOOD_MATURATION = CONVEYOR OUTFLOW
OUTFLOWS:
REDWOOD_DEATH = IF FIRE?=1 THEN (0.5 +
0.5*RANDOM(0,1))*REDWOOD ELSE
REDWOOD_SHADE_TOLERANCE*REDWOOD
REDWOOD_SAPLINGS(t) = REDWOOD_SAPLINGS(t - dt) +
(REDWOOD_GERMINATION - REDWOOD_MATURATION -
REDWOOD_SAPLING_DEATH) * dt
INIT REDWOOD_SAPLINGS = 2
        TRANSIT TIME = 25
        INFLOW LIMIT = ∞

```
        CAPACITY = ∞
INFLOWS:
REDWOOD_GERMINATION = (.00004 +
(ENERGY_REACHING_GROUND/15E10) + (RAINFALL/11E6)) *
REDWOOD_SEED
OUTFLOWS:
REDWOOD_MATURATION = CONVEYOR OUTFLOW
REDWOOD_SAPLING_DEATH = LEAKAGE OUTFLOW
        LEAKAGE FRACTION = IF FIRE?=1 THEN 1 ELSE IF
ENERGY_REACHING_GROUND < 1E5 THEN (1.2-
(ENERGY_REACHING_GROUND/100000)) ELSE IF
(ENERGY_REACHING_GROUND < 2E5 AND
ENERGY_REACHING_GROUND ≥ 1E5) THEN (0.1-
((ENERGY_REACHING_GROUND-2E5)/1E6)) ELSE 0.1
        NO-LEAK ZONE = 0

REDWOOD_SEED = 1000*REDWOOD
REDWOOD_SHADE_TOLERANCE =
GRAPH(REDWOOD_ENERGY/REDWOOD)
(0.00, 1.00), (20000, 0.54), (40000, 0.31), (60000,
0.195), (80000, 0.095), (100000, 0.045), (120000,
0.025), (140000, 0.02), (160000, 0.015), (180000,
0.01), (200000, 0.005)
```

SPRUCE MODULE
```
SPRUCE(t) = SPRUCE(t - dt) + (SPRUCE_MATURATION -
SPRUCE_DEATH) * dt
INIT SPRUCE = 2  {Initial Spruce trees in cell}
INFLOWS:
SPRUCE_MATURATION = CONVEYOR OUTFLOW
OUTFLOWS:
SPRUCE_DEATH = IF FIRE?=1 THEN (0.7 +
0.3*RANDOM(0,1))*SPRUCE ELSE
SPRUCE_SHADE_TOLERANCE*SPRUCE

SPRUCE_SAPLINGS(t) = SPRUCE_SAPLINGS(t - dt) +
(SPRUCE_GERMINATION - SPRUCE_MATURATION -
SPRUCE_SAPLING_DEATH) * dt
INIT SPRUCE_SAPLINGS = 3
        TRANSIT TIME = 15
        INFLOW LIMIT = ∞
        CAPACITY = ∞
```

```
INFLOWS:
SPRUCE_GERMINATION = (.00004 +
(ENERGY_REACHING_GROUND/150000000000) +
(RAINFALL/11000000)) * SPRUCE_SEED
OUTFLOWS:
SPRUCE_MATURATION = CONVEYOR OUTFLOW
SPRUCE_SAPLING_DEATH = LEAKAGE OUTFLOW
       LEAKAGE FRACTION = IF FIRE? = 1 THEN 1 ELSE IF
ENERGY_REACHING_GROUND < 100000 THEN (1.2-
(ENERGY_REACHING_GROUND/100000)) ELSE IF
(ENERGY_REACHING_GROUND<200000 AND
ENERGY_REACHING_GROUND≥100000) THEN (0.1-
((ENERGY_REACHING_GROUND-200000)/1000000)) ELSE 0.1
       NO-LEAK ZONE = 0

SPRUCE_SEED = 3000*SPRUCE
SPRUCE_SHADE_TOLERANCE = GRAPH(SPRUCE_ENERGY/SPRUCE)
(0.00, 0.995), (5000, 0.605), (10000, 0.42), (15000,
0.31), (20000, 0.255), (25000, 0.2), (30000, 0.15),
(35000, 0.105), (40000, 0.08), (45000, 0.05), (50000,
0.02)
```

18

Predator-Prey Models

He shall have his life for a prey, and shall live.
—Jeremiah 38:2

18.1 Basic Model of Predator-Prey Interactions

In this chapter we continue to model multispecies assemblages. But in contrast to the previous chapter, the two-species model developed here consists of a predator population and a prey population (Fig. 18.1). Both are present within a given area with a specified carrying capacity. The birth rates for predators and prey are given exogenously, yet the number of births depends on predator-prey interaction and the carrying capacity of their ecosystem:

$$\text{BIRTHS PREDATOR} = \text{BIRTH RATE PREDATOR}$$
$$* (\text{PREDATORS} - \text{DEATHS PREDATOR}) \quad (1)$$
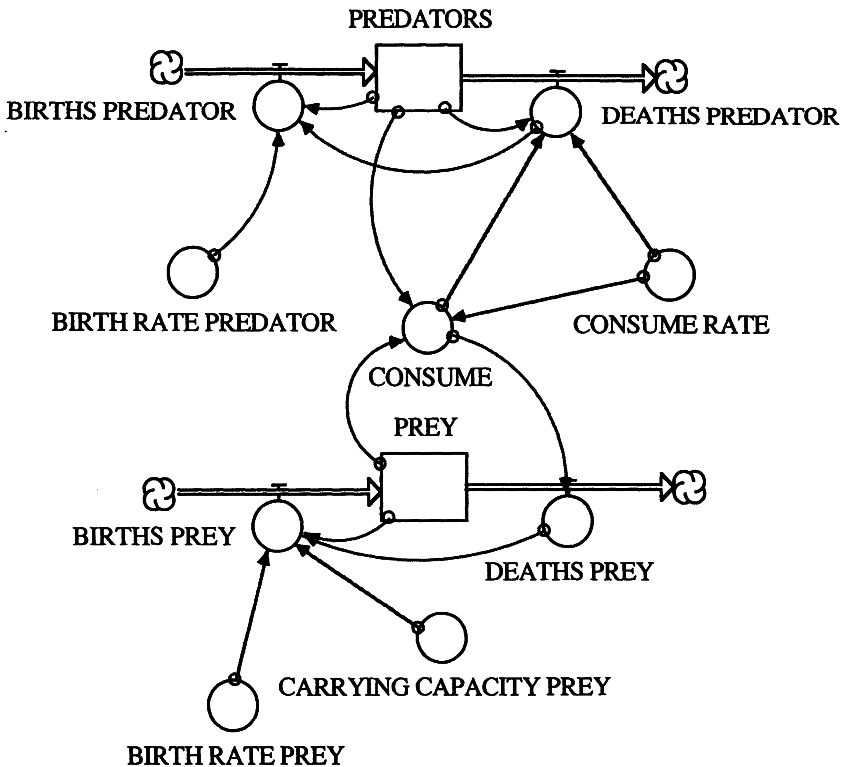
FIGURE 18.1

$$\text{BIRTHS PREY} = (\text{PREY-DEATHS PREY}) * \text{BIRTH RATE PREY}$$
$$* (1-\text{PREY/CARRYING CAPACITY PREY}) \qquad (2)$$

Similarly, the deaths of predators and prey depend on their interaction—the prey consumed by predators.

$$\text{DEATHS PREDATOR} = \text{PREDATORS} - \text{CONSUME/CONSUME RATE} \quad (3)$$

$$\text{DEATHS PREY} = \text{CONSUME} \qquad (4)$$

To ensure that the predators of our model do not CONSUME more prey than available, we make use of STELLA's built-in function MIN, which generates the smaller of two numbers. In our case

$$\text{CONSUME} = \text{MIN}(\text{PREY,CONSUME RATE} * \text{PREDATORS}) \qquad (5)$$

Run the model for different initial population sizes, birth and death rates, and carrying capacities. Can you save the populations from crashing?[1] In the model run below (Fig. 18.2) we have apparently not been able to avoid the demise of the two populations.

What are we missing in this model that allows real-world predator-prey systems to continue to exist? We'll provide one answer in the following section of this chapter. Remove the DEATHS PREDATOR connection to BIRTHS PREDATOR and rerun the model. Do you get a difference in the result? Why (not)?
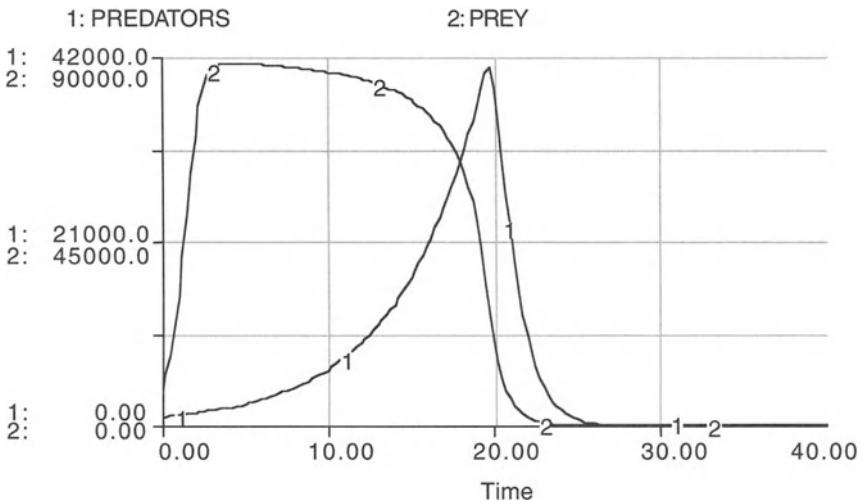
FIGURE 18.2

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[1] For modifications of the predator-prey models with oscillations around population equilibria and stable limit cycles see Swart, J. 1990. A System Dynamics Approach to Predator-Prey Modeling, *System Dynamics Review,* Vol. 6, pp. 94–99.

BASIC PREDATOR PREY MODEL

```
PREDATORS(t) = PREDATORS(t - dt) + (BIRTHS_PREDATOR -
DEATHS_PREDATOR) * dt
INIT PREDATORS = 900 {Individuals}
INFLOWS:
BIRTHS_PREDATOR = BIRTH_RATE_PREDATOR*(PREDATORS-
DEATHS_PREDATOR) {Individuals per Time Period}
OUTFLOWS:
DEATHS_PREDATOR = PREDATORS-CONSUME/CONSUME_RATE
{Individuals per Time Period}

PREY(t) = PREY(t - dt) + (BIRTHS_PREY - DEATHS_PREY) *
dt
INIT PREY = 9000 {Individuals}
INFLOWS:
BIRTHS_PREY = (PREY-DEATHS_PREY)*BIRTH_RATE_PREY*(1-
PREY/CARRYING_CAPACITY_PREY) {Individuals per Time
Period}
OUTFLOWS:
DEATHS_PREY = CONSUME {Individuals per Time Period}

BIRTH_RATE_PREDATOR = .2 {Individuals per Individuals
per Time Period}
BIRTH_RATE_PREY = 2 {Individuals per Individuals per
Time Period}
CARRYING_CAPACITY_PREY = 90000 {Individuals}
CONSUME = MIN(PREY,CONSUME_RATE*PREDATORS)
{Individuals per Time Period}
CONSUME_RATE = 1 {Number of Prey per Time Period per
Number of Predators}
```

18.2 Spatial Predator-Prey Model[2]

The model in the previous section had all predators and prey in the same place. There, the populations of predators and prey always crashed. How would our result differ if we broke up the same area into nine subcells, each with one-ninth of the original carrying capacity. All other model fea-

---

[2]This model is based on the project by one of our students, Jim Westervelt.

tures held the same, but now, we have predators and prey move among the cells, "chasing or fleeing from each other."

The prey migrate routinely regardless of their population in the starting or the receiving cells, and the predators migrate to a new cell when they begin to starve in their current cell. Once the migration quantity is established, a random process determines its distribution to adjacent cells. This simple idea enables the prey to "escape" to a neighboring cell where the predator population may be at a relatively low level. But the cellular model is more realistic than the single cell model; it adds geographic reality to the problem and it allows for differentiation of predator and prey behavior.

The STELLA model for the first cell of the nine-cell predator-prey model is shown in Figure 18.3. This first cell is in the upper left-hand corner of the square nine-cell model. Consequently, predators and prey can move to the
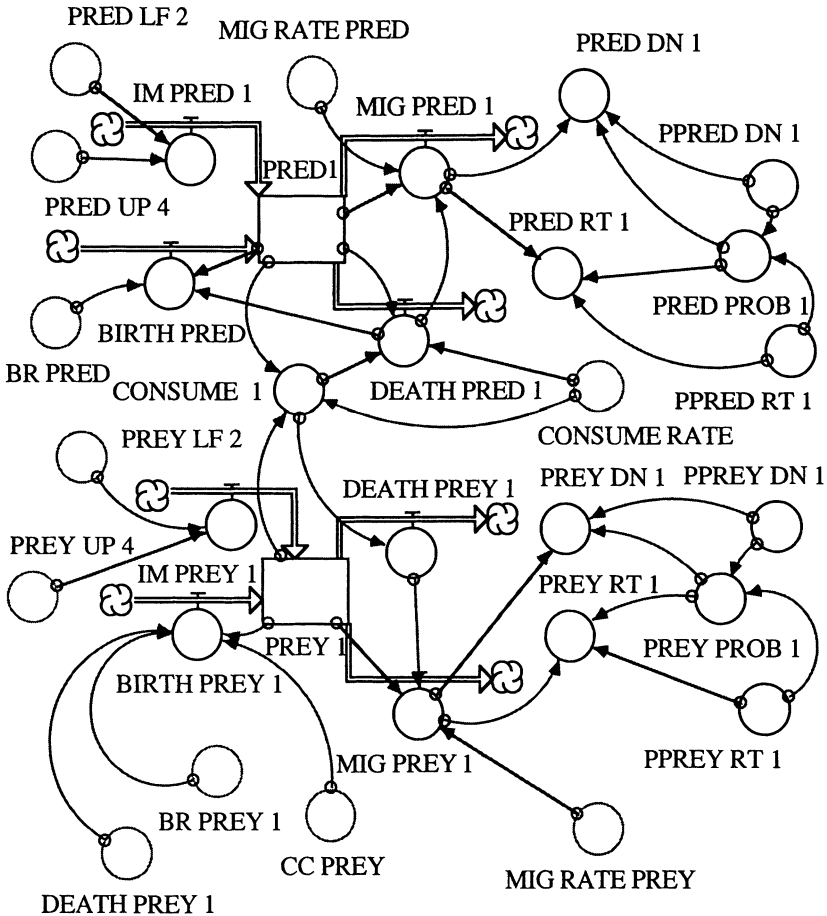
FIGURE 18.3

right (RT) and down (DN). The fraction of predators and prey that move is a random number between 0 and 1. The migration rate of prey, MIG RATE PREY, is specified exogenously. The number of prey consumed in cell 1 cannot exceed the number of prey in that cell and is at least 0 and at most the consumption rate times the number of predators in that cell. Thus,

$$\text{CONSUME 1} = \text{MIN(PREY 1, CONSUME RATE * PRED 1)} \qquad (6)$$

The consumption rate, in turn, is also given exogenously.

The migration rate of predators, MIG RATE PRED, depends on the availability of prey in the cell. For simplicity, we assume that the number of predators that migrate is the product of the number of starvation deaths and the migration rate of predators.

A listing of the STELLA equations for the first cell of the nine-cell model is given at the end of this chapter. The graphical representation of that cell is shown in Figure 18.3. In order to build the nine-cell model, the diagram is just duplicated and slightly modified. The modification necessary includes, besides a change of variable names, rules that enable movement in the proper directions.

The nine parts of the spatial predator-prey model can be combined to visually represent the predator and prey populations of each cell at a point in time (Fig. 18.4). Develop "Ghosts" in STELLA and animate them so that
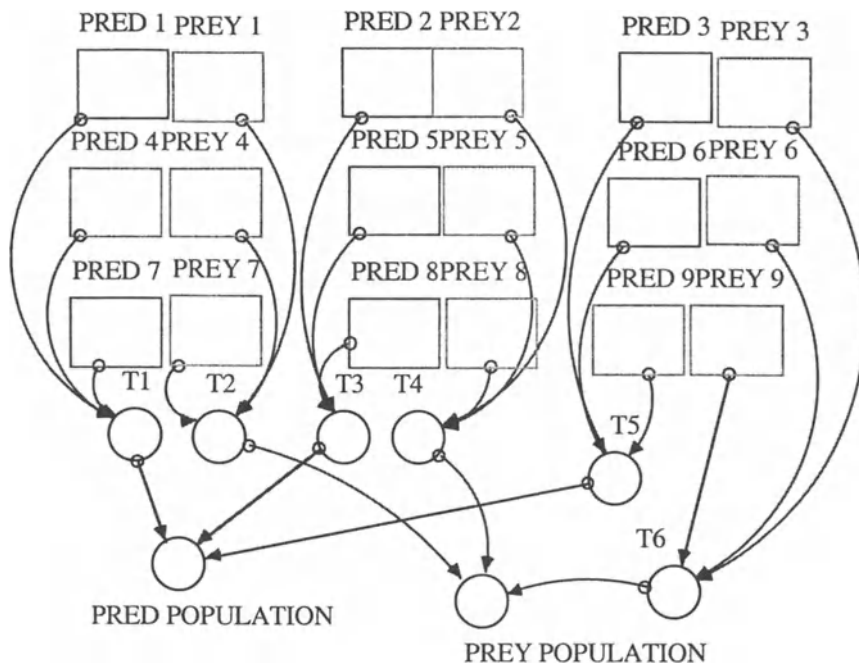
FIGURE 18.4
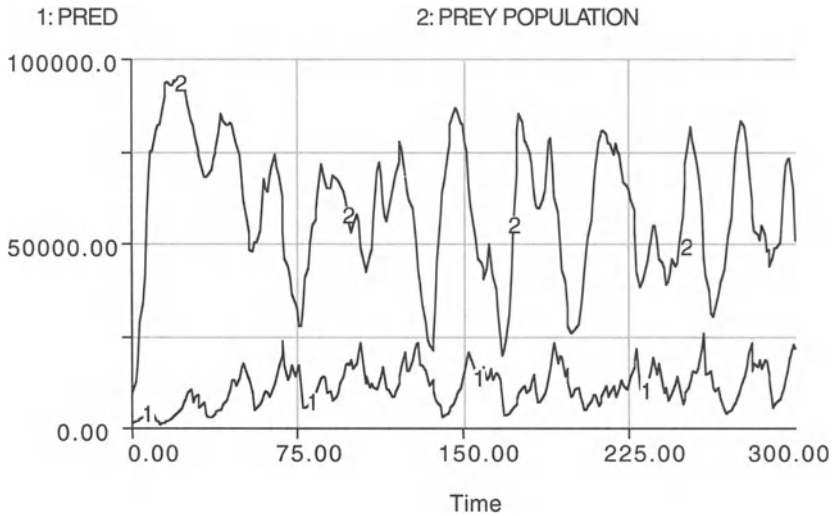
FIGURE 18.5

when you run the model you can see the population sizes change for each cell. Also, calculate the total populations of predators and prey and plot them over time.

Run this model and you will find that it will crash only very rarely. The results of one model run are shown in Figure 18.5. Try changing the parameters again and you will find that it is difficult to make it crash! Again, remove the DEATHS PREDATOR connection to BIRTHS PREDATOR and rerun the model. Do you get a difference in the result? Why (not)?

Trim the number of cells to four, spread the initial population evenly and see what happens. Try the four cells in a row and then in a square. Do you find a difference in survival times? Does this step give you any ideas about the shape and size of residue habitat that can maximize survival probability. Now try five in a row versus one cell with four cells arranged one on each of its sides. Run your models with different DT. How do the results differ, and why? If the model were expanded to 33×33 cells, would we expect to see patches with only prey, others with only predators?

By noting the largest range of the predators in a natural area, one could test the assemblages needed to meet prescribed probabilities of survival. Presumably, this largest range would enable the remainder of the living components of the system to survive easily.

What are the basic variables here? Cellular configuration, initial predator-prey population, and the distribution of the variation of the carrying capacity. Try to eliminate the latter two in order to see what real difference the configuration of the cell can make to the survival of the predator-prey process. Reconstruct the model making use of STELLA's array features.

FIRST CELL OF NINE CELL PREDATOR-PREY MODEL

```
PRED_1(t) = PRED_1(t - dt) + (BIRTH_PRED + IM_PRED_1 -
MIG_PRED_1 - DEATH_PRED_1) * dt
INIT PRED_1 = 1000
        INFLOWS:
                BIRTH_PRED = BR_PRED*(PRED_1-
DEATH_PRED_1)
                IM_PRED_1 = PRED_LF_2+PRED_UP_4
        OUTFLOWS:
                MIG_PRED_1 = MIN(PRED_1-DEATH_PRED_1,
        MIG_RATE_PRED*DEATH_PRED_1)
                DEATH_PRED_1 = PRED_1-
CONSUME_1/CONSUME_RATE)

PREY_1(t) = PREY_1(t - dt) + (BIRTH_PREY_1 + IM_PREY_1
- DEATH_PREY_1 - MIG_PREY_1) * dt
INIT PREY_1 = 10000
        INFLOWS:
                BIRTH_PREY_1 = (PREY_1-
DEATH_PREY_1)*BR_PREY_1*(1-
PREY_1/CC_PREY)
                IM_PREY_1 = PREY_LF_2+PREY_UP_4
        OUTFLOWS:
                DEATH_PREY_1 = CONSUME_1
                MIG_PREY_1 = MIG_RATE_PREY*(PREY_1-
DEATH_PREY_1)

BR_PRED = .2
BR_PREY_1 = 2
CC_PREY = 10000

CONSUME_RATE = 1
CONSUME_1 = MIN(PREY_1,CONSUME_RATE*PRED_1)
MIG_RATE_PRED = .05 {this times the number of
starvation deaths is the number that migrate}
MIG_RATE_PREY = .1 {this is the proportion of the prey
that migrate}

PPREY_DN_1 = RANDOM(0,1)
PPREY_RT_1 = RANDOM(0,1)

PRED_DN_1 = MIG_PRED_1*PPRED_DN_1/PRED_PROB_1
```

```
PRED_POPULATION  =  T5+T3+T1
PRED_PROB_1  =  PPRED_RT_1+PPRED_DN_1
PRED_RT_1  =  MIG_PRED_1*PPRED_RT_1/PRED_PROB_1


PREY_DN_1  =  MIG_PREY_1*PPREY_DN_1/PREY_PROB_1


PREY_POPULATION  =  T2+T4+T6
PREY_PROB_1  =  PPREY_RT_1+PPREY_DN_1


PREY_RT_1  =  MIG_PREY_1*PPREY_RT_1/PREY_PROB_1


T1  =  PRED_7+PRED_4+PRED_1
T2  =  PREY_7+PREY_4+PREY_1
T3  =  PRED_2+PRED_5+PRED_8
T4  =  PREY2+PREY_5+PREY_8
T5  =  PRED_9+PRED_6+PRED_3
T6  =  PREY_3+PREY_6+PREY_9
```

19

Epidemic Modeling

> There are epidemics of opinion as well as of disease.
> —Sir B. Brodie, *Psychological Inquiries*

In this chapter we model the spread of a disease through a population. Epidemics, such as the one modeled here, are of great concern to human societies. The complex interrelationships of biological, social, economic, and geographic relationships that drive or constrain an epidemic make dynamic models an invaluable tool for the analysis of particular diseases. The model developed here is fairly idealized but can be applied easily to real populations affected by a disease.[1]

An initial population of 1,000,000 (per 100 square miles) is not immune to a contagious disease. The rate at which they become sick is assumed to be a function of the product of the nonimmune population times the contagious plus sick population. The contagious population is assumed to become the sick population for a week and then, with a survival rate of .9, the survivors join the immune population. The nonimmune population is augmented with a constant birth rate of 5000 per week. The people in this model do not die of old age.

Setting the contagion rate proportional to the product of the nonimmune and the contagious and sick population is arbitrary. The form is suspiciously similar to the chemical reaction one we studied earlier! In the chemistry model, we used the law of mass action to calculate the concentration (number of moles of a substance per cubic meter of air) of one substance as it reacts with others and forms a new substance. Here, we convert the currently nonimmune population into a sick population. Similarly to the chemistry models using the law of mass action, you may want to change the "reaction rate" by, e.g., introducing an exponent α such that

$$\text{SICK RATE} = \text{CONTACT RATE} * (\text{CONTAGIOUS}+\text{SICK})^{\alpha}$$
$$*\text{NON IMMUNE} \tag{1}$$

---

[1]See Spain, J.D. 1982. *BASIC Microcomputer Models in Biology*, Addison-Wesley, Reading, MA, p. 118. For some realism, see the data on the Black Death in the 1300s in Italy (Curtis H. and N. Barnes. 1985. *Invitation to Biology*, Worth Publishers, New York.) These data show a declining peak as either people became aware of the vector or those most likely exposed to the vector died off or the naturally immune were selected for and that immunity was inheritable. The four occurrences of the plague in that century had a period of about 11 years. For chaotic epidemics, see: Schaffer, W. 1985. Can Nonlinear Dynamics Elucidate Mechanisms in Ecology and Epidemiology? *IMA Journal of Mathematics Applied in Medicine and Biology*, Vol. 2, pp. 221–252. Schaffer shows how a cyclic contact coefficient can produce chaos in this form of epidemic model.
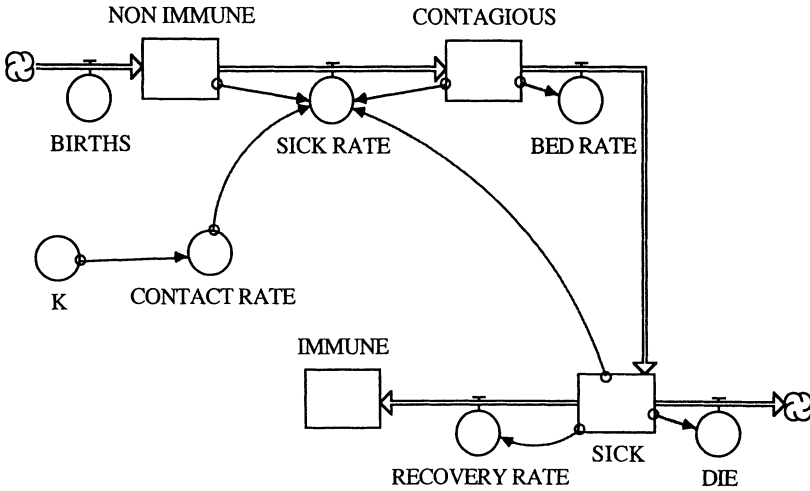
FIGURE 19.1

The structure of the complete model is shown in Figure 19.1. Initialize this model with the parameter values and initial conditions listed in the equations at the end of this chapter. Vary α in consecutive runs, for example, set $\alpha = .9$, $\alpha = 1.1$, $\alpha = 1.3$. Presumably, the basic form ($\alpha = 1.0$) has shown some historical veracity, i.e., the form has been sufficiently fit with historical data.

The reaction is an interesting one for such a simple model (Fig. 19.2). The initial epidemic is the most severe and converts 90% of the population to an immune condition. Ensuing epidemics occur with regular frequency
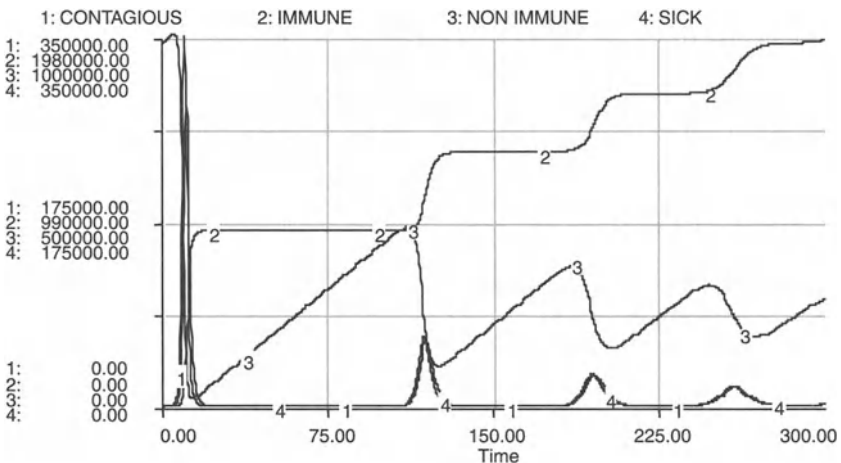
FIGURE 19.2

and are increasingly less severe, finally reaching a steady-sized nonimmune population. At this steady state, the immune population is growing at the birth rate, and the contagion rate is constant and equal to the birth rate.

Try running the model with different time steps and explain why the results differ. Could you connect the birth rate with the immune population and try to reach a steady state immune population? Does the disease die out of the population? Is it possible to wipe out the population with a variation in the parameters in this form of the model?

Can you introduce an optimum (minimum number of sick) vaccination program to stabilize the disease in this latter form of the model? Can you model how the disease can frustrate the vaccination program through mutation? Can you break the population into age groups with different k, death rates, birth rates, initial population, and disease-induced death rates for each? Show how some of these folks seem to be more resistant to the disease and skip from contagious to immune directly. Show how the result changes when immunity is slowly lost. Note that in reality the immunes mingle with the nonimmunes and therefore dilute the original effect of the contact rate coefficient. Can you fix this problem?

Split the model into one that captures the same population but distinguishes two regions. Immune and contagious people can travel, but not the sick ones. People from the two regions have different contact rates and are affected by the disease differently, i.e., the recovery rates differ for the two subgroups of the population. What are the implications for an optimal vaccination program that does not restrict travel between the regions? For additional examples of models of epidemics see Hannon and Ruth.[2]

EPIDEMIC MODEL

```
CONTAGIOUS(t) = CONTAGIOUS(t - dt) + (SICK_RATE -
BED_RATE) * dt
INIT CONTAGIOUS = 1 {Individuals}
INFLOWS:
SICK_RATE = CONTACT_RATE*(CONTAGIOUS+SICK)*NON_IMMUNE
{Individuals per Time Period}
OUTFLOWS:
BED_RATE = CONTAGIOUS {Individuals per Time Period}


IMMUNE(t) = IMMUNE(t - dt) + (RECOVERY_RATE) * dt
INIT IMMUNE = 0 {Individuals}
```

---

[2]Hannon, B. and M. Ruth (1997) *Modeling Dynamic Biological Systems*, Springer-Verlag, New York.

```
INFLOWS:
RECOVERY_RATE = .9*SICK {Individuals per Time Period}


NON_IMMUNE(t) = NON_IMMUNE(t - dt) + (BIRTHS -
SICK_RATE) * dt
INIT NON_IMMUNE = 1000000 {Individuals}
INFLOWS:
BIRTHS = 5000 {Individuals per Time Period}
OUTFLOWS:
SICK_RATE = CONTACT_RATE*(CONTAGIOUS+SICK)*NON_IMMUNE
{Individuals per Time Period}


SICK(t) = SICK(t - dt) + (BED_RATE - RECOVERY_RATE -
DIE) * dt
INIT SICK = 0 {Individuals}
INFLOWS:
BED_RATE = CONTAGIOUS {Individuals per Time Period}
OUTFLOWS:
RECOVERY_RATE = .9*SICK {Individuals per Time Period}
DIE = .1*SICK {Individuals per Time Period}


CONTACT_RATE = 0*K*(COS(TIME/PI)) +K
K = .000002 {1/(Number of Contagious+Sick)*Nonimmune)
per Time Period}
```

20

Reestablishment of Wolves

> What good, honest, generous men at home, will be wolves and foxes on change!
> —Ralph Waldo Emerson, *The Conduct of Life, Fate Wks.*

Originally, the territory of red wolves in the United States of America ranged from Pennsylvania in the north to Florida in the south, and as far west as Texas. Due to land development and large-scale predator control programs by the 1930s, only a few populations existed in the wild (in the Ozarks, Louisiana, and Texas). By 1967, the red wolf was listed as an endangered species. Also by this time, extensive hybridization had taken place between coyotes and red wolves as the coyotes invaded the former red wolf range. By 1975 it was believed that in order to save the species from extinction, it was necessary to capture a few of the remaining red wolves and breed them in captivity. Forty-three animals that were assumed to be red wolves were placed in captivity, but genetic analysis revealed that only 17 of these were "genetically pure" (nonhybrid) red wolves. A captive breeding program began, and in 1987, red wolves were released at the Alligator River National Wildlife Refuge in North Carolina. In 1991, reintroduction began at Cades Cove, a part of the Great Smoky Mountains National Park (GSMNP). The expressed goal of the U.S. Fish and Wildlife Service (USFWS) was to eventually have 220 red wolves living in the wild at three different locations, with 320 wolves breeding in captivity to sustain the wild populations.[1]

Today, the red wolf is the most endangered of all canids.[2] A crucial question therefore is whether the current population of red wolves in the GSMNP is a viable one, i.e., whether it will survive for at least 100 generations. Let us address this question with a dynamic model that starts with a population of 25 red wolves at the Cades Cove area of the GSMNP. The size of this area is 5000 acres.

In order to determine whether an initial red wolf population of 25 in the Smokies is viable, we need to model the basic, essential components of the red wolves' world. This means representing the wolf population itself, prey populations, management activities (release and removal of wolves), and the major factors in mortality of the wolves. Wolves are really very opportunistic in terms of their diet. They will eat rodents, rabbits, raccoons, deer—just about anything that moves. They'll even eat a few things that

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[1]U.S. Fish and Wildlife Service. 1995. *Red Wolf.* U.S. Department of the Interior, USFWS, Washington, DC.

[2]Alderton, D. 1994. Foxes, Wolves, and Wild Dogs of the World. Facts on File, New York.

don't move, including carrion, and berries on occasion. It seems sufficient for the purposes of determining population viability to simply represent the basics of the wolf diet.

Disease and starvation play major roles in wolf mortality, particularly in pup mortality. Red wolves, like all canids, are vulnerable to several parasites, including heartworms, mange mites, hookworms, ticks, and other intestinal worms. They can also become infected with rabies, canine distemper virus, and canine parvovirus. A parasite monitoring program of red wolves at the Alligator River National Wildlife Refuge determined that of all the parasites that might possibly infect the wolves, "only hookworms occurred frequently enough to be of immediate concern to red wolf reintroductions."[3] Thus, we model hookworms infection as a source of wolf mortality.

In the model (Fig. 20.1) the red wolf population is divided into two main categories (pups and mature wolves) because these groups have different
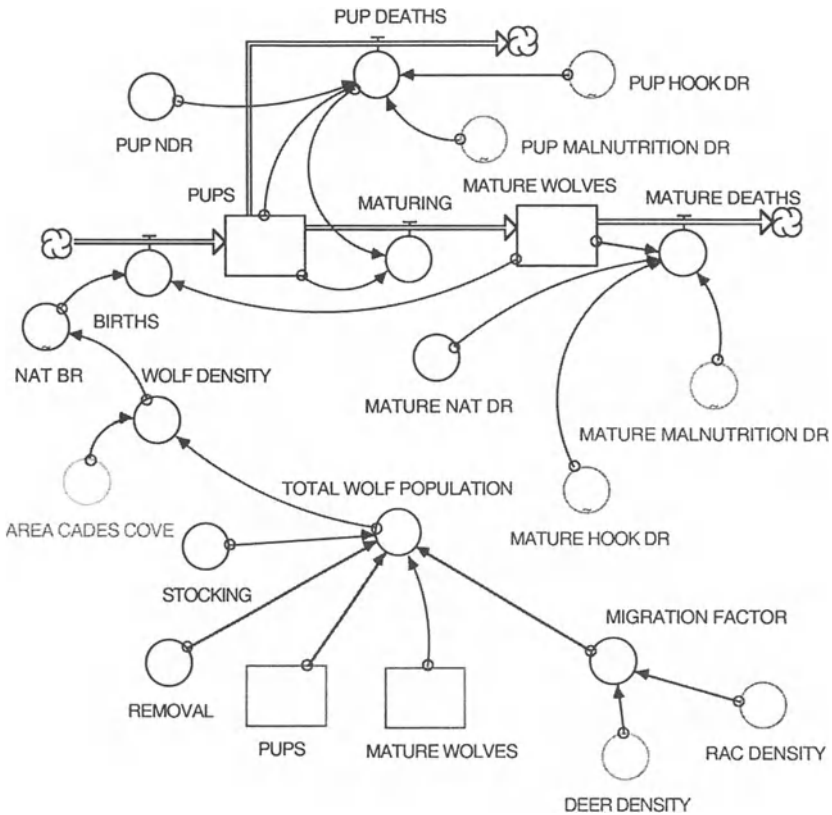
FIGURE 20.1

[3]Phillips, M.K. and J. Scheck. 1991. Parasitism in Captive and Reintroduced Red Wolves, *Journal of Wildlife Diseases*, Vol. 27, pp. 498–501.

death rates, and of course, since wolves aren't born reproductively mature. The initial stock of pups is 0, and that for the adult wolf population is 25. The variable MATURING represents the growth of pups into adults. It is defined as

$$\text{MATURING} = \text{DELAY} ((\text{PUPS} - \text{PUP DEATHS}),2) \qquad (1)$$

because it takes 2 years for pups to reach reproductive maturity. The natural death rate, NDR, of pups, irrespective of hookworm infection or starvation, is assumed to be .2.

$$\begin{aligned}\text{TOTAL WOLF POPULATION} = (\text{MATURE WOLVES} + \text{PUPS} \\ + \text{STOCKING} - \text{REMOVAL})^*\text{MIGRATION FACTOR} \qquad (2)\end{aligned}$$

The MIGRATION FACTOR is a percentage by which we multiply the total population to reflect that at all times a fraction of the wolf population disperses out of the area of Cades Cove. The migration factor is set as

$$\begin{aligned}\text{MIGRATION FACTOR} = \text{IF (DEER DENSITY} + \text{RAC DENSITY}) \\ < .2 \text{ THEN RANDOM } (.5, .8) \text{ ELSE } .9 \qquad (3)\end{aligned}$$

to reflect the impact that the density of the deer and raccoon prey has on red wolf migratory behavior.

STOCKING represents the addition of new wolves into the area by managers. It is very difficult to predict when captively bred animals will be available for release. Before any such animal can be released, it must be shown to be as genetically close to the existing wild population as possible, and then it must undergo "training" to learn how to live in the wild. To enhance their success rate, wolves are typically released as mated pairs, or occasionally as small family groups. For these reasons, the conditional statement for STOCKING is based on a random time function, and a random number of wolves released between 2 and 4:

$$\begin{aligned}\text{STOCKING} = \text{IF TIME} = \text{RANDOM } (0, 200) \\ \text{THEN RANDOM } (2,4) \text{ ELSE } 0 \qquad (4)\end{aligned}$$

Removal of wolves by managers usually occurs because the wolves are too tolerant of human beings. The natural birth rate, NAT BR, decreases as density increases. In captivity, wolves may live up to 14 years, but in the wild, their absolute maximum life expectancy is probably 8 years, and even that is likely generous.[4] Consequently, we assumed that 12 to 13% of the population would die each year, thus MATURE NDR =.125.

PUP DEATHS and MATURE DEATHS are specified using a law-of-mass-action-type relationship, with the population in question being multiplied by the various different death rates affecting that population:

$$\begin{aligned}\text{PUP DEATHS} = \text{PUPS} * \text{PUP NDR} \\ * \text{PUP MALNUTRITION DR} * \text{PUP HOOK DR} \qquad (5)\end{aligned}$$

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[4]Busch, R.H. 1995. *The Wolf Almanac.* Lyons & Burford, New York.

$$\text{MATURE DEATHS} = \text{MATURE WOLVES} * \text{MATURE NAT DR}$$
$$* \text{MATURE MALNUTRITION DR}$$
$$* \text{MATURE HOOK DR} \tag{6}$$

with the PUP HOOK DR and MATURE HOOK DR as the hookworm-induced death rates of pups and adults, respectively.

Hookworm larvae are found on the ground, and they infect animals when those animals come into contact with the soil during walking or lying down. The larvae need moisture to survive in the soil and they use rain as a dispersal agent. So, in the model infection rates increase with increasing yearly rainfall.[5] The corresponding relationship is specified as a graph. The minimum and maximum values for infection rate on the graph are based on reported infection rates at Alligator River National Wildlife Refuge (67%) and infection rates in Texas and Louisiana (82%).[6] RAINFALL, in turn, is set to be a random number between 45 and 55 inches, resulting in a long-term mean of 50 inches, as observed in that area. PUP HOOK DR is a graph based on the INFECTION RATE, increasing as infection rate increases (Fig. 20.2).

The deer population module (Fig. 20.3) is set up in basically the same way as the wolf population, with one stock each for juveniles and adults. As we did with wolves, we set the initial juvenile population at 0. The deer population in Cades Cove typically fluctuates from between 400 and 800 deer. We set the initial ADULT DEER population at 400.

The rate at which deer mature, D MATURING, is modeled analogously to wolves, with the survivors of the juvenile population flowing into the adult population stock after a delay of 1.5 years, the time it takes for deer to reach reproductive maturity.[7] Again, density is based on the total population size—the sum of the two subpopulations—divided by the area of

PUP HOOK DR

MATURE HOOK DR

INFECTION RATE

RAINFALL
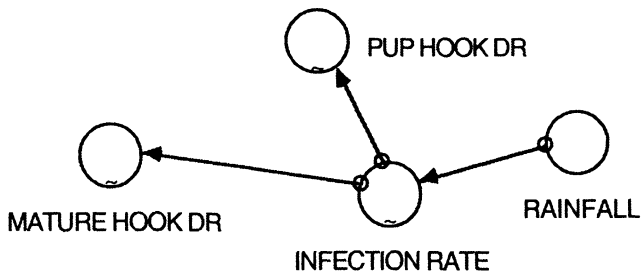
FIGURE 20.2

[5]Chandler, A.C. and C.P. Read. 1961. *Introduction to Parasitology.* 10th ed. John Wiley and Sons, New York.
[6]Phillips, M.K. and J. Scheck. 1991. Parasitism in Captive and Reintroduced Red Wolves, *Journal of Wildlife Diseases,* Vol. 27, pp. 498–501; Chandler, A.C. and C.P. Read. *Introduction to Parasitology.* 10th ed. John Wiley and Sons, New York, 1961.
[7]Halls, L.K. (ed.). 1984. *White-Tailed Deer: Ecology and Management.* Stackpole Books, Harrisburg, PA.
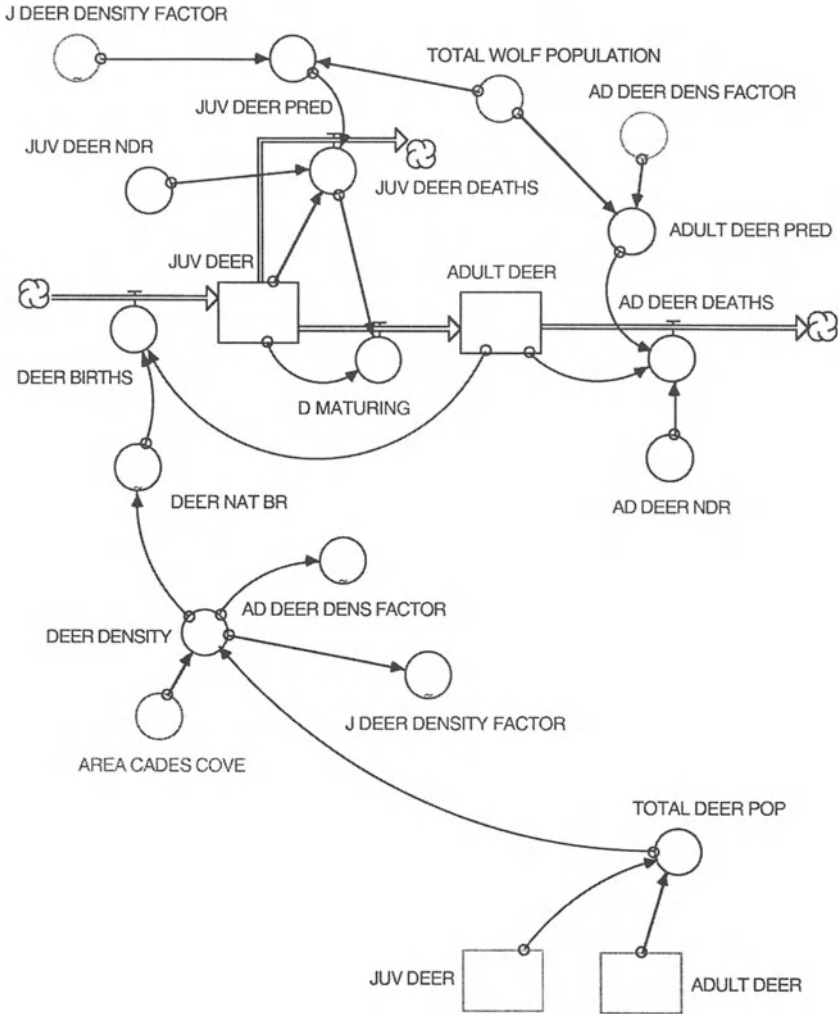
FIGURE 20.3

Cades Cove, and the natural birth rate is a graph showing decreasing birth rates with increasing density.

Maximum deer density on this graph is based on a generous estimate of the .2 deer per acre. Following data in the book by Halls,[8] we set the maximum birth rate of deer, DEER NAT BR =.68, the natural juvenile death rate JUV DEER NDR = .4, and the natural adult death rate AD DEER NDR = .2. JUV DEER DEATHS is based on the juvenile deer population times the juvenile natural death rate times the juvenile deer predation by wolves (JUV

[8]Halls, L.K. (ed.). 1984. *White-Tailed Deer: Ecology and Management.* Stackpole Books, Harrisburg, PA.

DEER PRED). JUV DEER PRED is the total wolf population times J DEER DENSITY FACTOR, which is the percentage of juvenile deer preyed upon per wolf per year. This is a graph based on the deer density, where percentage prey caught is increasing with increasing density. The maxima and minima for this graph are guesses based on the following assumptions: Deer in the Smoky Mountains area range in weight from approximately 88 to 300 lb,[9] so we decided to use 200 lb as our weight of a typical deer. 5 lb of food/day * 365 days/year = 1825 lb/year. 1825 lb/200 lb corresponds to approximately 9 deer per wolf per year. We originally divided this number by our initial deer population of 400 to get a predation factor of .023. Realizing this was a gross overestimate, since wolves don't prey on deer alone, we cut this figure approximately in half to give .012. Then, we split the population into juveniles and adults, and since wolves prey more heavily on juveniles than adults, we assumed the bulk of this factor .012 would be for juvenile predation. Thus, we set the minimum of the graph for the range of predation on the juvenile deer population at .01 and the maximum at .05 as arbitrary. The corresponding specifications for the adult portion of the population—AD DEER DEATHS and ADULT DEER PRED—are set up in the same way these factors were set up for the juvenile subpopulation. AD DEER DENS FACTOR is again a graph based on deer density, and the range of this percentage of adult deer preyed upon per wolf per year is set from 0 to .01 as arbitrary guesses; we purposefully made the range on this graph smaller than that on the JUV DEER PRED graph since wolves prey more heavily on juveniles than adults.

To model raccoon population dynamics, the model follows the same pattern as we used for wolves and deer (Fig. 20.4). Initial juvenile raccoon population is set equal to 0. To specify the initial population size of adult raccoon, we make use of the knowledge that when food is abundant, raccoon densities can be around 100 raccoons per square mile.[10] Consequently, 1,000 raccoons would be a generous estimate of the maximum population size. Since the model starts out on the low end of the possibilities for the deer population, the same assumption is made here, setting the initial ADULT RAC population at 400.

RAC MATURING represents the growth of raccoon juveniles into adults. That maturation typically takes one year. The natural birth rate of raccoon is a graph based on density, with birth rates decreasing as density increases. The maximum of 2 on this graph is based on the assumption that there are 4 cubs/litter, so 4 cubs/2 adults/yr = 2. Juvenile mortality is higher than adult mortality, with the maximum death rate of cubs in the winter around .56.[11] We therefore set the average for JUV RAC NDR to .4. Making the relationship

[9]Silva, M. and J.A. Downing. 1995. *CRC Handbook of Mammalian Body Masses.* CRC Press, Boca Raton, FL.

[10]MacClintock, D. 1981. *A Natural History of Raccoons.* Charles Scribners' Sons, New York.

[11]MacClintock, D. 1981. *A Natural History of Raccoons.* Charles Scribners' Sons, New York.
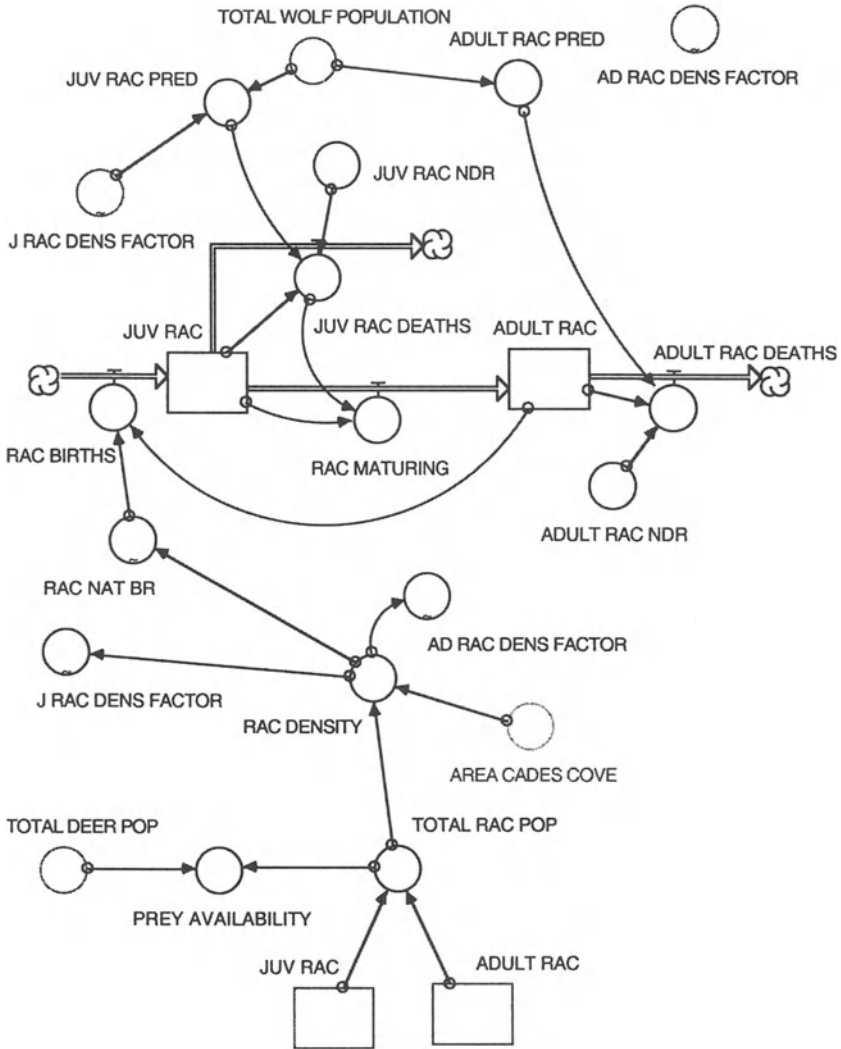
FIGURE 20.4

between juvenile and adult mortality similar to that in the deer population, we set the ADULT RAC NDR = .2.

J RAC DENS FACTOR and AD RAC DENS FACTOR are the percentages of each subpopulation preyed upon per wolf per year. Raccoons range in weight from 10 to 30 lb, and we assume here a weight of 18 lb for a raccoon. The range on the JUV RAC DENS FACTOR is arbitrarily placed between .03 and .06. For the AD RAC DENS FACTOR, this range falls in our model between .01 and .03—purposefully lower than that for the juveniles.

To represent the death of wolves by starvation, we created a parameter called PREY AVAILABILITY, which is the sum of total deer population and

FIGURE 20.5

total raccoon population (Fig. 20.5). We then create MATURE MALNUTRI-
TION DR and PUP MALNUTRITION DR, which are based on graphs of
PREY AVAILABILITY, with malnutrition death rates increasing as prey avail-
ability decreases. It is generally accepted that starvation plays a major role
in pup mortality, so we estimated that at least 10% of the pup population
would die of starvation each year, and arbitrarily set the maximum PUP
MALNUTRITION DR at 30%. The range of the MATURE MALNUTRITION
DR graph is set lower than that for the pups (.01–.10).

The results of our model are shown in Figure 20.6. It seems that according
to our model, at least, an initial population of 25 wolves in the GSMNP is in-
deed viable, in the sense that it will survive for at least 100 generations, and
that the minimum viable population of red wolves in the Smokies is 2. The
wolf population shows a general trend of growth (with fluctuations from year
to year based on the dynamics of the prey populations) to a maximum of
around 150 wolves. Run the model with an initial wolf population of 2 and
you will see that even that is enough to maintain wolves in the area for 100
generations. In our simple model there are no problems with inbreeding.
How would you change the model to accommodate that possibility?

Given all the guesses of parameter values that we made to specify and run
the model, a key question obviously is: How realistic is this model, and thus,
how much faith can be placed in its results? It seems that the results are
overly optimistic because we have underestimated the death rates for the
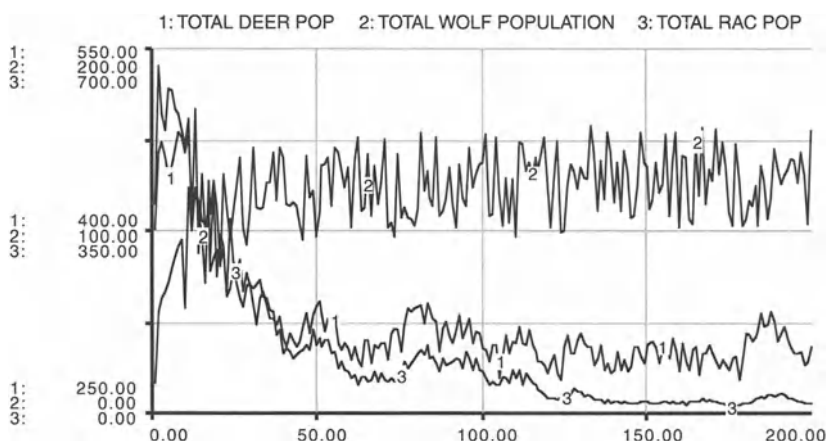
FIGURE 20.6

populations involved. In particular, since we did not incorporate all of the diseases that contribute to red wolf mortality, this model predicts that the wolf population will be more prosperous more quickly than it is likely to be in real life. Though you may have a population of 150 canids that look like red wolves, there is no way of knowing without genetic analyses just how many of those canids are really pure red wolves and how many are hybrids. One of the major factors that contributed to the decline of the red wolf species in the first place was genetic dilution of the species due to hybridization with coyotes. Crossbreeding is probably due to reduction in habitat size and the unchecked spread of coyotes. It is an established fact that when their numbers are small, red wolves will mate with coyotes. Coyotes are currently present in the Smokies, and while red wolves are released into the wild with their mates from captivity, there is no way to keep wolves born in the wild from choosing coyotes as mates. Therefore, genetic dilution of the species will certainly continue to occur, and it is difficult to predict whether the canid population that will survive in the Smokies will be composed of "pure" red wolves or one composed primarily of hybrids, as the wild red wolf population was prior to the initiation of captive breeding programs to save the species from extinction. Finally, the fate of the red wolf population depends on human activity in the area—a key issue that we have not modeled here.

To assess the viability of the population under alternative parameter assumptions, conduct a thorough sensitivity analysis and note the parameters that are critical for our results to hold. Within which ranges can these parameters vary and the wolf population still persist over 100 generations? How would the results of this model change if the seasonality of death rates for each of the populations was taken into account? How could dilution of the red wolf gene pool by hybridization with coyotes be incorporated into this model? How would viability of the wolf population be affected if random chaotic events led to large reductions in the prey populations?

```
                REESTABLISHMENT OF WOLVES

ADULT_DEER(t) = ADULT_DEER(t - dt) + (D_MATURING -
AD_DEER_DEATHS) * dt
INIT ADULT_DEER = 400
INFLOWS:
D_MATURING = DELAY((JUV_DEER-JUV_DEER_DEATHS),1.5)
OUTFLOWS:
AD_DEER_DEATHS =
ADULT_DEER*AD_DEER_NDR*ADULT_DEER_PRED

ADULT_RAC(t) = ADULT_RAC(t - dt) + (RAC_MATURING -
ADULT_RAC_DEATHS) * dt
INIT ADULT_RAC = 400
INFLOWS:
```

```
RAC_MATURING = DELAY((JUV_RAC-JUV_RAC_DEATHS),1)
OUTFLOWS:
ADULT_RAC_DEATHS =
ADULT_RAC*ADULT_RAC_NDR*ADULT_RAC_PRED

JUV_DEER(t) = JUV_DEER(t - dt) + (DEER_BIRTHS -
D_MATURING - JUV_DEER_DEATHS) * dt
INIT JUV_DEER = 0
INFLOWS:
DEER_BIRTHS = ADULT_DEER*DEER_NAT_BR
OUTFLOWS:
D_MATURING = DELAY((JUV_DEER-JUV_DEER_DEATHS),1.5)
JUV_DEER_DEATHS = JUV_DEER*JUV_DEER_NDR*JUV_DEER_PRED

JUV_RAC(t) = JUV_RAC(t - dt) + (RAC_BIRTHS -
RAC_MATURING - JUV_RAC_DEATHS) * dt
INIT JUV_RAC = 0
INFLOWS:
RAC_BIRTHS = ADULT_RAC*RAC_NAT_BR
OUTFLOWS:
RAC_MATURING = DELAY((JUV_RAC-JUV_RAC_DEATHS),1)
JUV_RAC_DEATHS = JUV_RAC*JUV_RAC_NDR*JUV_RAC_PRED

MATURE_WOLVES(t) = MATURE_WOLVES(t - dt) + (MATURING -
MATURE_DEATHS) * dt
INIT MATURE_WOLVES = 25
INFLOWS:
MATURING = DELAY((PUPS-PUP_DEATHS), 2)
OUTFLOWS:
MATURE_DEATHS =
MATURE_WOLVES*MATURE_NAT_DR*MATURE_MALNUTRITION_DR*
MATURE_HOOK_DR

PUPS(t) = PUPS(t - dt) + (BIRTHS - MATURING -
PUP_DEATHS) * dt
INIT PUPS = 0
INFLOWS:
BIRTHS = MATURE_WOLVES*NAT_BR
OUTFLOWS:
MATURING = DELAY((PUPS-PUP_DEATHS), 2)

PUP_DEATHS =
PUPS*PUP_NDR*PUP_MALNUTRITION_DR*PUP_HOOK_DR
ADULT_DEER_PRED =
TOTAL_WOLF_POPULATION*AD_DEER_DENS_FACTOR
```

```
ADULT_RAC_NDR = .1
ADULT_RAC_PRED =
TOTAL_WOLF_POPULATION*AD_RAC_DENS_FACTOR
AD_DEER_NDR = .2
AREA_CADES_COVE = 5000
DEER_DENSITY = TOTAL_DEER_POP/AREA_CADES_COVE
JUV_DEER_NDR = .4
JUV_DEER_PRED =
@BTX:TOTAL_WOLF_POPULATION*J_DEER_DENSITY_FACTOR
JUV_RAC_NDR = .25
JUV_RAC_PRED = TOTAL_WOLF_POPULATION*J_RAC_DENS_FACTOR
MATURE_NAT_DR = .125
MIGRATION_FACTOR = IF (DEER_DENSITY+RAC_DENSITY) < .2
THEN RANDOM ( .5, .8)
ELSE .9
PREY_AVAILABILITY = TOTAL_DEER_POP + TOTAL_RAC_POP
PUP_NDR = .2
RAC_DENSITY = TOTAL_RAC_POP/AREA_CADES_COVE
RAINFALL = RANDOM (45, 55)
REMOVAL = IF TIME = (RANDOM (0,200) ) THEN (RANDOM
(0, 1) ) ELSE 0
STOCKING = IF TIME = (RANDOM ( 0, 200) ) THEN (RANDOM
(2, 4)) ELSE 0
TOTAL_DEER_POP = ADULT_DEER+JUV_DEER
TOTAL_RAC_POP = ADULT_RAC+JUV_RAC
TOTAL_WOLF_POPULATION = (MATURE_WOLVES+PUPS+STOCKING-
REMOVAL)*MIGRATION_FACTOR
WOLF_DENSITY = TOTAL_WOLF_POPULATION/AREA_CADES_COVE
AD_DEER_DENS_FACTOR = GRAPH(DEER_DENSITY)
(0.00, 0.00015), (0.02, 0.00055), (0.04, 0.0012),
(0.06, 0.00175), (0.08, 0.0026), (0.1, 0.0033), (0.12,
0.00415), (0.14, 0.0049), (0.16, 0.00605), (0.18,
0.00725), (0.2, 0.0097)
AD_RAC_DENS_FACTOR = GRAPH(RAC_DENSITY)
(0.00, 0.01), (0.02, 0.0106), (0.04, 0.0111), (0.06,
0.0117), (0.08, 0.0127), (0.1, 0.0141), (0.12, 0.0158),
(0.14, 0.0176), (0.16, 0.0198), (0.18, 0.0231), (0.2,
0.03)
DEER_NAT_BR = GRAPH(DEER_DENSITY)
(0.00, 0.68), (0.02, 0.466), (0.04, 0.296), (0.06,
0.214), (0.08, 0.167), (0.1, 0.112), (0.12, 0.0816),
(0.14, 0.051), (0.16, 0.0238), (0.18, 0.0068), (0.2,
0.00)
```

```
INFECTION_RATE = GRAPH(RAINFALL)
(45.0, 0.67), (46.0, 0.676), (47.0, 0.681), (48.0,
0.688), (49.0, 0.698), (50.0, 0.708), (51.0, 0.717),
(52.0, 0.729), (53.0, 0.747), (54.0, 0.773), (55.0,
0.82)
J_DEER_DENSITY_FACTOR = GRAPH(DEER_DENSITY)
(0.00, 0.0104), (0.02, 0.0116), (0.04, 0.0132), (0.06,
0.0154), (0.08, 0.0186), (0.1, 0.022), (0.12, 0.025),
(0.14, 0.0286), (0.16, 0.0336), (0.18, 0.0398), (0.2,
0.05)
J_RAC_DENS_FACTOR = GRAPH(RAC_DENSITY)
(0.00, 0.03), (0.02, 0.0315), (0.04, 0.0331), (0.06,
0.0342), (0.08, 0.0355), (0.1, 0.0372), (0.12, 0.039),
(0.14, 0.0414), (0.16, 0.0454), (0.18, 0.051), (0.2,
0.06)
MATURE_HOOK_DR = GRAPH(INFECTION_RATE)
(0.67, 0.01), (0.685, 0.0104), (0.7, 0.0122), (0.715,
0.0142), (0.73, 0.0174), (0.745, 0.022), (0.76,
0.0268), (0.775, 0.0314), (0.79, 0.036), (0.805,
0.0416), (0.82, 0.0494)
MATURE_MALNUTRITION_DR = GRAPH(PREY_AVAILABILITY)
(0.00, 0.0987), (200, 0.0699), (400, 0.0559), (600,
0.0469), (800, 0.0406), (1000, 0.0339), (1200, 0.0289),
(1400, 0.0235), (1600, 0.0186), (1800, 0.0136), (2000,
0.01)
NAT_BR = GRAPH(WOLF_DENSITY)
(0.00, 2.48), (0.002, 1.71), (0.004, 1.21), (0.006,
0.838), (0.008, 0.55), (0.01, 0.4), (0.012, 0.238),
(0.014, 0.163), (0.016, 0.075), (0.018, 0.05), (0.02,
0.00)
PUP_HOOK_DR = GRAPH(INFECTION_RATE)
(0.67, 0.203), (0.685, 0.208), (0.7, 0.219), (0.715,
0.247), (0.73, 0.283), (0.745, 0.338), (0.76, 0.401),
(0.775, 0.497), (0.79, 0.591), (0.805, 0.67), (0.82,
0.75)
PUP_MALNUTRITION_DR = GRAPH(PREY_AVAILABILITY)
(0.00, 0.297), (200, 0.26), (400, 0.217), (600, 0.183),
(800, 0.155), (1000, 0.143), (1200, 0.132), (1400,
0.121), (1600, 0.112), (1800, 0.104), (2000, 0.1)
RAC_NAT_BR = GRAPH(RAC_DENSITY)
(0.00, 2.00), (0.02, 1.55), (0.04, 1.29), (0.06, 0.95),
(0.08, 0.69), (0.1, 0.49), (0.12, 0.33), (0.14, 0.21),
(0.16, 0.1), (0.18, 0.03), (0.2, 0.00)
```

21

Lyme Disease[1]

The best laid schemes o' Mice an' Men, / Gang aft agley.
—Robert Burns, *To a Mouse*

Over the past 20 years since Lyme disease was first diagnosed, it has been identified as the most common vector-borne disease in the United States. The repopulation of white-tailed deer in the United States has been associated with the emergence of this disease. The tick vector, *Ixodes scapularis,* harbors *Borrelia burgdorferi* (B.b.), the organism responsible for Lyme disease.[2] The larval and nymphal stages feed on intermediate hosts, which are mostly small mammals and birds. Occasionally they will feed on carnivores such as dogs and humans. The adult tick prefers to feed on deer, but will also feed on dogs and people. The main intermediate host in the northeastern United States is the white-footed mouse. In the Midwest, however, it has been determined that the eastern chipmunk may be equally important as an intermediate host. The ticks appear to follow the migration of deer. Deer appear to be only an amplification host for the tick population since they are able to clear infection from B.b. within a few days. Mice and chipmunks may serve as reservoirs for B.b. in nature since they maintain active infections for at least 3 to 4 months.

To investigate the ecology of Lyme disease we model a 1-hectare (0.1 square kilometer) oak forest and run that model on a weekly time interval, beginning with the first week of spring (T = 0), and DT = 0.5. In our model we will concentrate on the interrelationships among acorn production, white-footed mice (*Peromyscus leucopus*), white-tailed deer (*Odocoileus virginianus*), and the black-legged tick (*Ixodes scapularis*). Specifically, we will explore a hypothesis recently put forth by Ostfeld et al.[3] that states that the risk for human Lyme disease, due to increases in nymphal tick activity, should be greater approximately 2 years following a large mast event.

This hypothesis was developed based on the white-footed mouse population research performed by Wolff[4] showing that summer mouse populations correlate with acorn production. Accordingly, if the summer mouse

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[1]Thanks to M. Roberto Cortinas for helping develop the model of this chapter.
[2]Randolph, S.E. 1995. General Framework for Comparative Quantitative Studies on Transmission of Tick-Borne Diseases Using Lyme Borreliosis in Europe as an Example, *Journal of Medical Entomology,* Vol. 32, pp. 767–77.
[3]Ostfeld, R.S., C.G. Jones, and J.O. Wolff. 1996. Of Mice and Mast, *BioScience,* Vol. 46, pp. 323–330.
[4]Wolff, J.O. 1996. Population Fluctuations of Mast-Eating Rodents Are Correlated with Production of Acorns. *Journal of Mammalogy,* Vol. 77,pp. 850–856.

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density increases following a large masting event, then larval ticks should not have any difficulty finding mice on which to obtain a blood meal—the authors seem to assume that the mouse density establishes the carrying capacity for larval ticks. Thus, many larvae will be able to complete molting into the nymphal stage and reappear the following spring (1.5 years after the masting event).

Unfed larvae generally do not carry the Lyme disease bacterial organism B.d. and so the active larvae season (summer into fall) is not correlated with human cases of Lyme disease. Larvae generally get the infection from mice that had been infected earlier in the year by the other generation's active nymphal stage (spring into summer). After winter stasis and early spring molting, the unfed nymph is ready for a blood meal and is likely to transmit the Lyme disease agent to an intermediate host or to a dead-end host, such as humans. Additionally, the unfed nymphal stage is the most dangerous for it is the infected stage most likely to feed on human blood. Based on these facts, we should expect that if the population density of mice is high and many larvae can acquire a blood meal, the following spring should see a significant increase in active nymphs and an increase in human Lyme disease cases.

One key component of the ecology of Lyme disease is the production of acorns (Fig. 21.1). We assume that acorns can be produced by two different species of oak trees, the white oak (Quercus alba) and the black oak (Quercus velutina). Based on the work by Sork et al.[5] in east-central Missouri, we have been able to simulate the average yearly acorn production for each tree species, as well as simulate the production of the large acorn mast event associated with the two species. The acorn production is normally distributed and nonnegative. The white oak has large acorn production about every 3 years, whereas the black oak has a large mast event every 2 years.

Based on the amount of total acorns produced, we have developed an assumption about ACORN MAST INDEX—an index is actually calculated on the basis of field observations by counting the number of acorns on a series of randomly selected branches on a series of randomly selected trees. However, we did not want to extrapolate how many branches a tree has. Instead, we assumed that if acorn production is 0, then the mast index is 0. Additionally, acorn mast indices above 200 seem to be rare, so we set the highest acorn production possible in our model to correlate with a mast index of 200. We assumed a linear relationship between acorn production and mast index. Because of this relationship and because the mast index is not a measure of a particular tree but a measure applied to several branches of several trees, we do not require a real value for the number of trees in our model, but a proportion of trees. Figure 21.2 shows an example model run for a 50% composition of both oak species in our forest patch.

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[5]Sork, V.L., J. Bramble, and O. Sexton. 1993. Ecology of Mast-Fruiting in Three Species of North American Deciduous Oaks. Ecology, Vol. 74, pp. 528–541.
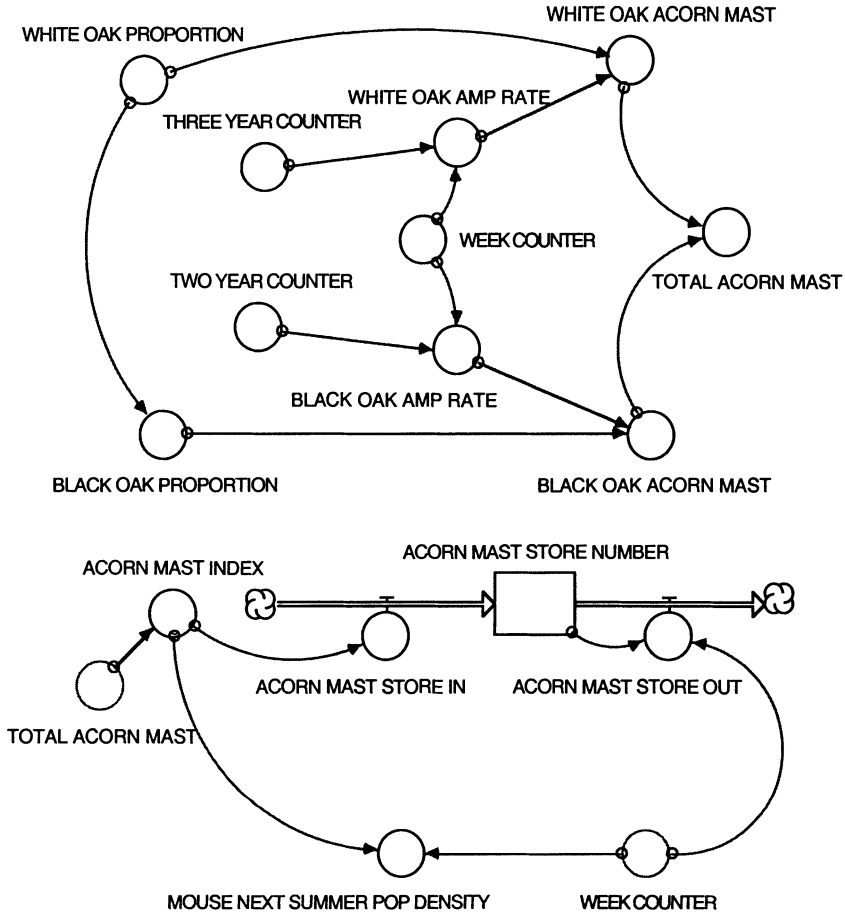
FIGURE 21.1

Based on the work by Wolff, we used a linear regression equation that relates the summer white-footed mouse density (mice per hectare) to the acorn mast index:

$$\text{SUMMER MOUSE POPULATION DENSITY}$$
$$= 7.28 + 0.60 * \text{ACORN MAST INDEX} \qquad (1)$$

Thus, the minimum mouse population density is 7.28 and mouse population density increases with an increase in the acorn mast index.

The mouse population dynamics are based on the acorn mast index (Fig. 21.3). In order to use the above equation, we assumed the summer mouse population density to be the carrying capacity (K) for mice. Additionally, the reason for the increase in population (N) during the summer following a large mast event is due to an additional litter being born in the winter. White-footed mouse females generally have two litters per year, but
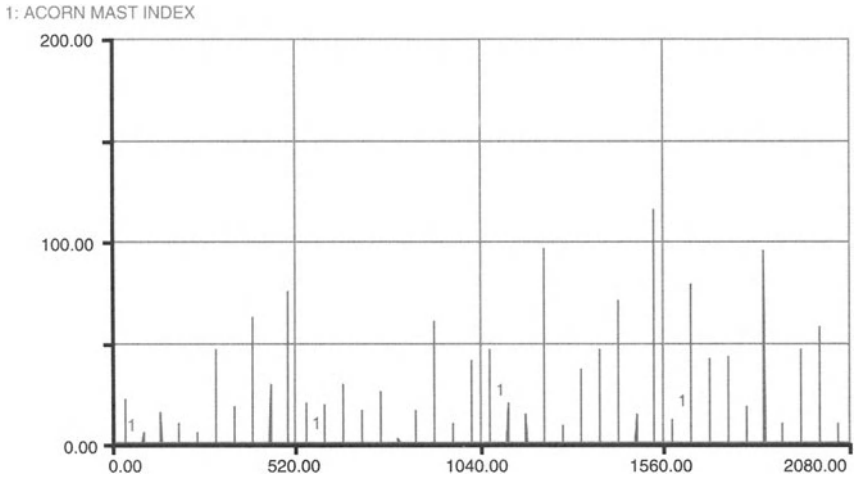
FIGURE 21.2
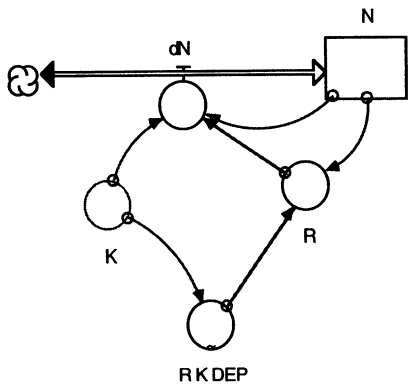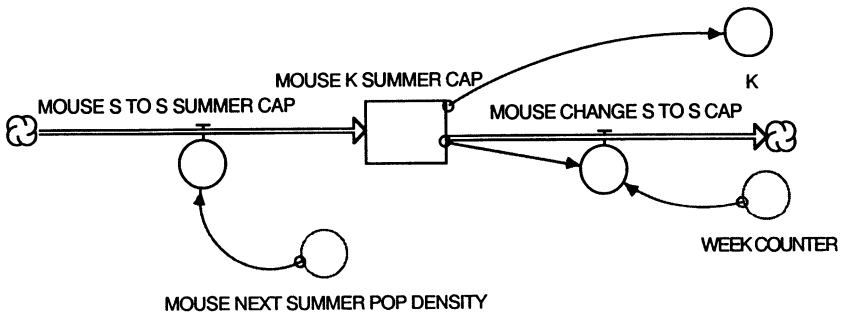
FIGURE 21.3

when the acorn mast is substantial some may have three. Because of this additional litter event, the natural rate of increase (R) for the population can vary based on the acorn mast production. Thus, we have assumed R to vary with changes in K. Since K is mast index-derived, R will also be driven by mast index due to its relationship with K. Nevertheless, the R value when K=20 is based on Wolff's work.

Though we have made many assumptions that may not reflect the actual conditions in the mouse population, we have modeled a population that reaches the predicted density the summer following a mast event. Figure 21.4 shows the time-varying behavior of the mouse population size, N, the carrying capacity, and the acorn mast index using Wolff's summer mouse population correlation equation. Figure 21.5 displays the relationship of R and the acorn mast index.

Changes in deer population dynamics are handled differently in the model from the ways in which we specified changes in mouse population (Fig. 21.6). Based on the work by McShea and Schwede,[6] we assumed deer respond to the variation in acorn crops by spending more or less time in particular parts of their habitats. The authors showed that deer are more likely to use the oak forest habitat if acorn production is good, and the percentage of habitat use is correlated with acorn production. Based on McShea and Schwede's data, we assumed an average 3.5 deer per hectare. Their study area was approximately 60% nonforested and 40% forested. Thus, we could assume for the purpose of this model that if our model was placed in 1 square kilometer of their study area, 4 hectares would be for-
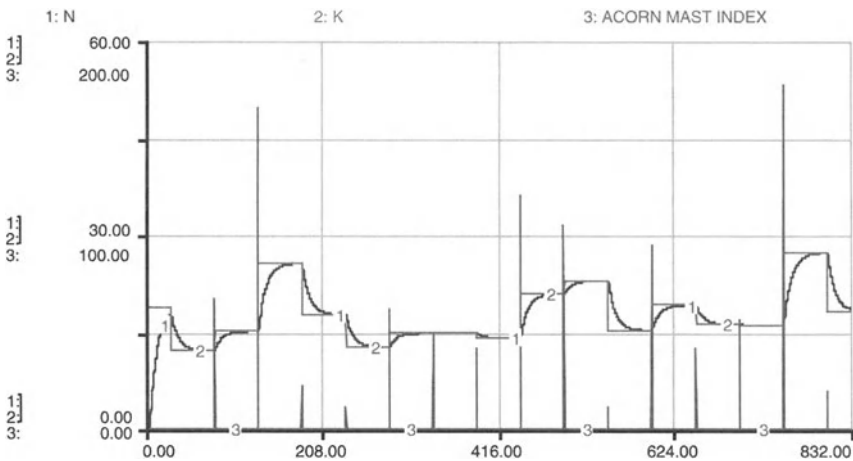
FIGURE 21.4

[6]McShea, W.J., and G. Schwede. 1993. Variable Acorn Crops: Responses of White-Tailed Deer and Other Mast Consumers, *Journal of Mammology,* Vol. 74, pp. 999–1006.
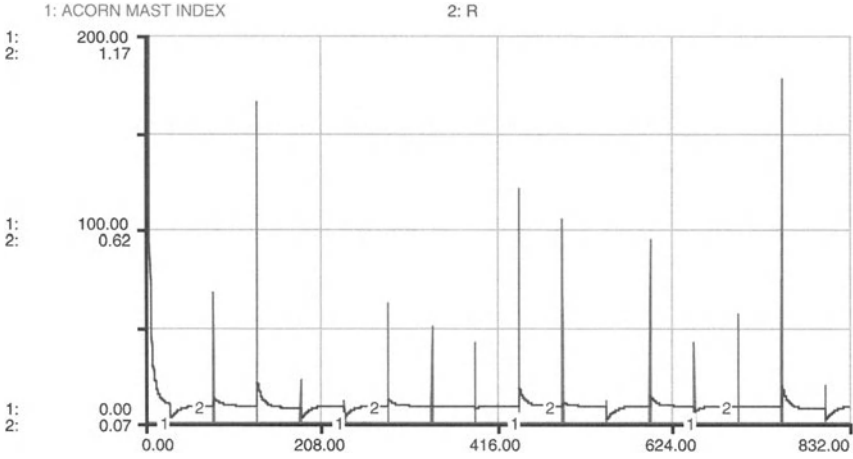
FIGURE 21.5

ested and one of those hectares would be our model. Assume a deer had 50% habitat use in the forest. Since our model is 1/4 of the total forest, we assumed that the deer's percent habitat use in our model is 12.5% (1/2 * 1/4 = 1/8). We were able to roughly correlate the increases in home range and forest habitat use with the acorn mast indices reported by Wolff. Both studies took place in Virginia at the same time, so it was good to see that
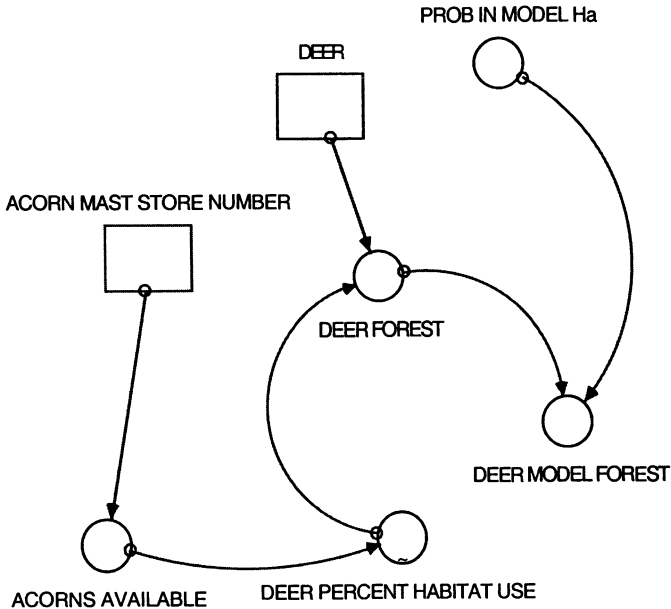
FIGURE 21.6

when Wolff reported high acorn mast years, McShea and Schwede's data indicated increases in percent habitat use.

Because deer percent habitat use is a function of the presence of acorns in the forest, we modeled acorn fall by using a sine function. As soon as the acorn production is randomly selected in the oak-acorn sector (during the last week of summer) and the acorn mast index is generated, a sine wave of acorn availability develops that peaks during late September into October and then declines. Since we roughly correlated the percent habitat use with the acorn mast index, we have assumed that acorn availability is proportional to the acorn mast index.

Based on the appearance and disappearance of acorns, the deer increase and then decrease their numbers in our model forest during the fall. Additionally, the maximum deer number is based on acorn production. In large mast years, more deer should come into our model. Figure 21.7 displays the acorn mast index in the late summer, and the subsequent autumnal acorn availability. Figure 21.8 shows the relationship between acorn mast index and deer wandering through our model.

Tick burdens on mice and deer are important, for we assumed that they represent the carrying capacity for our tick stage populations. Tick burdens on mice have been assumed to be related to the number of tick larvae and nymphs found on mice during the spring and summer seasons. We have used data collected from the 1990 and 1992 to 1997 seasons in Castle Rock State Park, Illinois. The data provided a mean number of larvae and nymphs on mice for a particular sampling day during a particular month. In order to ascertain what the tick burden would be for a particular week, we had to consider the biology of tick feeding. For example, larvae require about 4 days to successfully feed, whereas nymphs require about 4.5 days.
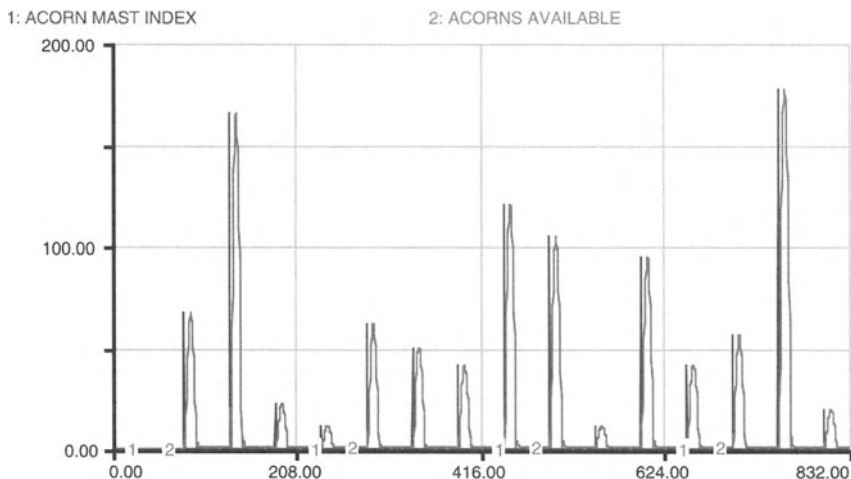
FIGURE 21.7
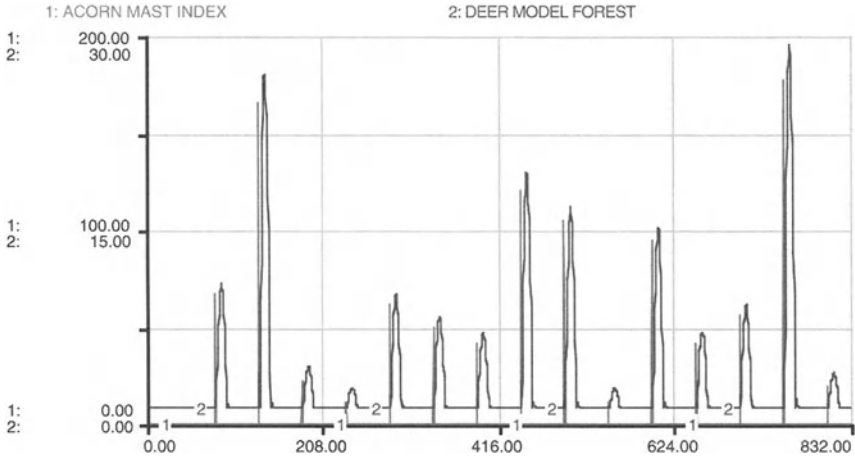
FIGURE 21.8

In order to acquire the weekly larval tick burden for this part of the model (Fig. 21.9) we specified the following relationship:

$$\text{WEEKLY LARVAL TICK BURDEN} = N * \text{DAILY LARVAL TICK BURDEN} * 7 / 4 \qquad (2)$$

For a daily larval tick burden of 5 larval ticks and 20 mice, 175 larval ticks could successfully feed per week (instead of 700 if the feeding time was one day). The same mathematical assumption was applied to the weekly nymphal tick burden per mouse (see Fig. 21.9).

Figure 21.10 displays the relationship between mouse population density and the weekly larval and nymphal tick burdens. Note the seasonal distribution of nymphs and larvae. We have used this seasonal information to regulate the appearance of unfed larvae and unfed nymphs. To make the model more realistic, we could have included an environmental basis for
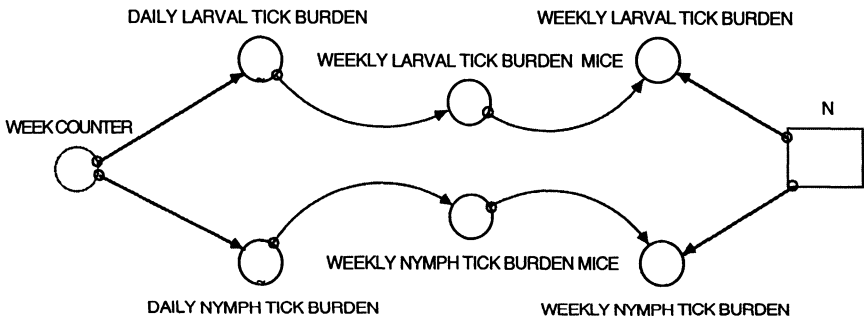
FIGURE 21.9
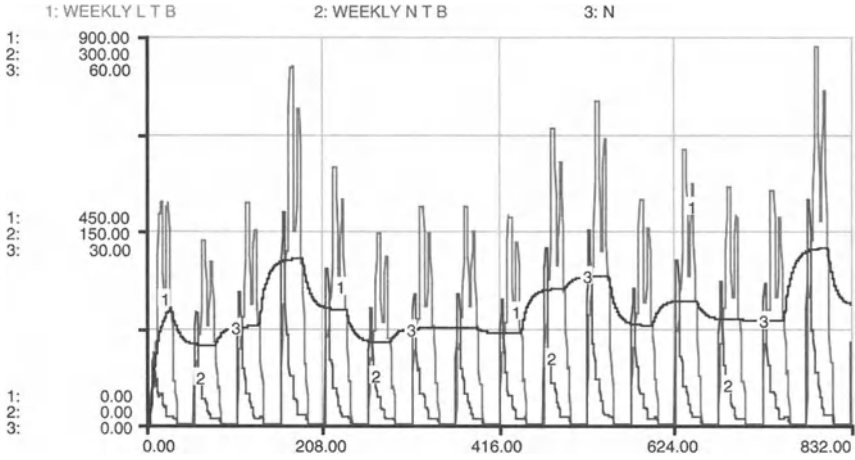
FIGURE 21.10
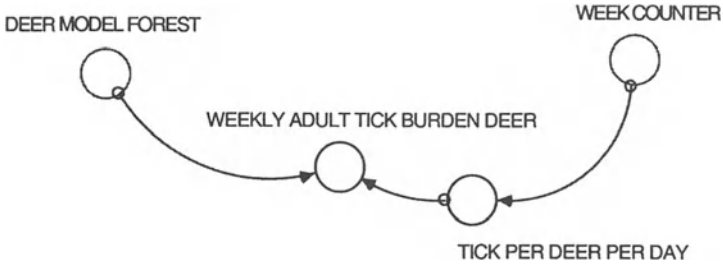
FIGURE 21.11
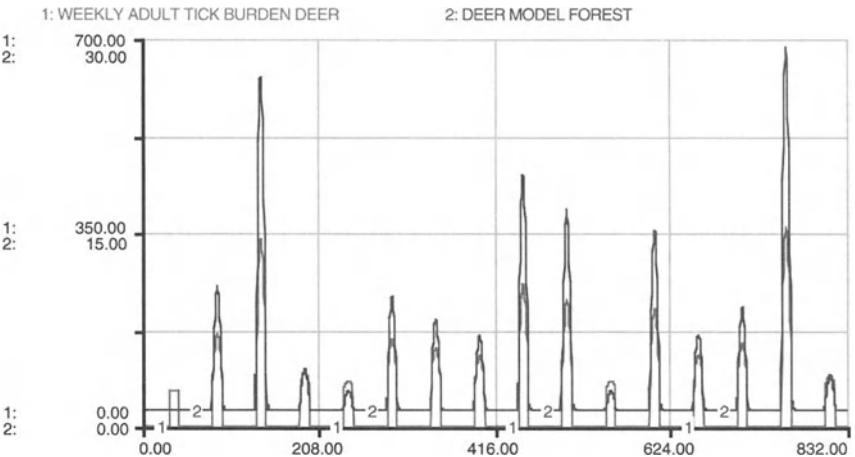
FIGURE 21.12

the emergence of the nymphal and larval stages, based on the microclimate of the litter layer, particularly humidity. This has not been done.

We assume that the number of adult ticks entering and leaving our model is equal, so we make no concessions to tick migration on deer. Additionally, we assume that deer are host to approximately 30 adult ticks per day, and since the feeding period for an adult female is 7 days, we assume 30 adult ticks per week on a deer (Fig. 21.11)

Figure 21.12 shows the seasonal variation in adult tick burdens in our 1-hectare forest.

The population dynamics of the black-legged tick illustrate the two different populations which are active during particular seasons (Figs. 21.13 and 21.14). We have assumed that the ticks cannot immigrate or emigrate from the model (or immigration = emigration). In nature, ovipositing and
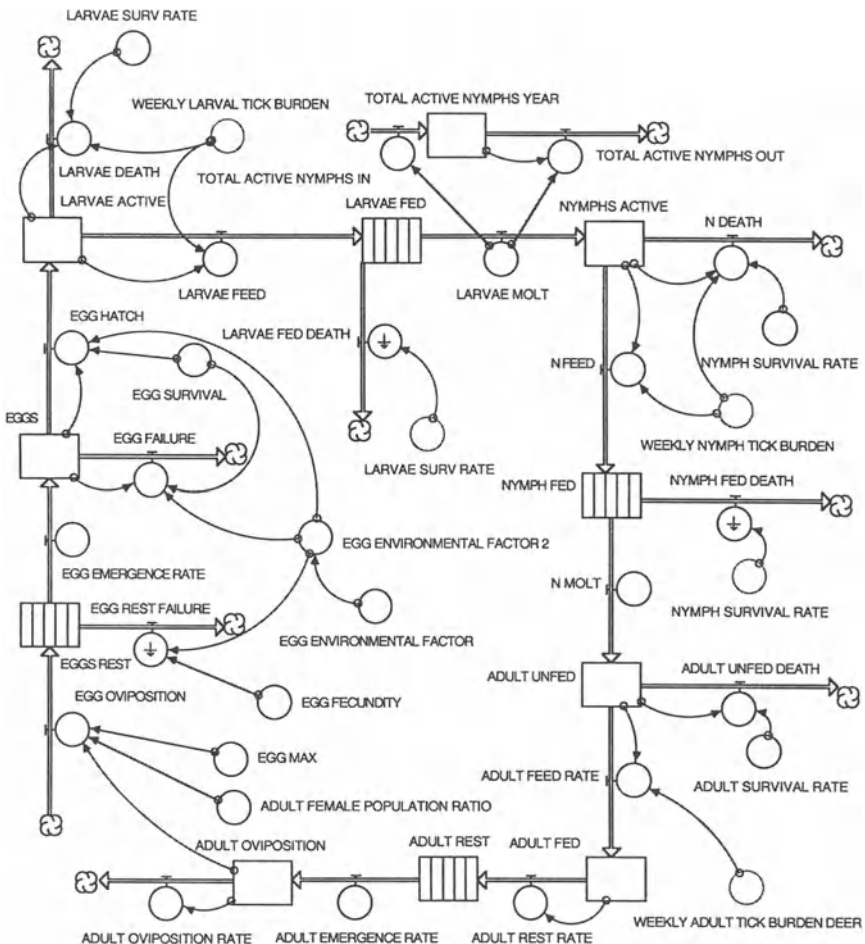
FIGURE 21.13
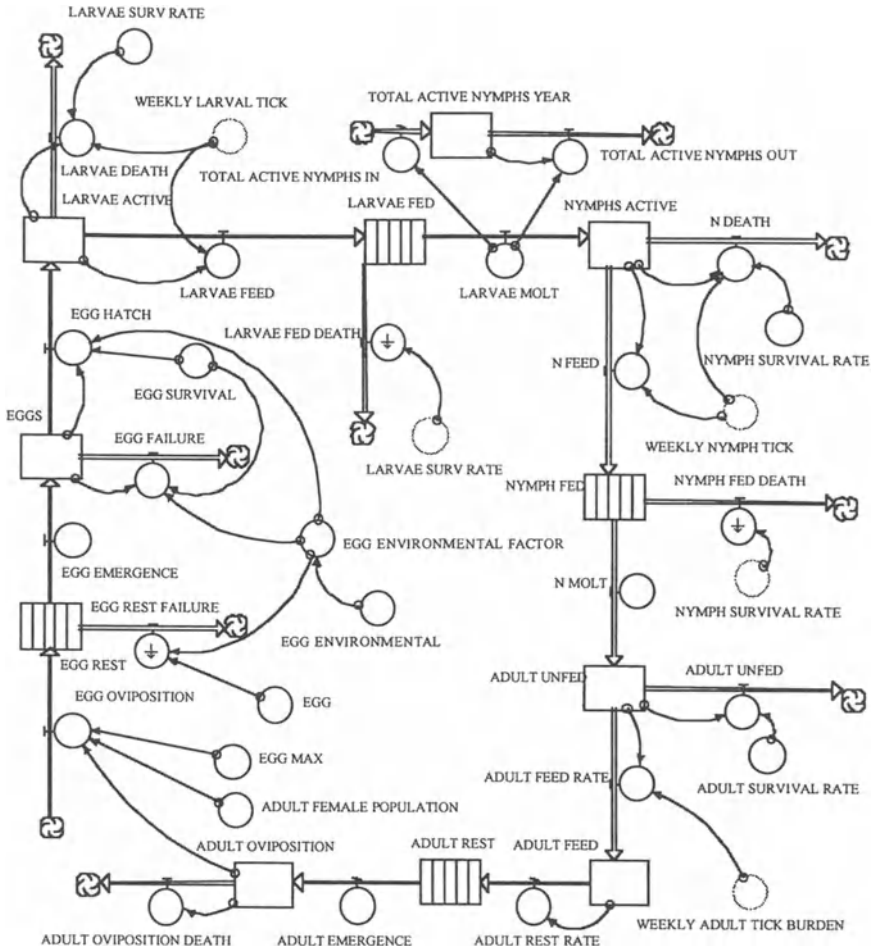
FIGURE 21.14

egg development are based on cumulative degree days. In lieu of good data to model temperature dependence, we have approximated the development of eggs by employing a sine wave function, with its average area under the curve equal to the average success of hatching and its length (in weeks) appropriate to the occurrence of hatching in the wild.

Ticks move in sequential fashion from egg to larva to nymph to adult during a 2-year period, in which they are in resting stages prior to molting to the next life stage. Conveyors have been used to simulate the rest periods. Additionally, we have assumed that intermediate and definitive host densities account for the carrying capacity and feeding success of the tick stages dependent on a blood meal to molt.
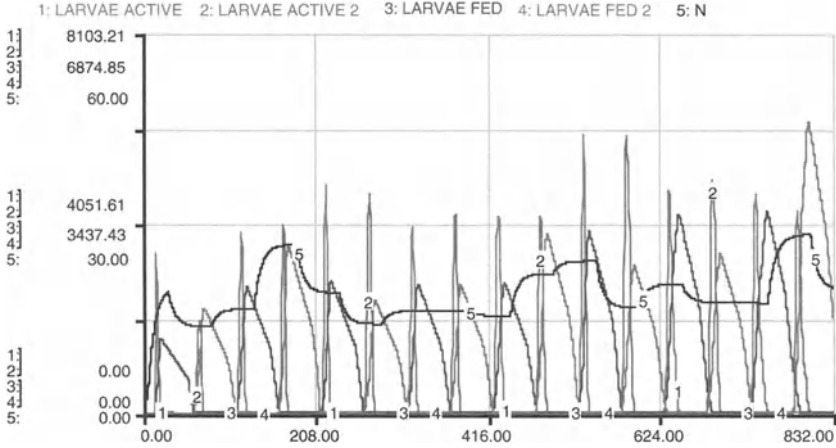
FIGURE 21.15

Figures 21.15 to 21.17 show, respectively, the populations of larvae, nymphs, and adult ticks. The last indicates that the adult tick population numbers are not yet limited by deer numbers.

Now that all the pieces of the model are in place, we can calculate the number of active tick nymphs. By accumulating the number of ticks successfully molting from larval to nymphal form, we have set up a reservoir that can provide the number of active nymphs per year. This variable can then be compared with the magnitude of the acorn mast index 2 years prior to see if the hypothesis holds true.
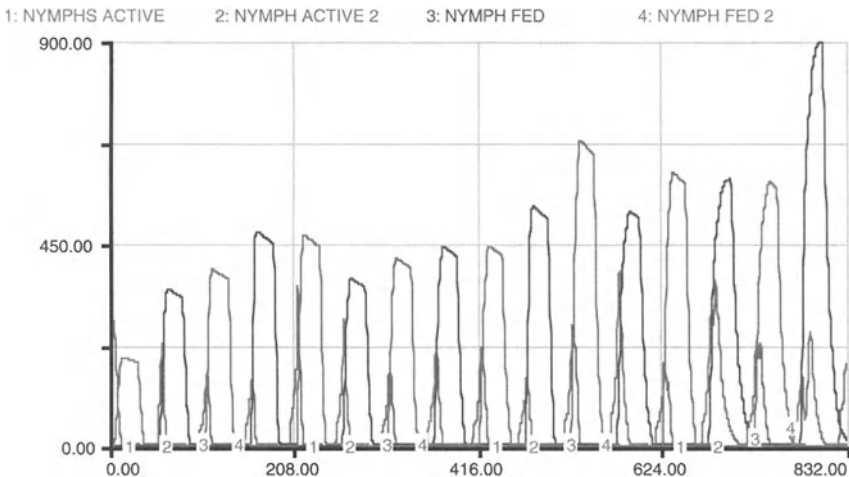
FIGURE 21.16
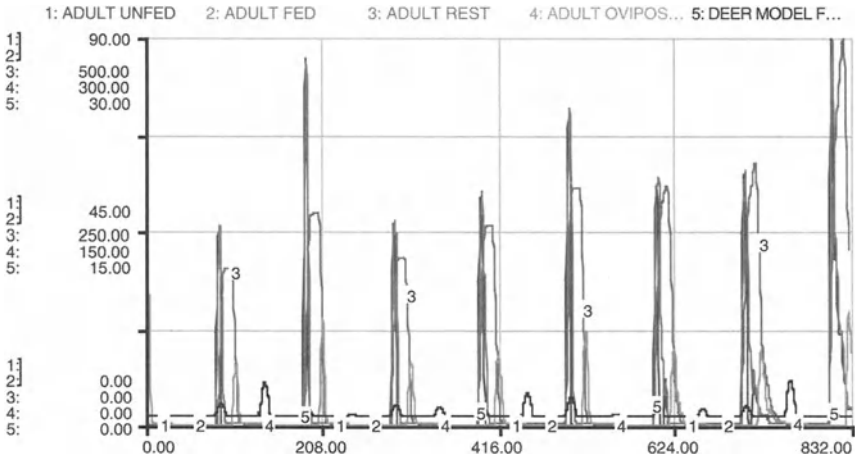
FIGURE 21.17

Based on the results shown in Figure 21.18, it seems that nymphal activity increases almost 2 years following a large mast event. However, the increase is proportional to the difference between the mast event responsible for the increase and the mast event that occurred almost 4 years prior to the nymphal increase. Thus, we could make predictions about the danger of Lyme disease associated with a masting event.

Another interesting question we can address with our model is whether the oak species composition has a population effect on the two different tick populations. *I. scapularis* has a life span of 2 years in the wild. If the populations are responsive to acorn mast production, and excessive mast acorn pro-
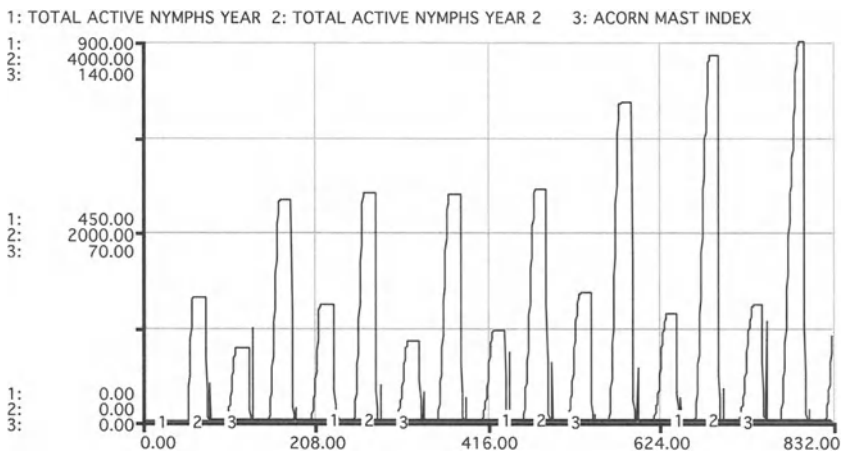
FIGURE 21.18

duction happens only every 2 years, one population of ticks may crash or remain very low whereas the other population maintains high numbers. As you run the model you can see that population 1 benefits from the black oak mast whereas population 2 maintains lower population numbers or crashes.

  How about a combination? Because black oak masts are larger than white oak masts, a 50-50 species composition may still favor the tick population synchronous with the 2-year black oak mast. Even with white oak at 75%, population 2 crashes. At 100% white oak composition, you can see that the 3-year cycle of large masts benefits both tick populations. However, also note the beneficial effect on active nymphs not only 2 years after the mast event but also 3 years after the mast event. Could there exist residual benefits from the large mast event that carried over because the tick burden carrying capacities were not limiting? What would happen if mice and/or deer were excluded from the model forest? What if the deer population density changed in response to disease, predation, and harvesting? What if mice were allowed to emigrate/immigrate into the model? The tick burdens on mice for our model are based on Illinois data, and mouse tick burdens are higher on the East Coast. What if the model employed higher daily tick burdens on mice? These are just a few questions that can be addressed with the model of this chapter, and we encourage you to do so.

```
                         LYME DISEASE

  Deer Movement
  DEER(t) = DEER(t - dt)
  INIT DEER = 35 {deer per 10 ha}

  ACORNS_AVAILABLE =
  ABS(SINWAVE(ACORN_MAST_STORE_NUMBER,(38-25)*2))*2
  DEER_FOREST = DEER*DEER_PERCENT_HABITAT_USE/100
  DEER_MODEL_FOREST = IF PROB_IN_MODEL_Ha < 0 THEN 0
  ELSE DEER_FOREST * PROB_IN_MODEL_Ha
  PROB_IN_MODEL_Ha = 0.25 {Deer are in 10 ha = 1 sq. km,
  6 ha nonforested, 4 ha forested. Model is in 1 of the 4
  forested hectares, so there's a 1 in 4 (25%)
  probability of being in the model forest.}
  DEER_PERCENT_HABITAT_USE = GRAPH(ACORNS_AVAILABLE)
  (0.00, 7.00), (10.0, 12.5), (20.0, 18.0), (30.0, 24.0),
  (40.0, 30.5), (50.0, 37.0), (60.0, 44.0), (70.0, 50.5),
  (80.0, 57.5), (90.0, 64.0), (100, 70.5)

  Mouse Population Dynamics

  MOUSE_K_SUMMER_CAP(t) = MOUSE_K_SUMMER_CAP(t - dt) +
  (MOUSE_S_TO_S_SUMMER_CAP - MOUSE_CHANGE_S_TO_S_CAP) * dt
```

```
INIT MOUSE_K_SUMMER_CAP = 20
INFLOWS:
MOUSE_S_TO_S_SUMMER_CAP =
MOUSE_NEXT_SUMMER_POP_DENSITY
OUTFLOWS:
MOUSE_CHANGE_S_TO_S_CAP = IF (TIME<25) THEN 0 ELSE IF
(WEEK_COUNTER = 25) THEN MOUSE_K_SUMMER_CAP ELSE 0

N(t) = N(t - dt) + (dN) * dt
INIT N = 2
INFLOWS:
dN = R*N*(1-(N/K)) {Individuals per Week}

K = MOUSE_K_SUMMER_CAP
R = R_K_DEP/N
R_K_DEP = GRAPH(K)
(0.00, 0.00), (12.0, 1.40), (24.0, 2.80), (36.0, 3.50),
(48.0, 4.20), (60.0, 4.90), (72.0, 5.50), (84.0, 6.00),
(96.0, 6.50), (108, 6.90), (120, 7.30)

Oak - Acorn Production
ACORN_MAST_STORE_NUMBER(t) = ACORN_MAST_STORE_NUMBER(t -
dt) + (ACORN_MAST_STORE_IN - ACORN_MAST_STORE_OUT) * dt
INIT ACORN_MAST_STORE_NUMBER = 0
INFLOWS:
ACORN_MAST_STORE_IN = ACORN_MAST_INDEX
OUTFLOWS:
ACORN_MAST_STORE_OUT = IF WEEK_COUNTER>38 THEN
ACORN_MAST_STORE_NUMBER ELSE 0

BLACK_OAK_ACORN_MAST = If (BLACK_OAK_AMP_RATE < 0)
THEN 0 ELSE
BLACK_OAK_PROPORTION*BLACK_OAK_AMP_RATE
BLACK_OAK_AMP_RATE = IF (TWO_YEAR_COUNTER = 77) THEN
(NORMAL(3259,2983)) ELSE
IF (TWO_YEAR_COUNTER<>77) AND (WEEK_COUNTER=25) THEN
(NORMAL(1059,834)) ELSE 0
BLACK_OAK_PROPORTION = IF WHITE_OAK_PROPORTION > 1 OR
WHITE_OAK_PROPORTION < 0 THEN 1/0 ELSE
(1-WHITE_OAK_PROPORTION)
MOUSE_NEXT_SUMMER_POP_DENSITY = IF (WEEK_COUNTER = 25)
THEN
.60*ACORN_MAST_INDEX + 7.28) ELSE 0 {Number of mice
the following summer in one hectare}
THREE_YEAR_COUNTER = COUNTER(0,156)
```

```
TOTAL_ACORN_MAST =
BLACK_OAK_ACORN_MAST+WHITE_OAK_ACORN_MAST
TWO_YEAR_COUNTER = COUNTER(0,104)
WEEK_COUNTER = COUNTER(0,52)
WHITE_OAK_ACORN_MAST = IF (WHITE_OAK_AMP_RATE < 0)
THEN 0 ELSE
WHITE_OAK_PROPORTION*WHITE_OAK_AMP_RATE
WHITE_OAK_AMP_RATE = IF (THREE_YEAR_COUNTER = 129)
THEN (NORMAL(2441,1526)) ELSE IF
(THREE_YEAR_COUNTER<>129) AND (WEEK_COUNTER = 25) THEN
(NORMAL(700,480)) ELSE 0
WHITE_OAK_PROPORTION = 1
ACORN_MAST_INDEX = GRAPH(TOTAL_ACORN_MAST)
(0.00, 0.00), (1000, 20.0), (2000, 40.0), (3000, 60.0),
(4000, 80.0), (5000, 100), (6000, 120), (7000, 140),
(8000, 160), (9000, 180), (10000, 200)

Tick Burden on Deer
TICK_PER_DEER_PER_DAY = IF (WEEK_COUNTER>=27) AND
(WEEK_COUNTER<=38) THEN 30 ELSE 0
WEEKLY_ADULT_TICK_BURDEN_DEER =
DEER_MODEL_FOREST*TICK_PER_DEER_PER_DAY*(7/7)

Tick Burden on Mice
WEEKLY_LARVAL_TICK_BURDEN =
N*WEEKLY_LARVAL_TICK_BURDEN__MICE
WEEKLY_LARVAL_TICK_BURDEN__MICE =
DAILY_LARVAL_TICK_BURDEN* (7/4) {convert successful
biting period per day to week}
WEEKLY_NYMPH_TICK_BURDEN =
N*WEEKLY_NYMPH_TICK_BURDEN_MICE
WEEKLY_NYMPH_TICK_BURDEN_MICE =
DAILY_NYMPH_TICK_BURDEN * (7/4.5) {convert successful
biting period per day to week}
DAILY_LARVAL_TICK_BURDEN = GRAPH(WEEK_COUNTER)
(0.00, 0.00), (2.00, 3.33), (4.00, 3.33), (6.00, 4.80),
(8.00, 4.80), (10.0, 9.90), (12.0, 9.90), (14.0, 9.90),
(16.0, 5.30), (18.0, 5.30), (20.0, 8.70), (22.0, 8.70),
(24.0, 4.40), (26.0, 4.40), (28.0, 1.80), (30.0, 1.80),
(32.0, 0.00), (34.0, 0.00), (36.0, 0.00), (38.0, 0.00),
(40.0, 0.00), (42.0, 0.00), (44.0, 0.00), (46.0, 0.00),
(48.0, 0.00), (50.0, 0.00), (52.0, 0.00)
DAILY_NYMPH_TICK_BURDEN = GRAPH(WEEK_COUNTER)
(0.00, 1.35), (2.00, 2.67), (4.00, 2.67), (6.00, 1.10),
(8.00, 1.10), (10.0, 0.6), (12.0, 0.6), (14.0, 0.6),
```

```
(16.0, 0.3), (18.0, 0.3), (20.0, 0.1), (22.0, 0.1),
(24.0, 0.1), (26.0, 0.1), (28.0, 0.1), (30.0, 0.1),
(32.0, 0.00),
(34.0, 0.00), (36.0, 0.00), (38.0, 0.00), (40.0, 0.00),
(42.0, 0.00), (44.0, 0.00), (46.0, 0.00), (48.0, 0.00),
(50.0, 0.00), (52.0, 0.00)


Tick Population Dynamics
ADULT_FED(t) = ADULT_FED(t - dt) + (ADULT_FEED_RATE -
ADULT_REST_RATE) * dt
INIT ADULT_FED = 0
INFLOWS:
ADULT_FEED_RATE = if
(WEEKLY_ADULT_TICK_BURDEN_DEER>=ADULT_UNFED) then
WEEKLY_ADULT_TICK_BURDEN_DEER else (ADULT_UNFED)
OUTFLOWS:
ADULT_REST_RATE = ADULT_FED


ADULT_OVIPOSITION(t) = ADULT_OVIPOSITION(t - dt) +
(ADULT_EMERGENCE_RATE -
ADULT_OVIPOSITION_RATE) * dt
INIT ADULT_OVIPOSITION = 300
INFLOWS:
ADULT_EMERGENCE_RATE = CONVEYOR OUTFLOW
OUTFLOWS:
ADULT_OVIPOSITION_RATE = ADULT_OVIPOSITION


ADULT_REST(t) = ADULT_REST(t - dt) + (ADULT_REST_RATE
- ADULT_EMERGENCE_RATE) * dt
INIT ADULT_REST = 0
      TRANSIT TIME = 18
      INFLOW LIMIT = ∞
      CAPACITY = ∞
INFLOWS:
ADULT_REST_RATE = ADULT_FED
OUTFLOWS:
ADULT_EMERGENCE_RATE = CONVEYOR OUTFLOW


ADULT_UNFED(t) = ADULT_UNFED(t - dt) + (N_MOLT -
ADULT_FEED_RATE - ADULT_UNFED_DEATH) * dt
INIT ADULT_UNFED = 0
INFLOWS:
N_MOLT = CONVEYOR OUTFLOW
OUTFLOWS:
```

```
ADULT_FEED_RATE = IF
(WEEKLY_ADULT_TICK_BURDEN_DEER>=ADULT_UNFED) THEN
WEEKLY_ADULT_TICK_BURDEN_DEER ELSE (ADULT_UNFED)
ADULT_UNFED_DEATH = (1-ADULT_SURVIVAL_RATE) *
ADULT_UNFED

EGGS(t) = EGGS(t - dt) + (EGG_EMERGENCE_RATE -
EGG_HATCH - EGG_FAILURE) * dt
INIT EGGS = 0
INFLOWS:
EGG_EMERGENCE_RATE = CONVEYOR OUTFLOW
OUTFLOWS:
EGG_HATCH =
(EGGS*EGG_SURVIVAL*EGG_ENVIRONMENTAL_FACTOR_2)
EGG_FAILURE = IF (EGG_ENVIRONMENTAL_FACTOR_2 = 0) THEN
EGGS ELSE (1-EGG_SURVIVAL)*EGGS

EGGS_REST(t) = EGGS_REST(t - dt) + (EGG_OVIPOSITION -
EGG_EMERGENCE_RATE - EGG_REST_FAILURE) * dt
INIT EGGS_REST = 0
        TRANSIT TIME = 9
        INFLOW LIMIT = ∞
        CAPACITY = ∞
INFLOWS:
EGG_OVIPOSITION =
ADULT_OVIPOSITION*ADULT_FEMALE_POPULATION_RATIO*
EGG_MAX
OUTFLOWS:
EGG_EMERGENCE_RATE = CONVEYOR OUTFLOW
EGG_REST_FAILURE = LEAKAGE OUTFLOW
        LEAKAGE FRACTION = IF
(EGG_ENVIRONMENTAL_FACTOR_2=0) THEN 1 ELSE (1-
EGG_FECUNDITY)
        NO-LEAK ZONE = 0

LARVAE_ACTIVE(t) = LARVAE_ACTIVE(t - dt) + (EGG_HATCH
- LARVAE_DEATH - LARVAE_FEED) * dt
INIT LARVAE_ACTIVE = 0
INFLOWS:
EGG_HATCH =
(EGGS*EGG_SURVIVAL*EGG_ENVIRONMENTAL_FACTOR_2)
OUTFLOWS:
LARVAE_DEATH = IF (WEEKLY_LARVAL_TICK_BURDEN=0) THEN
LARVAE_ACTIVE ELSE (1-LARVAE_SURV_RATE)*LARVAE_ACTIVE
```

```
LARVAE_FEED = IF (LARVAE_ACTIVE >=
WEEKLY_LARVAL_TICK_BURDEN) THEN
WEEKLY_LARVAL_TICK_BURDEN ELSE (LARVAE_ACTIVE)

LARVAE_FED(t) = LARVAE_FED(t - dt) + (LARVAE_FEED -
LARVAE_MOLT - LARVAE_FED_DEATH) * dt
INIT LARVAE_FED = 0
      TRANSIT TIME = 40
      INFLOW LIMIT = ∞
      CAPACITY = ∞
INFLOWS:
LARVAE_FEED = IF (LARVAE_ACTIVE >=
WEEKLY_LARVAL_TICK_BURDEN) THEN
WEEKLY_LARVAL_TICK_BURDEN ELSE (LARVAE_ACTIVE)
OUTFLOWS:
LARVAE_MOLT = CONVEYOR OUTFLOW
LARVAE_FED_DEATH = LEAKAGE OUTFLOW
      LEAKAGE FRACTION = (1-LARVAE_SURV_RATE)
      NO-LEAK ZONE = 0

NYMPHS_ACTIVE(t) = NYMPHS_ACTIVE(t - dt) +
(LARVAE_MOLT - N_DEATH - N_FEED) * dt
INIT NYMPHS_ACTIVE = 0
INFLOWS:
LARVAE_MOLT = CONVEYOR OUTFLOW
OUTFLOWS:
N_DEATH = IF (WEEKLY_NYMPH_TICK_BURDEN = 0) THEN
NYMPHS_ACTIVE ELSE (1-NYMPH_SURVIVAL_RATE) *
NYMPHS_ACTIVE
N_FEED = IF (NYMPHS_ACTIVE >=
WEEKLY_NYMPH_TICK_BURDEN) THEN
WEEKLY_NYMPH_TICK_BURDEN ELSE (NYMPHS_ACTIVE)

NYMPH_FED(t) = NYMPH_FED(t - dt) + (N_FEED - N_MOLT -
NYMPH_FED_DEATH) * dt
INIT NYMPH_FED = 0
      TRANSIT TIME = 26
      INFLOW LIMIT = ∞
      CAPACITY = ∞
INFLOWS:
N_FEED = IF (NYMPHS_ACTIVE >=
WEEKLY_NYMPH_TICK_BURDEN) THEN
WEEKLY_NYMPH_TICK_BURDEN ELSE (NYMPHS_ACTIVE)
OUTFLOWS:
N_MOLT = CONVEYOR OUTFLOW
```

```
NYMPH_FED_DEATH = LEAKAGE OUTFLOW
      LEAKAGE FRACTION = (1-NYMPH_SURVIVAL_RATE)
      NO-LEAK ZONE = 0


TOTAL_ACTIVE_NYMPHS_YEAR(t) =
TOTAL_ACTIVE_NYMPHS_YEAR(t - dt) +
(TOTAL_ACTIVE_NYMPHS_IN - TOTAL_ACTIVE_NYMPHS_OUT) * dt
INIT TOTAL_ACTIVE_NYMPHS_YEAR = 0
INFLOWS:
TOTAL_ACTIVE_NYMPHS_IN = LARVAE_MOLT
OUTFLOWS:
TOTAL_ACTIVE_NYMPHS_OUT = IF LARVAE_MOLT = 0 THEN
TOTAL_ACTIVE_NYMPHS_YEAR ELSE 0


ADULT_FEMALE_POPULATION_RATIO = 0.5
ADULT_SURVIVAL_RATE = .92
EGG_ENVIRONMENTAL_FACTOR = (SIN(2*PI/52*TIME))*1.632
EGG_ENVIRONMENTAL_FACTOR_2 = IF
(EGG_ENVIRONMENTAL_FACTOR<0) THEN 0 ELSE
EGG_ENVIRONMENTAL_FACTOR
EGG_FECUNDITY = 1
EGG_MAX = 3000 {Max number of eggs per oviposition}
EGG_SURVIVAL = .0336
LARVAE_SURV_RATE = .37
NYMPH_SURVIVAL_RATE = .92


Tick Population Dynamics 2
AADULT_FED_2(t) = AADULT_FED_2(t - dt) + (ADULT_FEED_2
- ADULT_REST_RATE_2) * dt
INIT AADULT_FED_2 = 0
INFLOWS:
ADULT_FEED_2 = IF
(WEEKLY_ADULT_TICK_BURDEN_DEER>=ADULT_UNFED_2) THEN
WEEKLY_ADULT_TICK_BURDEN_DEER ELSE (ADULT_UNFED_2)
OUTFLOWS:
ADULT_REST_RATE_2 = AADULT_FED_2


ADULT_OVIPOSITION_2(t) = ADULT_OVIPOSITION_2(t - dt) +
(ADULT_EMERGENCE_RATE_2 - ADULT_OVIPOSITION_DEATH_2) *
dt
INIT ADULT_OVIPOSITION_2 = 0
INFLOWS:
ADULT_EMERGENCE_RATE_2 = CONVEYOR OUTFLOW
OUTFLOWS:
ADULT_OVIPOSITION_DEATH_2 = ADULT_OVIPOSITION_2
```

```
ADULT_REST_2(t) = ADULT_REST_2(t - dt) +
(ADULT_REST_RATE_2 - ADULT_EMERGENCE_RATE_2) * dt
INIT ADULT_REST_2 = 0
        TRANSIT TIME = 18
        INFLOW LIMIT = ∞
        CAPACITY = ∞
INFLOWS:
ADULT_REST_RATE_2 = AADULT_FED_2
OUTFLOWS:
ADULT_EMERGENCE_RATE_2 = CONVEYOR OUTFLOW

ADULT_UNFED_2(t) = ADULT_UNFED_2(t - dt) +
(NYMPH_MOLT_2 - ADULT_FEED_2 - ADULT_UNFED_DEATH_2) *
dt
INIT ADULT_UNFED_2 = 0
INFLOWS:
NYMPH_MOLT_2 = CONVEYOR OUTFLOW
OUTFLOWS:
ADULT_FEED_2 = IF
(WEEKLY_ADULT_TICK_BURDEN_DEER>=ADULT_UNFED_2) THEN
WEEKLY_ADULT_TICK_BURDEN_DEER ELSE (ADULT_UNFED_2)
ADULT_UNFED_DEATH_2 = (1-ADULT_SURVIVAL_RATE) *
ADULT_UNFED_2

EGGS_2(t) = EGGS_2(t -dt) + (EGG_EMERGENCE_RATE_2 -
EGG_HATCH_2 - EGG_FAILURE_2) * dt
INIT EGGS_2 = 0
INFLOWS:
EGG_EMERGENCE_RATE_2 = CONVEYOR OUTFLOW
OUTFLOWS:
EGG_HATCH_2 =
(EGGS_2*EGG_SURVIVAL*EGG_ENVIRONMENTAL_FACTOR_2)
EGG_FAILURE_2 = IF (EGG_ENVIRONMENTAL_FACTOR_2=0) THEN
EGGS_2 ELSE (1-EGG_SURVIVAL)*EGGS_2

EGG_REST_2(t) = EGG_REST_2(t - dt) +
(EGG_OVIPOSITION_2 - EGG_EMERGENCE_RATE_2 -
EGG_REST_FAILURE_2) * dt
INIT EGG_REST_2 = 0
        TRANSIT TIME = 9
        INFLOW LIMIT = ∞
        CAPACITY = ∞
INFLOWS:
EGG_OVIPOSITION_2 = ADULT_OVIPOSITION_2*ADULT_FEMALE_
POPULATION_RATIO*EGG_MAX
```

```
OUTFLOWS:
EGG_EMERGENCE_RATE_2 = CONVEYOR OUTFLOW
EGG_REST_FAILURE_2 = LEAKAGE OUTFLOW
      LEAKAGE FRACTION  = IF
(EGG_ENVIRONMENTAL_FACTOR_2=0) THEN 1 ELSE (1-
EGG_FECUNDITY)
      NO-LEAK ZONE = 0


LARVAE_ACTIVE_2(t) = LARVAE_ACTIVE_2(t - dt) +
(EGG_HATCH_2 - L_DEATH_2 -LARVAE_FEED_2) * dt
INIT LARVAE_ACTIVE_2 = 0
INFLOWS:
EGG_HATCH_2 =
(EGGS_2*EGG_SURVIVAL*EGG_ENVIRONMENTAL_FACTOR_2)
OUTFLOWS:
L_DEATH_2 = IF (WEEKLY_LARVAL_TICK_BURDEN=0) THEN
LARVAE_ACTIVE_2 ELSE (1-
LARVAE_SURV_RATE)*LARVAE_ACTIVE_2
LARVAE_FEED_2 = IF (LARVAE_ACTIVE_2 >=
WEEKLY_LARVAL_TICK_BURDEN) THEN
WEEKLY_LARVAL_TICK_BURDEN ELSE (LARVAE_ACTIVE_2)


LARVAE_FED_2(t) = LARVAE_FED_2(t - dt) +
(LARVAE_FEED_2 - LARVAE_MOLT_2 - LARVAE_FED_DEATH_2) *
dt
INIT
LARVAE_FED_2 = 0
      TRANSIT TIME = 40
      INFLOW LIMIT = ∞
      CAPACITY = ∞
INFLOWS:
LARVAE_FEED_2 = IF (LARVAE_ACTIVE_2 >=
WEEKLY_LARVAL_TICK_BURDEN) THEN
WEEKLY_LARVAL_TICK_BURDEN ELSE (LARVAE_ACTIVE_2)
OUTFLOWS:
LARVAE_MOLT_2 = CONVEYOR OUTFLOW
LARVAE_FED_DEATH_2 = LEAKAGE OUTFLOW
      LEAKAGE FRACTION = (1-LARVAE_SURV_RATE)
      NO-LEAK ZONE = 0


NYMPH_ACTIVE_2(t) = NYMPH_ACTIVE_2(t - dt) +
(LARVAE_MOLT_2 - N_DEATH_2 - NYMPH_FEED_2) * dt
INIT NYMPH_ACTIVE_2 = 350
INFLOWS:
LARVAE_MOLT_2 = CONVEYOR OUTFLOW
```

```
OUTFLOWS:
N_DEATH_2 = IF (WEEKLY_NYMPH_TICK_BURDEN=0) THEN
NYMPH_ACTIVE_2 ELSE (1-NYMPH_SURVIVAL_RATE) *
NYMPH_ACTIVE_2
NYMPH_FEED_2 = IF (NYMPH_ACTIVE_2 >=
WEEKLY_NYMPH_TICK_BURDEN) THEN WEEKLY_NYMPH_TICK_BURDEN
ELSE (NYMPH_ACTIVE_2)

NYMPH_FED_2(t) = NYMPH_FED_2(t - dt) + (NYMPH_FEED_2 -
NYMPH_MOLT_2 - NYMPH_FED_DEATH_2) * dt
INIT NYMPH_FED_2 = 0
      TRANSIT TIME = 26
      INFLOW LIMIT = ∞
      CAPACITY = ∞
INFLOWS:
NYMPH_FEED_2 = IF (NYMPH_ACTIVE_2 >=
WEEKLY_NYMPH_TICK_BURDEN) THEN WEEKLY_NYMPH_TICK_BURDEN
ELSE (NYMPH_ACTIVE_2)
OUTFLOWS:
NYMPH_MOLT_2 = CONVEYOR OUTFLOW
NYMPH_FED_DEATH_2 = LEAKAGE OUTFLOW
      LEAKAGE FRACTION = (1-NYMPH_SURVIVAL_RATE)
      NO-LEAK ZONE = 0

TOTAL_ACTIVE_NYMPHS_YEAR_2(t) =
TOTAL_ACTIVE_NYMPHS_YEAR_2(t - dt) +
(TOTAL_ACTIVE_NYMPHS_IN_2 - TOTAL_ACTIVE_NYMPHS_OUT_2)
* dt
INIT TOTAL_ACTIVE_NYMPHS_YEAR_2 = 0
INFLOWS:
TOTAL_ACTIVE_NYMPHS_IN_2 = LARVAE_MOLT_2
OUTFLOWS:
TOTAL_ACTIVE_NYMPHS_OUT_2 = IF LARVAE_MOLT_2 = 0 THEN
TOTAL_ACTIVE_NYMPHS_YEAR_2 ELSE 0
```

22
Tragedy of the Commons

> In real life relevant information is essential. And the bother is that we
> never have all the information we would like. Nature has established
> patterns, but only for the most part. Theory, which abstracts from na-
> ture, is kinder: we either have the information we need or else we have
> no need for information.
>
> —Peter L. Bernstein, *Against the Gods*

Prior to the mid-1750s, the villages of England had large nearby areas that be-
longed to the village. These common areas were not held in private own-
ership. Rather, many members of the village raised sheep on their land and
often released their extra sheep upon the commons. They reasoned that even
if their animals on the commons did not grow well due to overconsumption
of the grass, any growth was a benefit that would have otherwise not materi-
alized. So the villagers keep putting out sheep until these sheep overcon-
sumed the grass, resulting in the collapse of the commons and the death of
the sheep. This process is called the tragedy of the commons and it exists in
many areas today, including air and water pollution. The air and water are still
generally thought of as "free" dumping places for the pollutants of utilities, in-
dustries, and autos, though the regulators and the economists are catching up!

Let us model this process of unimpaired access to the commons and the
resulting tragedy. Once we developed such a model, we can devise a
procedure based on cooperation in the village that leads to the maximum
number of sheep sustainable on the commons at a steady state.

Assume that the weather is constant and that the commons has a unit
area with an initial grass stock of 250. This grass stock can reach a maxi-
mum 1000 units. We model the growth of grass with a simple biological
function, shown in the graph in Figure 22.1. A similar version of this func-
tion was employed in the early models of this book.

Assume that the only way that the grass can be removed is through con-
sumption by sheep. The amount of grass eaten per head is a function of the
amount of grass, and the life span of the sheep is a function of the con-
sumption per head. The death rate is also specified graphically and shown
in Figure 22.2.

The biological functions are assumed to be given and cannot be inten-
tionally changed, i.e., they are not part of the decision rules of the model.
Though we cannot influence the biological functions, we have control over
the addition of sheep. The initial stock of sheep is 50 individuals and the in-
itial stock of grass is 250 tons. At first, let us model the add rate as a func-
tion of what we think that everyone else is doing. So we add sheep at a rate
proportional to the number of sheep on the commons. The model is shown
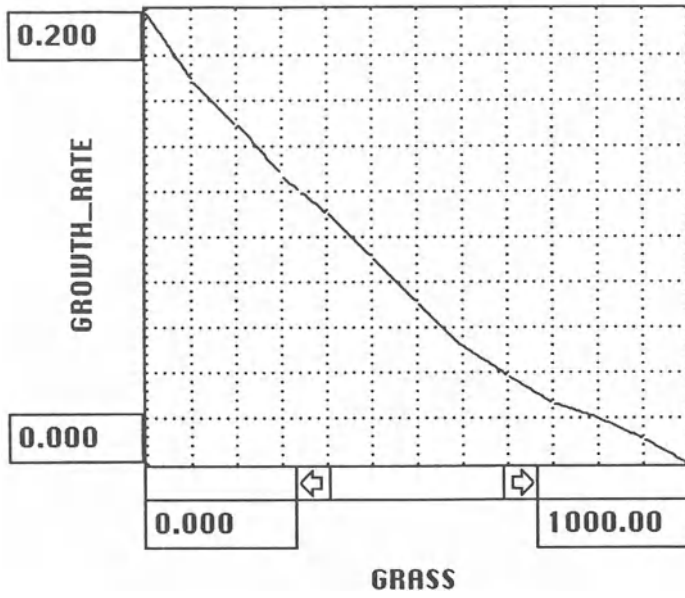
FIGURE 22.1
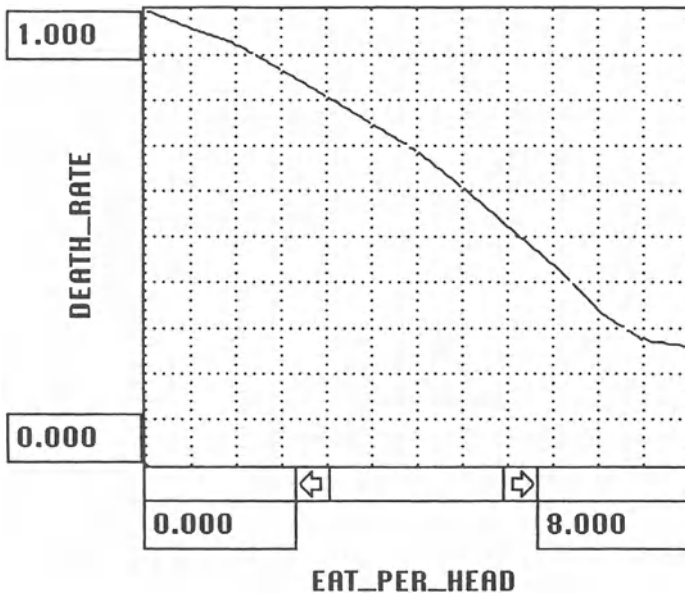
FIGURE 22.2
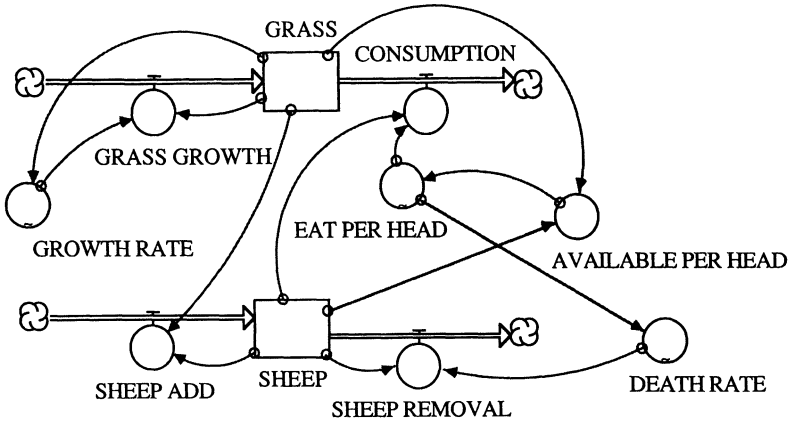
FIGURE 22.3

in Figure 22.3 and the results are plotted in Figure 22.4. Indeed, we observe the tragedy of the commons.

Alternatively to adding sheep proportional to the number of sheep on the commons, we may choose a decision rule that is less detrimental to the resource necessary to maintain the population of sheep. For example, we may want to add sheep proportional to the amount of grass on the commons. The result of this model is shown in Figure 22.5. The initial number of sheep is too large for the commons, resulting in a decline of the amount of grass available and a reduction in the size of the sheep population. As the stock of grass recovers, we are able to add more sheep.

A third decision rule would be to add sheep proportional to the number of sheep on the commons *and* proportional to the amount of grass. The result
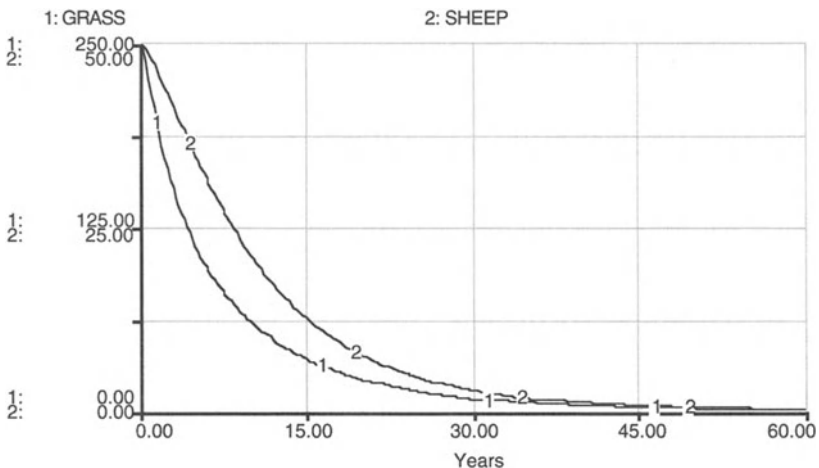
FIGURE 22.4
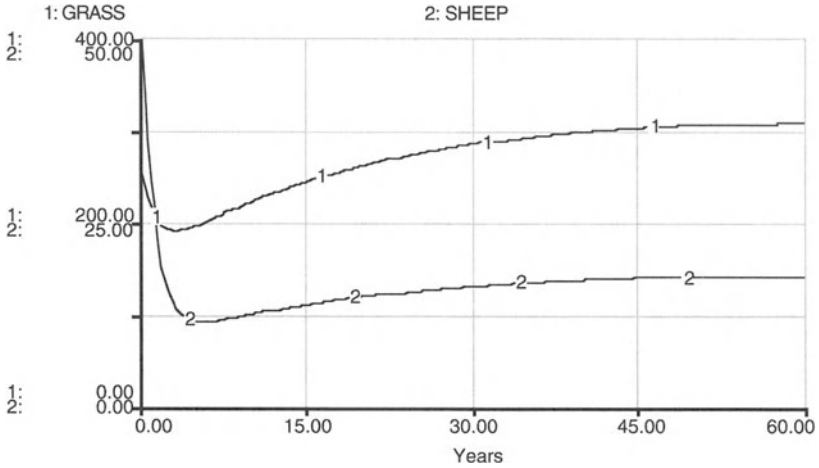
FIGURE 22.5

of this model is shown in Figure 22.6. Of course, with this decision rule we will not do as well in the long run as when we only consider the resource base for our decision. Again, we do not avoid the tragedy of the commons.

Now the problem becomes one of finding the largest steady state level of sheep that can be raised on this commons through cooperation. Try adding sheep only if the grass is above a certain level. Imagine that the Commons Board, appointed by the village, measures the grass regularly and issues permits on some equitable basis to the shepherds of the village. What level of grass is controlling and how many sheep to add, are the critical ques-
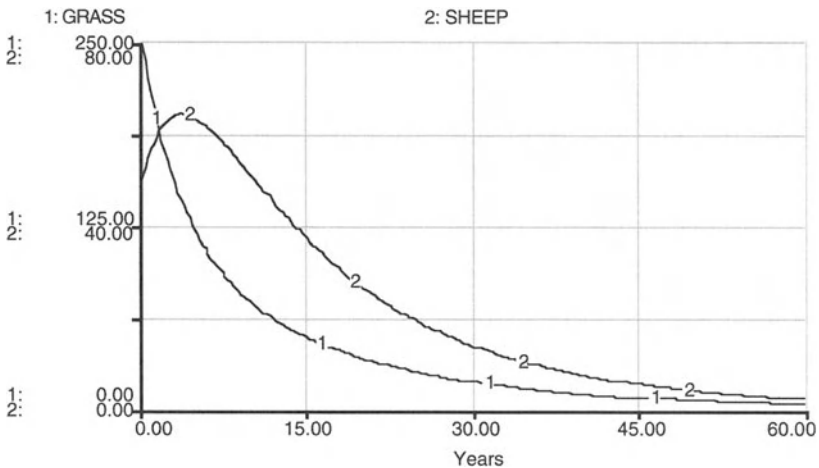
FIGURE 22.6

tions. Set up the model such that the Board takes one whole DT = 1 to report and issue the permits. In such a model extension, we have a lag in the system.

Now try making the weather cycle through the seasons, that is, have the grass senesce periodically and then regrow. Or perhaps more appropriately, have the grass growth rate cycle between 0 and some maximum representing the seasonal effects. Does this make it harder to avoid the tragedy?

TRAGEDY OF THE COMMONS

```
GRASS(t) = GRASS(t - dt) + (GRASS_GROWTH -
CONSUMPTION) * dt
INIT GRASS = 250

INFLOWS:
GRASS_GROWTH = GRASS*GROWTH_RATE
OUTFLOWS:
CONSUMPTION = SHEEP*EAT_PER_HEAD
SHEEP(t) = SHEEP(t - dt) + (SHEEP_ADD - SHEEP_REMOVAL)
* dt
INIT SHEEP = 50

INFLOWS:
SHEEP_ADD = .85*SHEEP*0 + .05*GRASS*1
OUTFLOWS:
SHEEP_REMOVAL = SHEEP*DEATH_RATE
AVAILABLE_PER_HEAD = GRASS/SHEEP
DEATH_RATE = GRAPH(EAT_PER_HEAD)
(0.00, 1.00), (0.667, 0.96), (1.33, 0.92), (2.00,
0.87), (2.67, 0.805), (3.33, 0.75), (4.00, 0.685),
(4.67, 0.61), (5.33, 0.525), (6.00, 0.44), (6.67,
0.34), (7.33, 0.275), (8.00, 0.26)
EAT_PER_HEAD = GRAPH(AVAILABLE_PER_HEAD)
(0.00, 0.00), (1.00, 0.23), (2.00, 0.56), (3.00, 0.87),
(4.00, 1.27), (5.00, 1.62), (6.00, 1.85), (7.00, 1.94),
(8.00, 1.98), (9.00, 2.00), (10.0, 2.00)
GROWTH_RATE = GRAPH(GRASS)
(0.00, 0.198), (83.3, 0.169), (167, 0.149), (250,
0.127), (333, 0.111), (417, 0.091), (500, 0.071), (583,
0.052), (667, 0.039), (750, 0.027), (833, 0.0195),
(917, 0.0105), (1000, 0.00)
```

Part 6

Economic Models

23

Introduction to
Modeling Economic Processes

> Now there can be no doubt that pleasure, pain, labor, utility, value,
> wealth, money, capital, etc. are all notions admitting of quantity; nay,
> the whole of our actions in industry and trade certainly depend upon
> comparing quantities of advantage and disadvantage.
> —Stanley W. Jevons, *The Theory of Political Economy*

We will now begin to see the importance of blending analytical and nu-
merical solutions. Analytical methods can carry us to a level of understand-
ing of system processes where further work would be either inefficient or
impossible. For example, for many of the nonlinear dynamic relationships
that underlie our models there are no analytical solutions at all. Numerical
techniques, such as those provided by STELLA, can take over from there.
Conversely, numerical solutions can be sped up significantly and increased
in accuracy if we make use of analytical methods. Analytics can give us
equations that will describe not just optimal operating points but whole op-
timal trajectories through time—a problem too difficult for numerics to for-
mulate and too difficult for analytics to completely solve. Therefore, we at-
tempt to merge both methods to build on the strength of each and arrive at
a solution superior to each individual method.

Let us now turn to some models in economics. When trying to learn a
new field, look for the unique principles that that field uses. We can iden-
tify three principles that underlie economics: substitution, opportunity cost,
and time value. When machinery and energy are used to replace labor in
agricultural production, it is an example of substitution. Completely differ-
ent inputs (tractors, gasoline, information) increase in use while the person-
hours of labor decrease at a given output.

A computer programmer who makes $25 per hour works as a carpenter
on her own home rather than hire a carpenter whom she would have to
pay only $15 per hour. The programmer could have worked extra time as a
programmer but chose to be a carpenter in her evenings instead. Assuming
that all other costs (or pleasures) associated with working at the home are
equal to those of working as a programmer, the person encounters an op-
portunity cost—the cost of those evening hours—of $25 per hour.

If someone loans you $100 dollars for a year, it will not suffice for you to
give that person back $100 one year from now. You must promise to pay
her a premium to keep her from using the $100 now. Foregoing consump-
tion is an expensive and uncertain venture. Why does that person insist on
the extra pay? You may not be able to repay the loan at all. Furthermore,

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TABLE 23.1

Term	Typical Units	Symbol
Input quantity	Tons/year	X
Production, output	Tons/year	Q
Total cost	Dollars/year	C
Average cost	Dollars/ton	AC
Marginal cost	Dollars/ton	MC
Price	Dollars/ton	P
Revenue	Dollars/year	R
Marginal revenue	Dollars/ton	MR
Profit	Dollars/year	R-C

she did have to wait. So you must pay her for the risk and the delay. The amount paid in excess of the $100 reflects time value.

These principles—substitution, opportunity cost, and time value—are explicitly dealt with later on in this part of the book. First, however, we must understand the ideas of competition, monopoly, and economic equilibrium. These provide the framework for our analyses. We use a series of three models to illustrate frequently made assumptions on the behavior of firms. For example, a standard assumption is that firms expand the level of their output until they maximize the level their profits. Another important assumption is that firms have perfect information about all current and future features of the economy and environment that are necessary for their decision making. All relevant information is correctly subsumed in the price of a good or service, thereby giving important signals to producers and consumers for their behavior. These assumptions are shown in the following three problems. It is convenient, however, to first define the terms we are using (Table 23.1).

24

The Competitive Firm

> The idea ... was to do good and net a profit, there being no competition to speak of.
>
> —James Joyce, *Ulysses*

Let us assume that there are a large number of firms that produce an identical product sold on the same market. The output generated by each firm is very small in comparison with the demand for that product. As a result, each firm cannot influence the price of its product: If the firm chooses to sell the product at a price slightly higher than that of its competitors, it will lose its customers. If the firm sells below the market price, demand for its product increases, driving up the cost of production, thereby forcing the firm to increase its price. In such a perfectly competitive setting, the price of a product is given for each firm. Perfect competition may be assumed for markets for final products or markets for production inputs, such as labor, capital, materials, energy, and information.

In addition to perfect competition we may assume that each firm attempts to maximize its profits. For simplicity, we assume that there is only a single input used in the production process. The price of the input, R1, is given for the firm. The production cost is directly proportional to the amount X of the input,

$$C = R1*X \qquad (1)$$

The input is used by the firm according to a given "recipe," the production function (see Chapter 3 for a discussion). The production function for our firm is assumed to be

$$Q = A*X^{\wedge}ALPHA \qquad (2)$$

where A is a scaling parameter and ALPHA a constant exponent.

Maximizing profits can be done by varying the quantity of inputs used in the production process. With increasing amounts of the input, output increases. This output can be sold on the market at the price P and generates a revenue, R = P*Q. However, increasing amounts of the input increases the cost of production. Thus, in order to maximize profits, the competitive firm must maximize the difference between revenues and cost:

$$PROFIT = R - C \qquad (3)$$

In Chapter 7 we noted that STELLA is most commonly used to solve differential equations. In the model of the competitive firm, the differential equation of interest is the equation that describes the change in profits. If the change in profits is positive with increasing input, then it is advantageous for

the firm to keep on expanding. If increases in input lead to a decrease in profits, then the optimal size of the firm (as measured by the amount of Q produced) is surpassed. Thus, there is an optimum amount Q at which profit is maximum.

This relationship between changes in Q and in PROFIT can be expressed mathematically as a derivative. We saw in Chapter 7 that STELLA can be used to calculate derivatives of functions, or equivalently the slopes of graphs representing the relationship among variables. Derivatives, or slopes, are expressed as a ratio of the change in one variable to a change in another variable. This relationship, however complicated, must contain only two variables, so that the ratio is unambiguous. Slopes are useful in finding the peaks or minima that one variable reaches as the other changes. Mathematically such changes are expressed as total, not partial, derivatives.

STELLA can easily handle the total derivatives but not the partial derivatives. Total derivatives are just the difference in a state variable. The functional form of the partial derivatives must be entered directly in the program.

How can we calculate the optimal output of the firm? In order to find the optimal output, the firm must successively adjust its input such that the peak in the PROFIT curve is reached. This adjustment process is done in the program at only one place, the flow CHANGE X. The model for this optimization problem is shown in Figure 24.1.
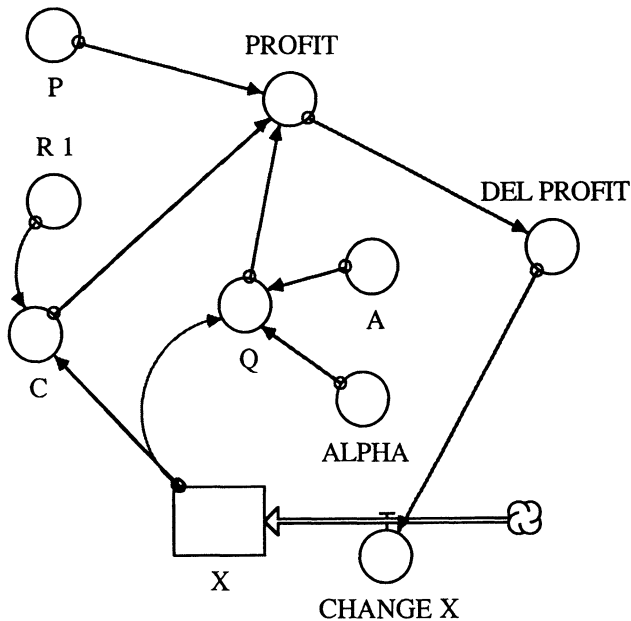
FIGURE 24.1

The optimization model is verified by comparing marginal cost with price, here held constant. This is one way in which economists build trust in a model—by checking its consistency with theory. Another, more ambitious approach would be to check against reality—more ambitious because it may require not just a rethinking of the structure of a model but also the reasonability of the assumptions on which the model is based. The latter could go at the heart of the economic theory itself and may help advance the discipline as a whole.

Economic theory tells us that in the optimum of a profit-maximizing firm in perfectly competitive markets, marginal cost and price are equal. This can be shown mathematically by taking the derivative of the PROFIT function with respect to Q:

$$\frac{\partial \text{PROFIT}}{\partial Q} = \frac{\partial R}{\partial Q} - \frac{\partial C}{\partial Q} = MR - MC = P - MC \tag{4}$$

This derivative must be equal to 0 for the firm to be in an optimum. Thus

$$P - MC = 0 \Rightarrow P = MC. \tag{5}$$

The STELLA program for the profit-maximizing firm contains the following part, used to verify the optimum by applying economic theory. With this model component we can calculate at each period the marginal cost and the marginal revenue. The latter is equal to the price. As the optimum is approached, the marginal cost curve should rise to the marginal revenue curve. Both should be equal in the optimum. Marginal cost and revenue are calculated with the module of Figure 24.2, and indeed the expected result holds, as Figures 24.3 and 24.4 show.
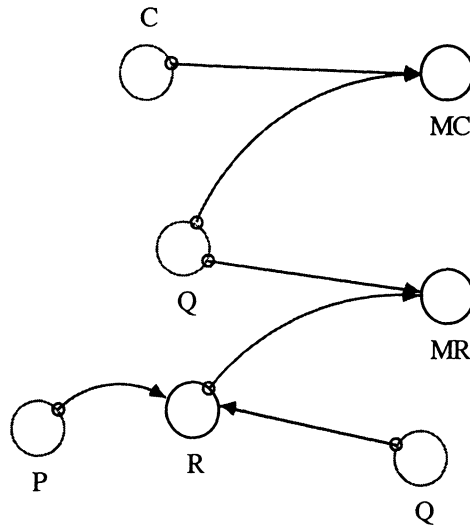
FIGURE 24.2
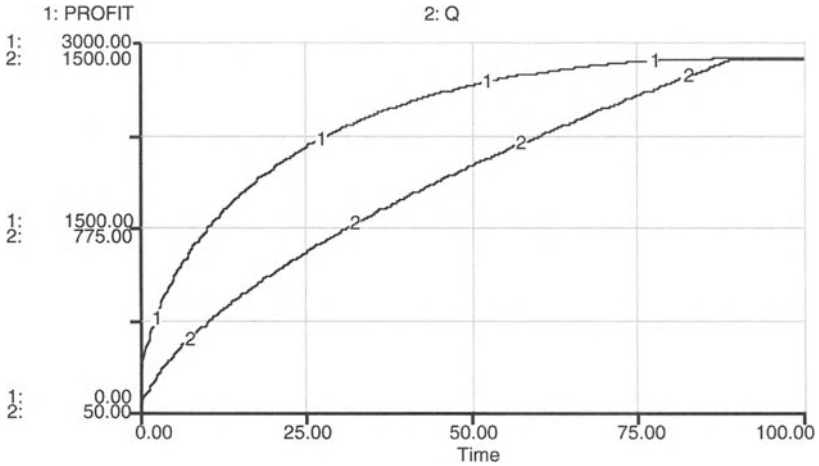
FIGURE 24.3

Profit maximizing is only one of several possible assumptions about the behavior of a firm. Can you put yourself in the place of our businesswoman and name several other objectives? How would you model alternative behavioral assumptions?

The slope of the profit with respect to Q is found in this problem by comparing the current profit with the delayed profit, the profit rate one small time step ago. If the change in profit becomes 0 or negative, CHANGE X changes from 3 to 0. When the profit rate is positive, the input
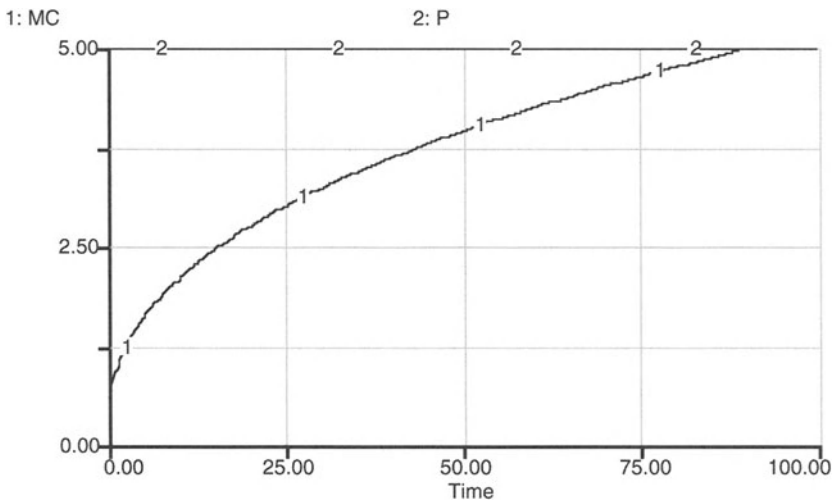
FIGURE 24.4

X is indexed upward through CHANGE X. The input X allows an increase in Q through the production function, and it also feeds to the cost calculation with the unit cost of the input X, R1. The marginal cost (MC) and the marginal revenue (MR) are calculated in the right-hand-side part of the model. MC is the change in cost divided by the change in Q. MR is the change in revenue (P*Q) divided by the change in Q.

The maximum profit for this single competitive firm is not 0 (2878) when its optimal Q (1437) is reached. This means that more firms will enter the market, driving the price down and their collective output Q up, until the profit rate of the last firm is 0. If we assume that all the firms in the market use identical technology (A and ALPHA are the same for each firm) and pay the same rate for a unit of input, R1, the problem can be solved in STELLA, and we will do so in Chapter 26. However, first we will deal with the firm that handles two rather than one input. The rate at which the profit rate is maximized is of course controlled by CHANGE X. In reality this rate is set by the rate at which a new plant can be designed, built, and brought on-line.

Note well that the rate of growth to the maximum profit point is arbitrary. The actual rate of growth of a firm to the approximate target size is the rate of investment in new and replacement capital, the availability of trained labor and natural resources, and the understanding of how to employ them with near minimum cost. In our model, try to change the rate at which the plant grows. Change its production function. What happens if you make ALPHA greater than 1 and what is the meaning of this? When ALPHA is less than 1, we are representing one of the principle truths of economics: during expansion, sooner or later, the law of diminishing returns sets in. Diminishing returns to scale means that an increase in inputs leads to an increase of output but at a decreasing rate.

Before we leave this problem, let's try to make it more realistic. Try to add a capital stock to the problem. Make the rate of growth of X a delayed function of the capital stock. Add the cost of the borrowed money to pay for the stock to the cost function. Add the productivity of the capital stock to the production function (Q). If we let the profit change function determine the rate of capital expansion, and the size of the capital stock determine (with a lag time) the rate of use of X, we have a reasonable model of a firm.

```
           PROFIT MAXIMIZING COMPETITIVE FIRM

  X(t) = X(t - dt) + (CHANGE_X) * dt
  INIT X = 1 {Units of X}
  INFLOWS:
  CHANGE_X = IF DEL_PROFIT > 0 THEN 3 ELSE 0 {Units of X
  per Time Period}
  A = 50 {Units of Q per Units of X}
```

```
ALPHA = .6
C = R_1*X {$}
DEL_PROFIT = PROFIT - DELAY(PROFIT,DT,.1) {$}
MC = (C - DELAY(C,DT,.1))/(Q - DELAY(Q,DT,.1)) {$ per
Units of Q}
MR = (R - DELAY(R,DT,.1))/(Q - DELAY(Q,DT,.1)) {$ per
Units of Q}
P = 5 {$ per Unit}
PROFIT = P*Q - C {$}
Q = A*X^ALPHA {Units of Q}
R = P*Q {$}
R_1 = 16 {$ per Unit X}
```

25

The Monopolistic Firm

Monopoly, that is, exclusion of customers, has certainly no tendency to produce increase of the number of traders.
—Jeremy Bentham, *Emancipate Your Colonies*

25.1 Basic Model

In this chapter we model the same firm as we saw in the problem of the previous chapter but with new management. The firm is now a monopoly, meaning that it can adjust its output and selling price to maximize its profit. This situation means that the profits will be higher than was the case with the competitive firm in this market and the output will be lower.

The only difference here from the model in the previous chapter is that price is now a function of the quantity consumed. The relationship between price and the quantity consumed is given by a demand curve. A linear demand curve is shown in the Figure 25.1.

Each unit of a good gives utility. If consumption increases, utility increases at a decreasing rate. Therefore, the demand curve is downward sloping, expressing the fact that consumers are willing to pay less the more of the good can be consumed. In our model, the demand curve was concocted to include the P = MC point of the model in the previous chapter. Additionally, we arbitrarily chose a price at which the quantity demanded becomes 0. This "choke-off price" is set in the model to 10 and corresponds to the intercept of the demand curve above with the vertical axis.

Price per
Unit (e.g.
$ per ton)

Quantity (e.g. tons per year)

FIGURE 25.1

267
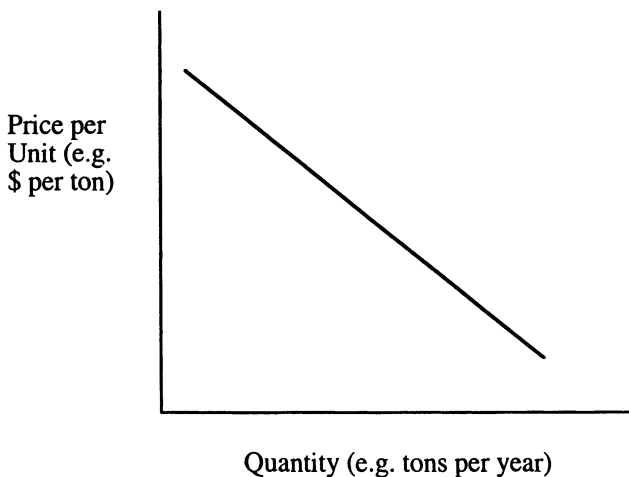
How does the monopoly choose its profit-maximizing-output level and what is the corresponding price? As in the analytic solution of the previous chapter, we take first the derivative of the profit function with respect to output. We arrive at the condition that marginal revenues must equal marginal cost if the firm wants to achieve a profit maximum:

$$\frac{\partial PROFIT}{\partial Q} = \frac{\partial R}{\partial Q} - \frac{\partial C}{\partial Q} = MR - MC = 0 \Rightarrow MR = MC \qquad (1)$$

However, MR ≠ P because the price depends on the demand curve. The difference between price and marginal cost is the monopoly rent rate. The solution to the monopolist's profit maximization problem is shown in Figure 25.2 as Q*, the corresponding price is P*. The supply curve is the monopolist's marginal cost curve.

Figure 25.2 illustrates that in the optimum, the monopolist produces less and charges a higher price than a firm in a perfectly competitive market. As a result, profits will be higher, too.

Let us model the process in which the monopolist identifies her profit-maximizing output level (Fig. 25.3). For the monopolist whose initial output level is below the profit-maximizing one, price will initially be high but drop with increased production.

The results of the model are shown in Figure 25.4. Profits increase steadily until a maximum is achieved. At that point marginal revenues equal marginal costs, and from then on price and output are held constant. Since output remains constant, the marginal cost and marginal revenue curves are no longer defined in our model, once the optimum is achieved: these curves are calculated by dividing through Q minus DELAY(Q,DT), which is
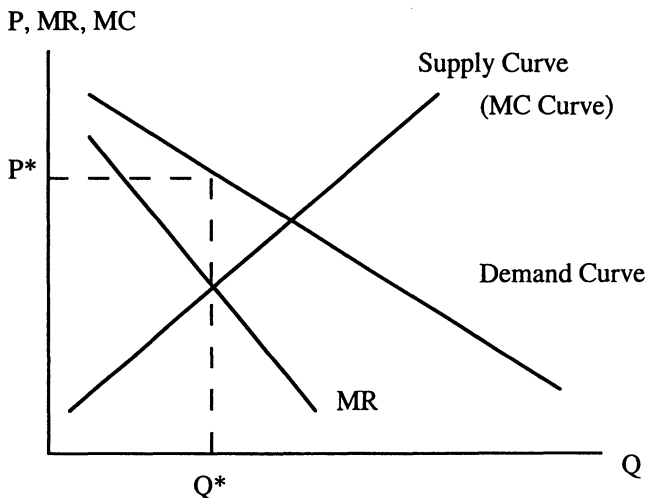
FIGURE 25.2
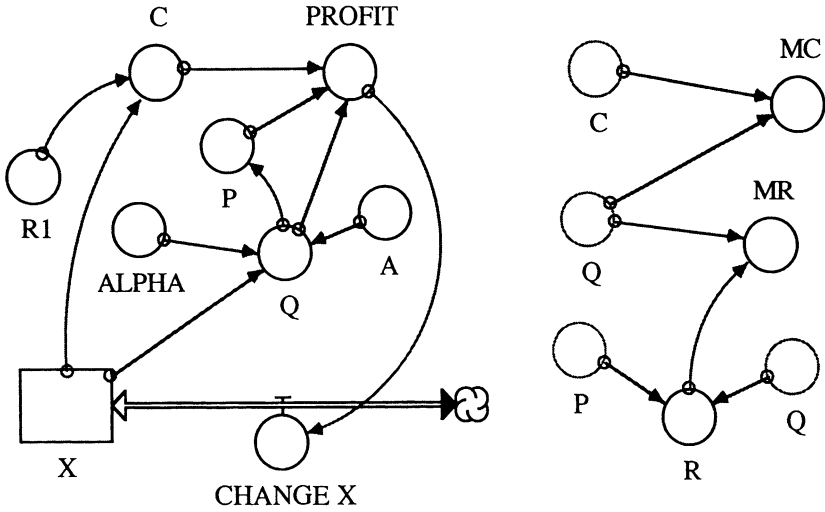
FIGURE 25.3

0. Use STELLA to calculate that monopoly rent rate and observe its change over time. Before you run your model, however, guess the likely outcome.

In Figure 25.4, the maximum profit is found to be $4287 for an output of 934 units, at a price of $6.84 per unit output. We find the marginal revenue is no longer strictly equal to the price, but becomes equal to the marginal cost when the profit is maximized. This is exactly what the analytic theory would predict.

How might you improve the CHANGE X portion of the model to reflect more realistic conditions? Should you hook it to the level of the profit rate
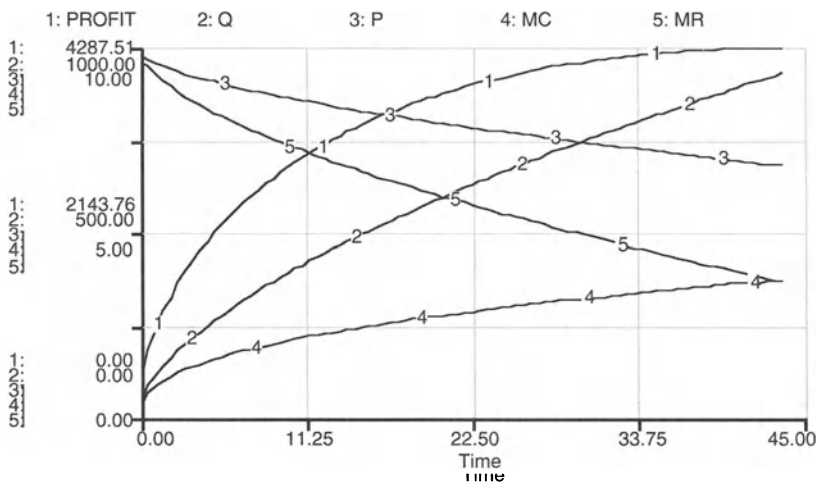
FIGURE 25.4

to show that it would be easier to borrow money (higher CHANGE X) if the profit rate is higher? How about sudden rises in the level of R1? Can the model handle it? If not, try averaging the CHANGE X variable to keep it from being too sensitive. What about inventory? How would you introduce into the model an inventory of produced goods?

```
                    BASIC MONOLISTIC MODEL

X(t) = X(t - dt) + (CHANGE_X) * dt
INIT X = 1 {Units of X}
INFLOWS:
CHANGE_X = IF (PROFIT - DELAY(PROFIT,DT,1)) > 0 THEN 3
ELSE 0 {Units of X per Time Period}

A = 50 {Units of Q per Units of X}
ALPHA = .6
C = R1*X {$}
MC = (C - DELAY(C,DT,.01))/(Q - DELAY(Q,DT,.1)) {$ per
Units of Q}
MR = (R - DELAY(R,DT,10))/(Q - DELAY(Q,DT,.1)) {$ per
Units of Q}
P = 10-.00338*Q {The selling price is a function of Q;
that is, the monopoly controls the output level AND
price, unlike the competitive firm who could not affect
price. This is the demand curve, experimentally
derived. $ per Unit Q}
PROFIT = (P*Q-C) {$}
Q = A*X^ALPHA {Units of Q}
R = P*Q {$}
R1 = 16 {$ per Units X}
```

25.2 Taxing Monopolies

The higher prices of monopolies have given rise to many activities in the U.S. Justice Department over the past century. Monopolies have been broken by the Department in an effort to restore competition, lower prices, and increase supplies.

Frequently, monopolies are price-regulated. The common monopoly might have its price regulated to where $P = MC$, if the regulators can get through the thicket of opposing lawyers and accountants. At this point, the monopoly rent rate is 0 and the firm is still making a profit. Should the regulators manage to reduce the price charged by the monopoly to the average cost of the firm, then profits become 0.

Normally, firms choose output levels corresponding to the rising portion of their marginal and average cost curves. There are natural monopolies, however, that operate in the region of declining average and marginal costs. Electric utilities belong to this group. It can be easily shown that the lowest price to regulate such a natural monopoly is where average cost equals average revenues, i.e.,

$$AC = \frac{R}{Q} = P \qquad (2)$$

If the utilities were regulated such that

$$MC = \frac{R}{Q} = P \qquad (3)$$

the firm fails.[1] Still, partial, or regional, monopolies can develop if the firm is facing a demand curve different from the overall market demand curve, i.e., if the market price is higher than the intersection of its marginal cost curve and its own demand curve.

Another approach, perhaps more tempting for the government than controlling price, is to let monopolies reign and then tax them. Thus, the government might try to avoid the costly antitrust litigation and the separate problem of raising taxes elsewhere.

What sort of monopoly-taxing strategies cause what kinds of impacts? We can easily evaluate three: a tax based on a fraction of the profits, a quantity-based tax, and an annual lump sum tax. In order to assess the impacts of these taxes, we only need to slightly modify the model of the previous section. These modifications are shown in Figure 25.5 and include a new calculation of profits and, in the case of a production tax, a new marginal revenue, NMR.

Figures 25.6 and 25.7 reveal the effect of the profit tax and the quantity-based production tax, respectively. Figure 25.8 shows that the effect of the lump sum tax on output prices is nonexistent. The firm's profit is lowered by the amount of the tax, but the profit-maximizing monopoly will not change its price or its output level. This is the pure tax-collecting role of the monopoly.

It is hard for the government to tell when it is taxing the monopoly too much or not enough. Profits are a very slippery number to accurately determine. Therefore, the most common way to tax a monopoly is on the basis of its output, i.e., to impose a tax on the firm per unit of its output. The graphs show the unfortunate results: consumer price is higher in the case of an output or production tax, and consequently, output is lower. Marginal cost and marginal revenue differ at the profit-maximizing level of

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[1]See Pindyck, R. and D. Rubinfeld. 1989. *Microeconomics*, Macmillan, New York, p. 354.
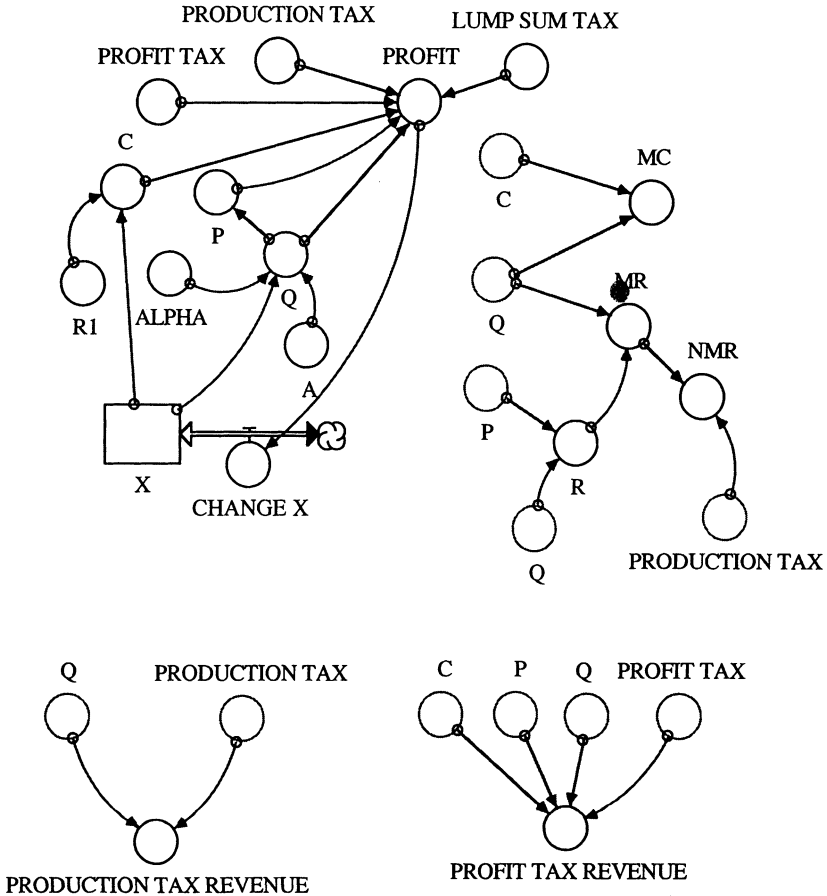
FIGURE 25.5
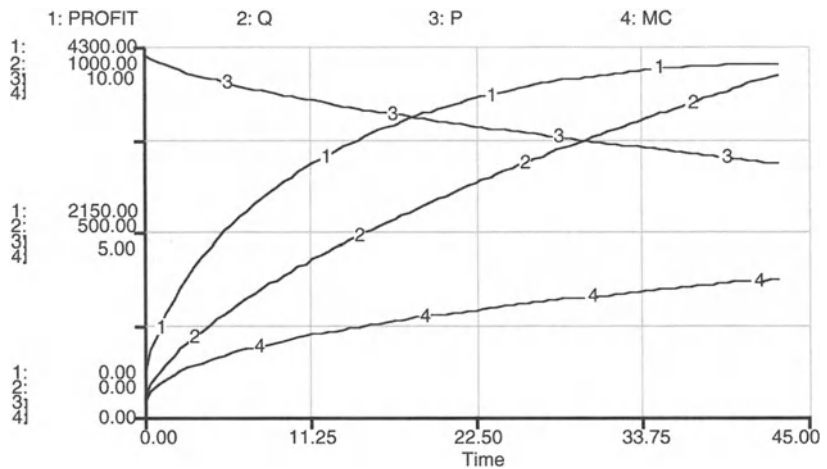
FIGURE 25.6
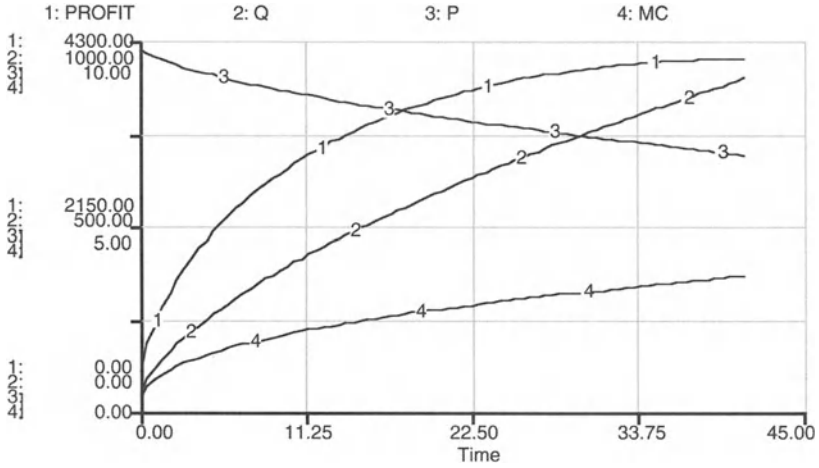
output by the amount of the tax. This is calculated in the model as NMR, the net marginal revenue.

When output-based taxes are placed on monopolies, these firms try to gain the right to use accounting depreciation rates that show losses that are approximately equal to their tax payments, and the revenue situation for the government becomes a draw. So, we as government agents, revert to taxing the average individual consumer where no such explicit optimizing occurs. Yet some individuals can and do employ the needed expertise to reduce their taxes while others with less disposable income find that the
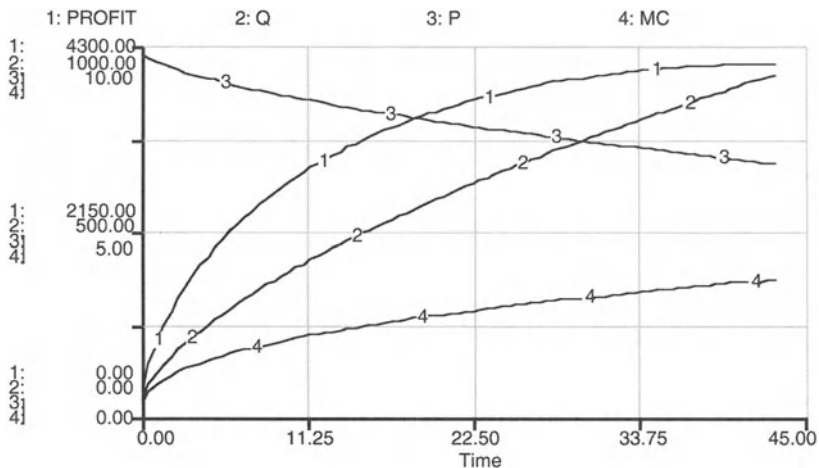
FIGURE 25.8

cost of trying such a procedure exceeds its benefits. In this case, we would be trading revenue-raising efficiency for regressive taxes.

The model was run for a production tax equal to $0.20 per unit output. The tax revenues collected by the government from the firm in its profit-maximizing production point are approximately $183. We can easily find the profit tax rate that yields the same tax revenue from the profit-maximizing firm. We find that the profit tax rate must be approximately $ 0.0425 per dollar of profits earned by the firm. Of course, the lump sum tax with the same tax is just $183.

The calculations of tax revenues in STELLA are not done with stocks but with translation variables. If we use stocks, we calculate the cumulative tax revenue that is collected by the government over time, and of course, depends on the speed at which the monopolist adjusts production to achieve a profit maximum. The faster the profit-maximizing output can be achieved, and everything else being equal, the lower the cumulative tax revenues. However, we are interested in the tax revenues in the firm's optimum, independent of the monopolist's ability to adjust her production. Thus, we need to calculate the tax revenues at each period of time. The use of translation variables enables us to do this.

```
                    MONOPOLY MODEL WITH TAXES

X(t) = X(t - dt) + (CHANGE_X) * dt
INIT X = 1 {Units of X}
INFLOWS:
CHANGE_X = IF (PROFIT - DELAY(PROFIT,DT,1)) > 0 THEN 3
ELSE 0 {Units of X per Time Period}

A = 50 {Units of Q per Units of X}
ALPHA = .6
C = R1*X {$ per Time Period}
LUMP_SUM_TAX = 183 {$ per Time Period}
MC = (C - DELAY(C,DT,.01))/(Q - DELAY(Q,DT,.1)) {$ per
Units of Q}
MR = (R - DELAY(R,DT,10))/(Q - DELAY(Q,DT,.1)) {$ per
Units of Q}
NMR = MR-PRODUCTION_TAX {$ per Units Q}
P = 10-.00338*Q {The selling price is a function of q;
that is, the monopoly controls the output level AND
price, unlike the competitive firm who could not affect
price. This is the demand curve, experimentally
derived. $ per Unit Q}
PRODUCTION_TAX = 0 {.2 is the value of the production
tax in the absence of other taxes; $ per Unit Q}
```

```
PRODUCTION_TAX_REVENUE = PRODUCTION_TAX*Q
PROFIT = ((P - PRODUCTION_TAX)*Q - C)*(1-PROFIT_TAX) -
LUMP_SUM_TAX {$ per Time Period}
PROFIT_TAX = 0 {0.0425 is the value if the other tax
rates are zero; $ per $ Profit}
PROFIT_TAX_REVENUE = (P*Q - C)*PROFIT_TAX
Q = A*X^ALPHA {Units of Q}
R = P*Q {$ per Time Period}
R1 = 16 {$ per Unit X}
```

26
Competitive Equilibrium

> The only stable foundation of most of the improvements in social life is
> Agriculture.
>   —Joseph Priestley, *Lecture on History*

In our discussion of competitive firms we assumed that there is a large
number of firms active in the market. In real economies, however, the
number of firms that can successfully compete is often rather limited. In this
chapter, we develop a model to determine the number of competitive
profit-maximizing firms that can be supported with a given demand curve.
Here we learn that the simple production function of the previous chapters
can no longer be used. That production function is incapable of providing
the necessary diminishing output with respect to more input and an aver-
age cost (AC = C/Q) curve with a minimum value. A more complicated
functional form is required even though most microeconomics is done with
the simpler version. The new production function is:

$$Q = A*X^2 - ALPHA*X^3 \tag{1}$$

This function meets the above requirements of decreasing returns to scale.

We know from economic theory that when a firm's production level is
forced down by competition to a level where the market price is equal to
its average cost, its profit is 0. It is on the verge of going out of business
even though, as seen from the theory, it is also maximizing profit at this
point—the maximum profit is 0. Therefore, at this Q only, price = aver-
age cost = marginal cost, for the firm. This can be shown analytically as
follows:

1. Setting the derivative of the AC with respect to Q equal to 0, we see that
   MC = AC at the minimum of the AC curve:

$$\frac{\partial AC}{\partial Q} = \frac{\partial (C/Q)}{\partial Q} = \frac{\frac{\partial C}{\partial Q} Q - C}{Q^2} = 0 \Rightarrow \frac{MC}{Q} = \frac{AC}{Q} \Rightarrow MC = AC \tag{2}$$

2. We know that P = MC for profit maximization.
3. Therefore, P = AC (=MC).
4. Consequently, PROFIT = 0 when MC = AC at the minimum of the AC
   curve. These relationships are shown schematically in Figure 26.1.

To complete the problem, let us assume that all N firms in this market
have identical technologies and face the same price for their single input,
R1. The demand curve is therefore: P = 10 − .00338*Q*N. The problem now
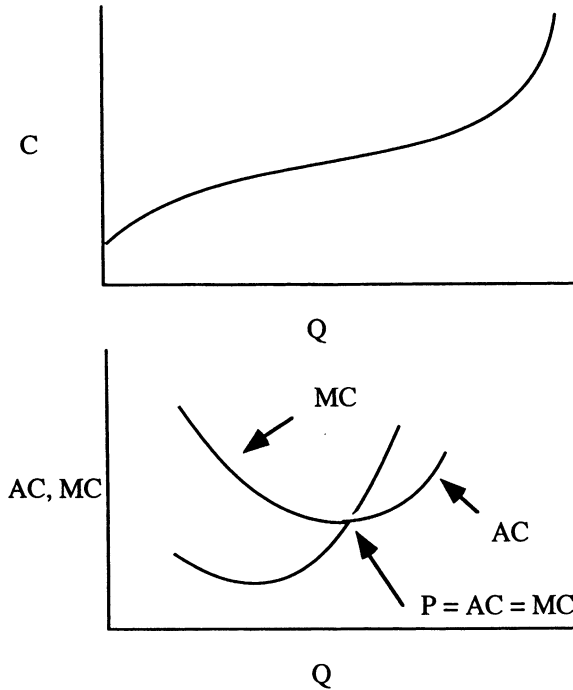
FIGURE 26.1

is one of driving the difference between average cost and marginal cost to 0 while insisting that each firm is maximizing profits (P = MC). By increasing X, we exhaust all profits through an increase in the number of firms in the market. Marginal cost is made equal to output price, in the equation for N, to guarantee maximum profits for every X. The model is shown in Figure 26.2.

Figure 26.3 shows the optimal number of firms is about 29, and it shows the continuous equality of price and MC—maximum possible profits for each firm—and the eventual equality of MC and AC when the number of firms entering the market has stabilized—maximum but 0 profit for all firms. This is the equilibrium for the market. Equilibrium of the firm is assumed from the beginning with condition P = MC. Run the model at a DT = .25. Plot Q versus X and Q versus C to understand the full nature of the production function.

Imagine that there exist five other firms that are the same as the ones shown in the diagram except that for these five firms, A = 75. These are the low-cost firms. Think of them as farms on fertile land, and the firms with A = 50 as farms on poorer land. The good farms are limited in number to five. How many of the poorer farms will exist in the market
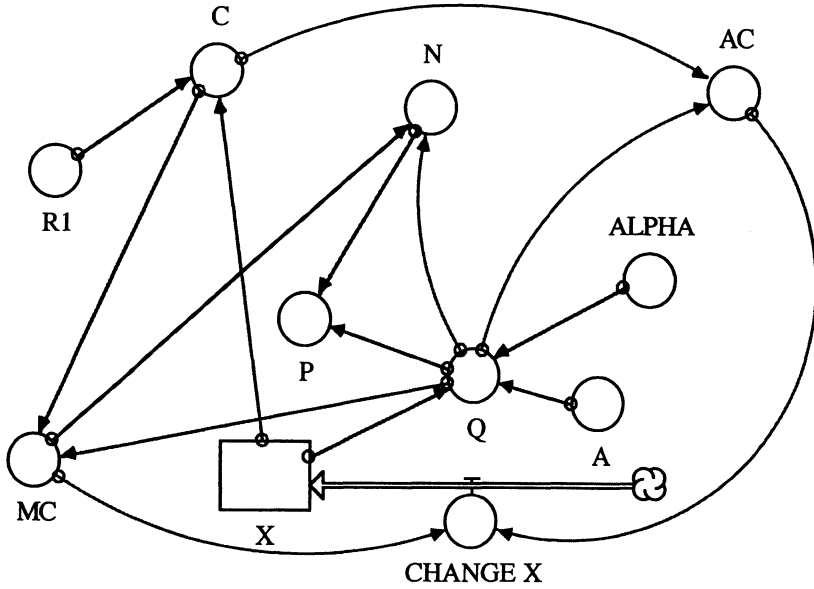
FIGURE 26.2

when equilibrium is reached? How much profit will a good farm make? This apparently complex problem can be solved in a few minutes using the model shown, but you must really understand the meaning in the mechanisms behind Figures 26.1 and 26.2.
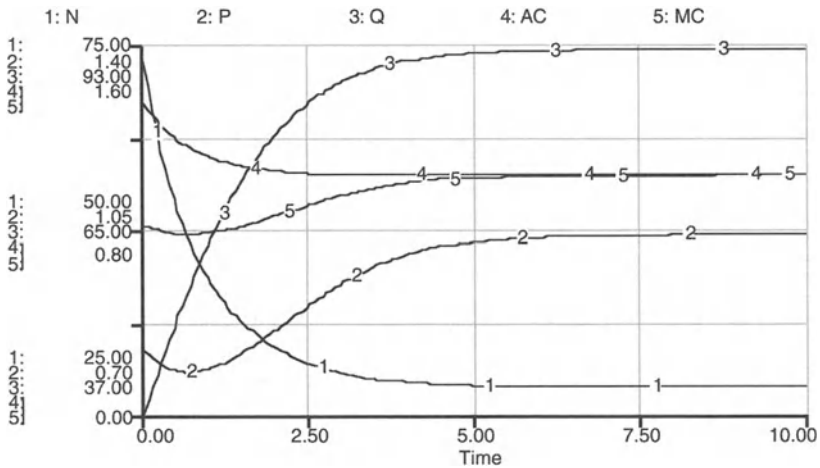
FIGURE 26.3

MARKET EQUILIBRIUM MODEL

```
X(t) = X(t - dt) + (CHANGE_X) * dt
INIT X = 1 {Units of X}
INFLOWS:
CHANGE_X = AC - MC {Units of X per Time Period}

A = 50 {Units of Output per Unit Input}
AC = C/Q {$ per Unit of Q}
ALPHA = 13 {Units of Output per Unit Input}
C = R1*X {$}
MC = (C-DELAY(C,DT,1))/(Q-DELAY(Q,DT,.001)) {$ per
Unit of Q}
N = (10-MC)/(.00338*Q) {Number of Firms}
P = 10-.00338*N*Q
Q = A*X^2-ALPHA*X^3
R1 = 50 {$ per Unit X}
```

27

Substitution

> This is the principle upon which the sacrifice of love acts, as distin-
> guished from the sacrifice of mere substitution.
> —Thomas Mozley, *Universal Sermons*

27.1 Isoquants

The firms of our models so far used only one input. Real production op-
erations, however, use multiple inputs and are able to substitute, within
limits, one input for another. This substitution in a competitive firm with
two inputs, X and Y, is captured in the models of this chapter. First, we
identify the realm of minimum X-Y combinations that yield a given output.
In the following section we identify then the profit-maximizing output level
together with the optimal combination of inputs.

Let us assume that the firm already identified its profit-maximizing output
level $\bar{Q}$. The production function for the firm is

$$Q = A*X\text{^}ALPHA * Y\text{^}BETA \tag{1}$$

Substituting $\bar{Q}$ for Q yields the recipe that generates the fixed output level $\bar{Q}$.

In general, the production function generates an increasing output for in-
creasing quantities of inputs X and Y. Furthermore, each input is essential
in the sense that no output can be achieved without that input. As inputs
increase, output increases at a decreasing rate if $0 < ALPHA + BETA < 1$.

The possibility for input combinations at a given output level $Q = \bar{Q}$ are
infinite. However, we are only interested in those input combinations that
just yield the output $\bar{Q}$. The realm of these minimum input combinations
that generate $\bar{Q}$ is called an isoquant. The isoquant for a given output level
can be readily calculated from the production function by solving it for one
of the inputs. Analytically, the isoquants are of the form

$$Y = \left(\frac{\bar{Q}}{A*X^{ALPHA}}\right)^{\frac{1}{BETA}} \tag{2}$$

Let us also assume that the prices of the two inputs X and Y are given for
the firm. These prices are R1 and R2, respectively. The cost function can
then be calculated as

$$C = R1*X + R2*Y \tag{3}$$

The model of Figure 27.1 generates an isoquant and cost function for an
output level of $\bar{Q} = 183.13$ units.
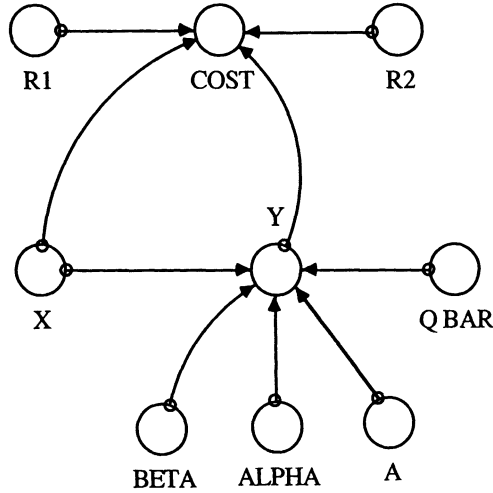
FIGURE 27.1

The resulting isoquant and cost function are shown in Figures 27.2 and 27.3. The cost function has a minimum that determines the firm's choice of a profit-maximizing input combination, assuming that the profit-maximizing output level was already determined. In fact, both decisions are carried out simultaneously by the firm. The model of the following section provides a solution to this problem of simultaneously choosing profit-maximizing input and output quantities.

FIGURE 27.2

1: X v. Y
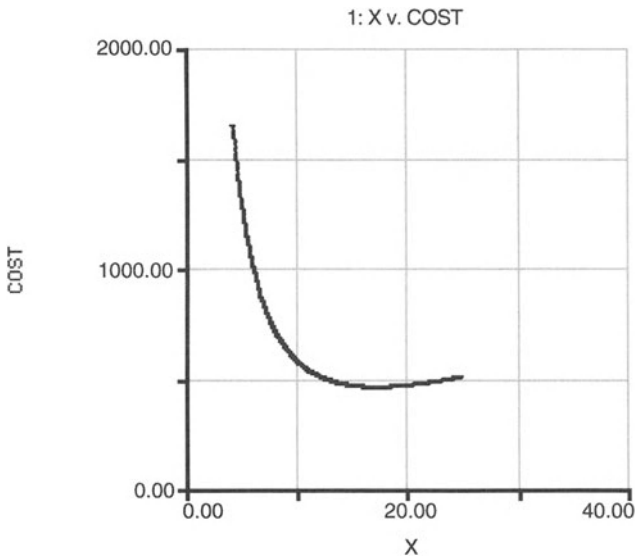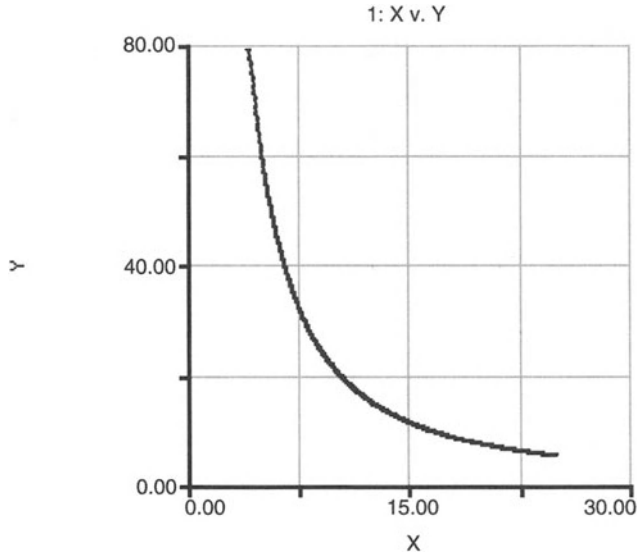
FIGURE 27.3

In the model developed here, the minimum costs are $459.99. In that minimum, the firm uses 17.25 units of X. The second graph shows the relationships between X and Y, the isoquant. Given the cost-minimizing X, the optimal choice of Y is 9.20 units.

```
                         ISOQUANTS

 A = 50 {Units of Output per Units of X per Units of Y}
 ALPHA = .3
 BETA = .2
 COST = R1*X + R2*Y {$}
 Q_BAR = 183.13 {Units of Output}
 R1 = 16 {$ Per Units of X}
 R2 = 20 {$ Per Units of Y}
 X = .1*(TIME + 1)
 Y = (Q_BAR/(A*X^ALPHA))^(1/BETA)
```

27.2 Finding the Profit-Maximizing Output Level and Input Combinations

As in the previous chapters, we assume that the firm seeks the profit-maximizing output level, given fixed input costs (R1, R2) and output price

P. The production function is assumed to be of the same form as in the previous chapter:

$$Q = A*X^{\wedge}ALPHA*Y^{\wedge}BETA \qquad (4)$$

A, ALPHA, and BETA are constants that are characteristic for the hypothetical production processes investigated here.

The problem is to find the cost-minimizing mix of the inputs, X and Y, and at the same time, maximize profits. It is a very difficult problem to solve with STELLA and so we mix in a little analysis before approaching the numerical solution. It can be shown with a bit of calculus that the cost-minimizing level of Y is R1*BETA*X/R2*ALPHA. To derive the conditions for profit-maximizing output and optimal input combinations, note that

$$PROFIT = P*Q - R1*X - R2*Y \qquad (5)$$

Taking the partial derivatives of the PROFIT function with respect to X and Y yields

$$\frac{\partial PROFIT}{\partial X} = P*ALPHA*Q/X - R1 = 0 \qquad (6)$$

$$\frac{\partial PROFIT}{\partial Y} = P*BETA*Q/Y - R2 = 0 \qquad (7)$$

These two equations can be combined to give to condition that in the optimum

$$Y = R1*BETA*X/R2*ALPHA \qquad (8)$$

This equation provides a relationship between the two inputs. Once one input (X) is determined for the profit-maximizing output level, we can readily calculate the corresponding, optimum quantity of the other input (Y). Now we can proceed as we did in the problem of Chapter 26, adjusting the amount of one input until we find the profit maximum. As in that model, a positive slope on the profit curve causes X to rise, which in turn causes Y to increase, and together, X and Y give an increase in Q. Revenue (P*Q) and cost combine to give the new profit, based on a cost-minimizing mix of the inputs. Any number of inputs can be handled in this way.

Note the independence of the output level in this result. Realize that the total derivative of the cost must be 0 and for fixed Q, the total derivative of the profit function must also be 0. These two necessary conditions yield the result shown here. The sufficient condition for a maximum is that ALPHA + BETA < 1.

The model for the two-input case shown in Figure 27.4 assumes initially a small value for the input X. Since the firm attempts to achieve a profit maximum, there is some amount of input Y that corresponds to X, according to Y = R1*BETA*X/R2*ALPHA. Given these inputs of X and Y, we are able to calculate the corresponding output Q, cost C, and thus, the profit realized by the firm. This profit need not be the maximum, given the prices P for its output, and R1 and R2 for its inputs. As long as profit maximum is
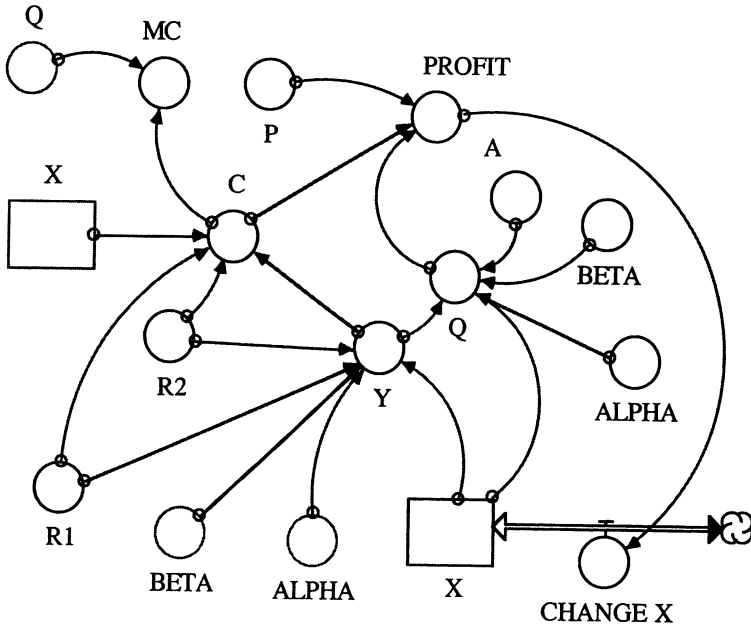
FIGURE 27.4

not achieved, i.e., as long as an increase in inputs X and Y yields an in-
crease in PROFIT over the profits achieved one small time step DT earlier,
X, and thereby Y, are increased.

The model shows that the profit-maximizing output level is Q = 183.13
units (Fig. 27.5). This is exactly the output level that we chose to model the
isoquant of the previous section. The optimum output level requires a cost-
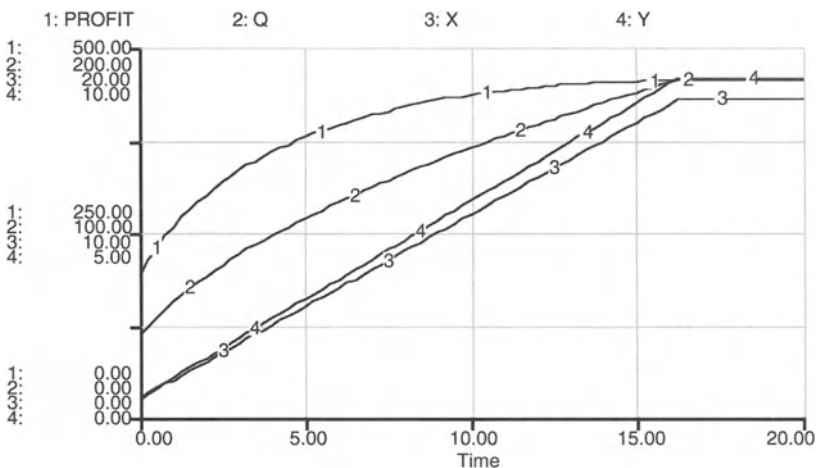
FIGURE 27.5
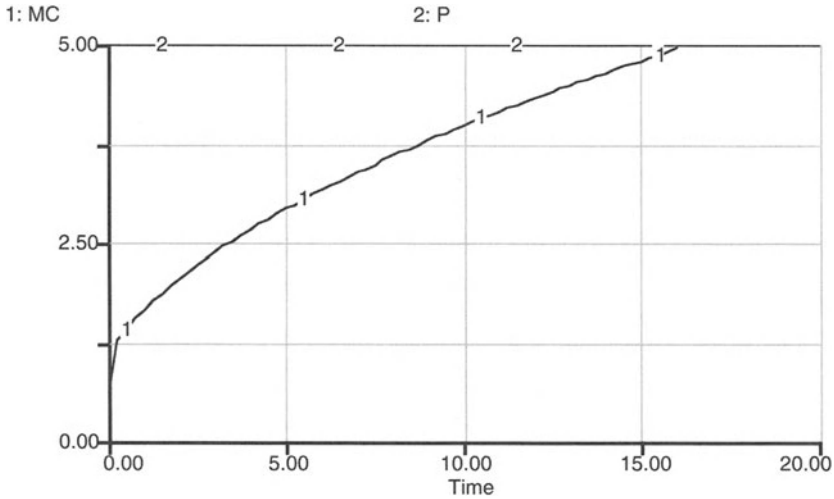
FIGURE 27.6

minimizing choice of X and Y. Our new model results coincide with those of the previous section, i.e., the profit-maximizing input combination is X = 17.25, Y = 9.20.

The graphs show the peak of PROFIT with associated Q, X, and Y. It is also shown that marginal cost, MC (the change in cost associated with a change in Q), rises to meet the price at the time when profit is a maximum (Fig. 27.6). Thus, these costs behave as the theory of microeconomics predicts.

A numerical solution using STELLA can be worked out completely by combining the ideas in the previous economic models with this approach. However, you will often find that the appropriate combination of analytic and numerical analysis will minimize your overall effort.

```
             COMPETITIVE FIRM WITH SUBSTITUTION

  X(t) = X(t - dt) + (CHANGE_X) * dt
  INIT X = 1 {Units of X}
  INFLOWS:
  CHANGE_X = if (PROFIT-Delay(PROFIT,dt,.1)) > 0 then 1
  else 0 {Units of X per Time Period}

  A = 50 {Units of Q per (Units of X)^Alpha per Units of
  Y}
  ALPHA = .3
  BETA = .2
  C = R1*X+R2*Y {$}
```

```
MC = (C-DELAY(C,dt,.1))/(Q-DELAY(Q,dt,.1)) {$ per
Units of Q}
P = 5 {$ per Unit Q}
PROFIT = P*Q - C {$}
Q = A*X^ALPHA*Y^BETA {Units of Q}
R1 = 16 {$ per Unit X}
R2 = 20 {$ per Unit Y}
Y = ((R1*BETA)/(R2*ALPHA)) * X {Units of Y}
```

28
Time Value

> I'm making good, all right, ... since I got married—and its only a
> question of time.
>
> —Eugene O'Neill, *Strange Interlude*

As we mentioned in Chapter 25, firms make their decisions not just with re-
gard to the production possibilities and demand at the present period of
time but rather attempt to optimize their activities over a longer time hori-
zon. How can these firms compare profits that occur at different periods of
time? Having a dollar today is surely different from having a dollar next
year. In order to compare profits that accrue at different time periods, we
may calculate their present value rather than their current value. In this
chapter we take up the issue of present versus current value in more detail
and provide the tools that are necessary for the following chapters to assess
the optimal behavior of firms over time.

In order to calculate the present value of profits, rather than the current
value, we must understand how to use STELLA in that effort. In Figure 28.1
the stock of present value ($100) is drawn down by a given percentage
of the remaining stock of value, in each time step. The percentage at which
the initial stock "looses value" to the decision maker over time is called the
discount rate.

Frequently, it is sufficient to take the interest rate as the relevant discount
rate. Choosing the interest rate for discounting the initial stock can be justi-
fied with the argument on opportunity costs made in Chapter 23. We have
always the opportunity to spend an initial endowment or invest it in a port-
folio that earns a certain interest. Thus, the interest rate reflects the oppor-
tunity costs and may therefore be an appropriate measure of the value we
assign to the initial endowment over time.

The model calculates the present value of $100 at some time, t, in the fu-
ture. For example, you can see from Figure 28.2 that the present value of
$100 in time period 30 is less than $25. Conversely, if you had invested that
present value today at the given interest rate, you would own $100 in 30

PRESENT VALUE

PRESVALUE RATE

INTEREST RATE

FIGURE 28.1

1: PRESENT VALUE
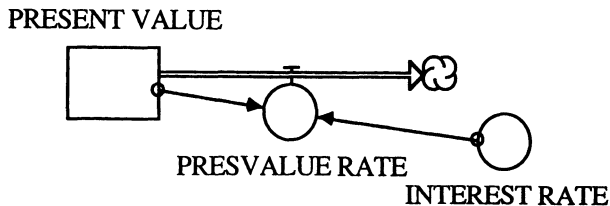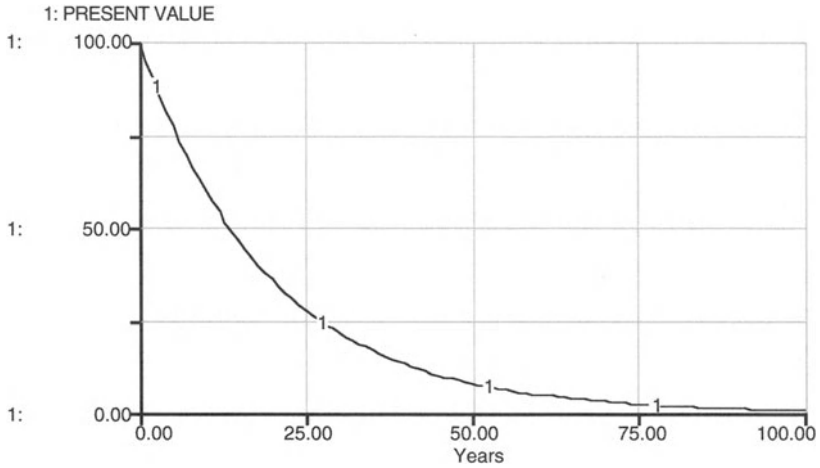
FIGURE 28.2

years from now. Compare this model with the model of diffusion discussed in Chapter 2.

If we are interested in the amount received t periods from now for an investment of $100 at a given interest rate, we just need to reverse the arrow in the STELLA model, thereby accumulating value. Such a calculation gives us the future value of our investment. Now the stock rises exponentially.

Often we do not invest a given amount of money in just one period but repeat the investment over a number of periods. Similarly, firms typically receive a stream of profits over time. In the model of Figure 28.3, the current value of an annual investment series ($100) is calculated. The result is displayed in Figure 28.4.

Investments may not be uniform over time. Similarly, streams of profits may follow cycles of high and low sales. For simplicity, in the model of Figure 28.5 we represent such cyclic investment patterns, or annual payments AP, by a sine wave (mean value = 1.0). The resulting present value is calculated and plotted in Figure 28.6. Here we had to calculate the present
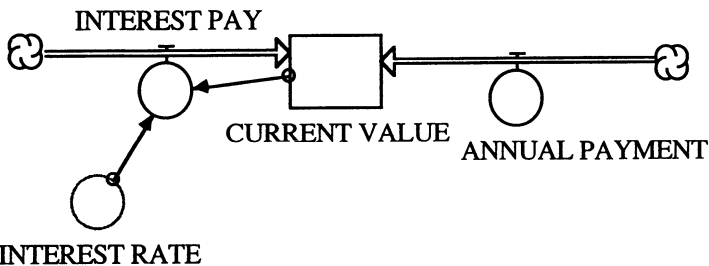
FIGURE 28.3
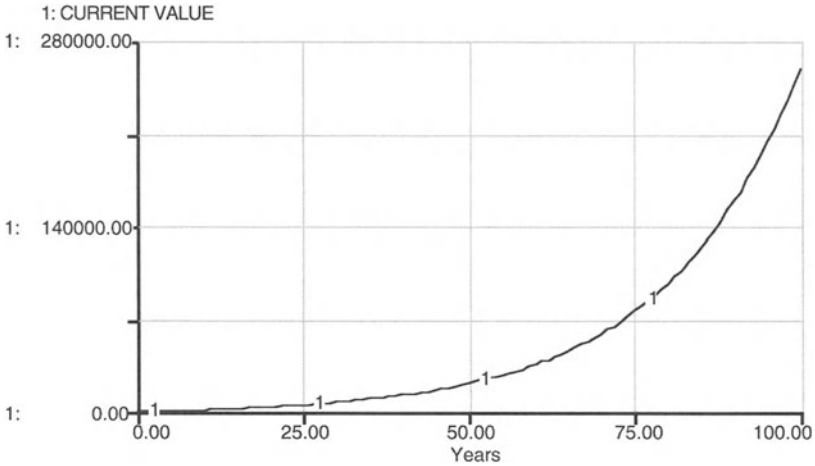
FIGURE 28.4

value and sum it, or accumulate it, for the combined present value of the series of values represented by the sine function.

In calculating the present value, one can use either EXP(–INTEREST RATE*TIME) or the factor (1 + INTEREST RATE)^(–TIME/DT). What is the difference? The first one is continuous (or infinitely often) negative compounding and the second is stepwise (DT) negative compounding. Run the two with DT = 1 and note the difference. The second rises to the same present value of some string of future values but it rises more quickly. Change DT to smaller values and run the model repeatedly. You should find the curves get closer and closer. The exponential discounting is just getting more accurate but the step discounting is actually changing to more and more frequent negative compounding, a conceptually different result.

Now return to the models of Chapters 24 and 25. In the two microeconomic problems shown in those chapters, the firm is thought to maximize
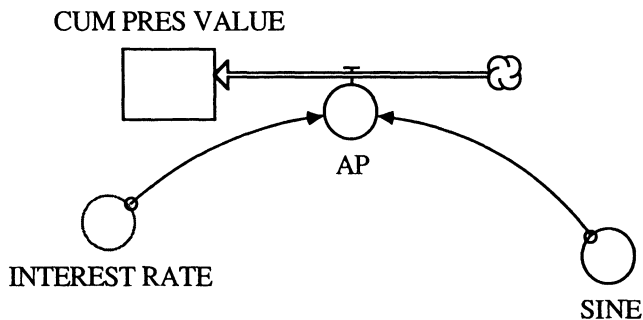
FIGURE 28.5
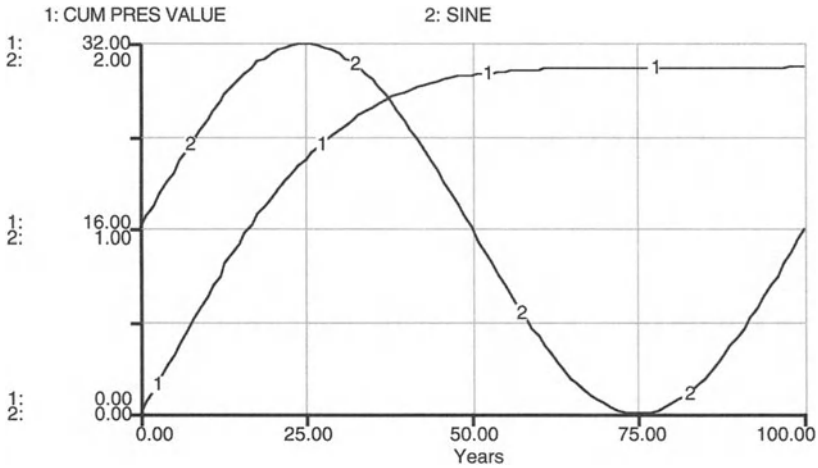
FIGURE 28.6

the current value of its profit. A more reasonable view is that future profits to the firm have a lower value right now. Consequently, the firm is more realistically thought to maximize the present value of its profits. This modification is easy to make by multiplying the PROFIT equation by

$$e^{-\text{discount rate}*t}$$

where e is Euler's number. Choose an interest rate of 5% as the relevant rate of discount.

In STELLA, the discounting is achieved by using the built-in function EXP for Euler's number, and TIME for t. The present value of profits is then

PRESENT VALUE PROFIT = PROFIT*EXP(–INTEREST RATE*TIME) (1)

The discount rate is usually assumed to be constant throughout the period under consideration. Modify the models of the competitive firm and the monopolist to calculate the present value of profits. How do the results differ from the calculation of current value of profits? How are the results different if you choose discrete discounting?

```
                    TIME VALUE MODELS


CUM_PRES_VALUE(t)  =  CUM_PRES_VALUE(t - dt) + (AP) * dt
INIT CUM_PRES_VALUE = 0 {$}
INFLOWS:
AP = SINE*EXP(-INTEREST_RATE*TIME) {$ per Time Period}


CURRENT_VALUE(t)  =  CURRENT_VALUE(t - dt) +
(INTEREST_PAY + ANNUAL_PAYMENT) * dt
```

```
INIT CURRENT_VALUE = 1.00{$}
INFLOWS:
INTEREST_PAY =
PULSE(CURRENT_VALUE*INTEREST_RATE*DT,0,DT) {$ per Time
Period}
ANNUAL_PAYMENT = PULSE(100,0,1) {$ per Time Period}

PRESENT_VALUE(t) = PRESENT_VALUE(t - dt) + (-
PRESVALUE_RATE) * dt
INIT PRESENT_VALUE = 100 {$}
OUTFLOWS:
PRESVALUE_RATE = PRESENT_VALUE*INTEREST_RATE {$ per
Time Period}

INTEREST_RATE = .05 {annual interest rate; $ Interest
per $ Principal per Time Period}
SINE =   SIN(.0628*TIME)+1   {PULSE(100,0,1);  $}
```

29

Opportunity Cost

> Then can I grieve at grievances foregone.
> —William Shakspeare, Sonnets, xxx

In the previous models of a firm we assumed that all production is imme-
diately sold on the market place. Profits accrue at the same time a good is
produced. However, real firms do not always know the future demand for
their products, and the level of production itself may vary due to shortages
in input supplies, machine breakdown, and other unforeseen factors. Thus,
firms hold inventories of their products to buffer short-term changes in de-
mand and production. Holding inventories during some period, however,
means foregoing some profits in that period. The goods may have been
sold but instead they are kept in a warehouse whose costs must be covered
by the firm. The loss of profits in a period is the opportunity cost of keep-
ing an inventory in that period.

We now show the profit-maximizing firm with the production and sales
arranged to be connected only through the warehouse inventory. Produc-
tion responds to inventory for which a desirable level must be set from the
experience of the entrepreneur. The sales level is adjusted to maximize
profits. Adjustments in production and sales are specified as graphical rela-
tionships shown in Figures 29.1 and 29.2. Adjustment in production is
achieved through a change in the amount of input X used in the production
process. The change in inputs, DEL X, in turn, is driven by the change in in-
ventory, DEL INVENTORY. Similarly, the change in sales, DEL SALES, is
driven by the change in profit, DEL PROFIT.

Two types of costs are presented in this model: one for the production
costs and one for the cost of inventory, the opportunity cost. Inventory is
production not yet sold. Since the value of the inventory is thought of as
"foregone sales" rather than "sunk costs," the price of the product times the
size of the inventory, times the rate of interest for the time step, gives the
money foregone because of keeping an inventory. Multiplying by the inter-
est rate for the time step is necessary to calculate the amount of money that
could have been made had the inventory been sold and the profits from it
be invested. This procedure takes into account time value, the principle dis-
cussed in more detail in the previous chapter.

There are many ways to give value to inventory and this is but one of
them. Can you think of more? Are they more appropriate?

Note that the price is a function of the rate of sales. This relationship is
expressed by the demand curve. As we have noted in previous chapters,
the demand curve is usually downward sloping—if the firm lowers the
price, the sales increase.

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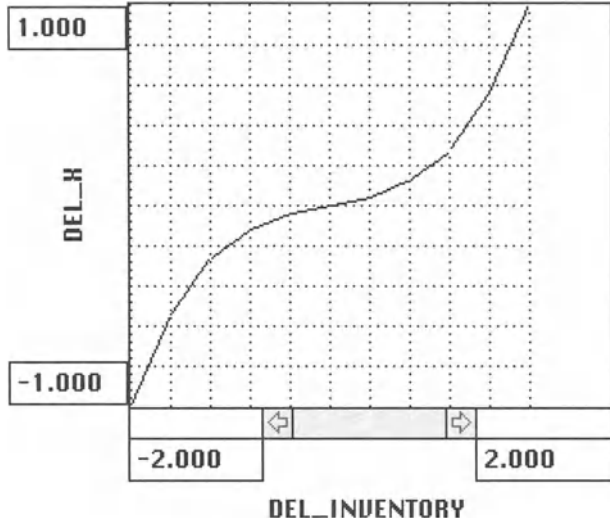
FIGURE 29.1

The profit is compared with its most recent value, and if it is still grow-
ing, sales are increased. Sales are slowed (fewer salesperson-years, less ad-
vertising) if the profit begins to diminish. We have averaged the rate of pro-
duction so that planning for production can be handled (worker hiring, rate
control, overtime). The real problem here is to set all the parameters and
find the optimal inventory. Recreate the model as in Figure 29.3 and find
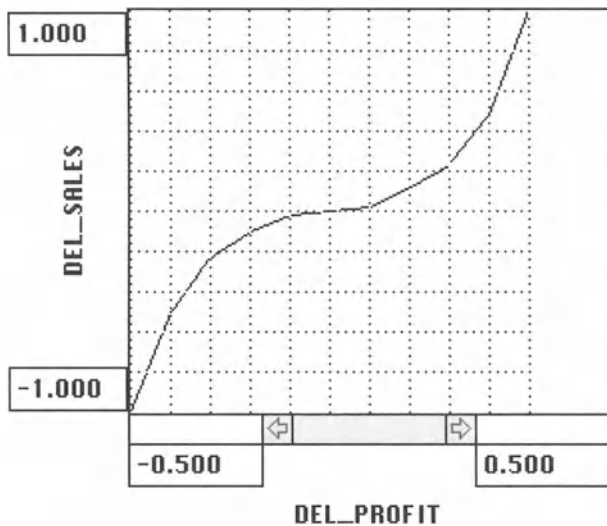
FIGURE 29.2
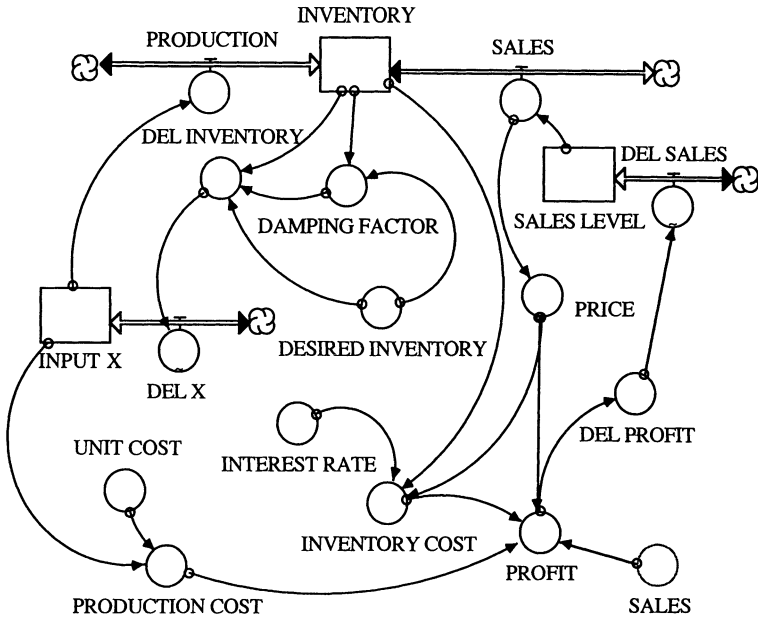
FIGURE 29.3

for yourself the optimal inventory. Hopefully, it is larger than that needed to meet the fluctuations in sales and labor force availability. Our results are shown in Figure 29.4.

Try to constrain this process by saying that labor is the input X and it is only available at certain times, e.g., 40 hours per week or 8 hours per day for five days per week.
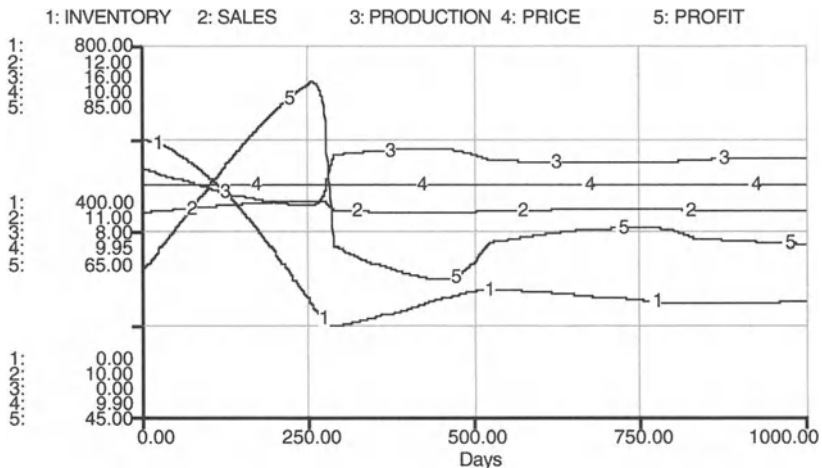
FIGURE 29.4

You can see why business people always talk about inventory control. The government is not powerless in this problem. They could allow portions of inventory costs to be called legitimate expenses and therefore deductible from net income and taxes. The federally financed interstate highway system allows (largely at the expense of the public and the environment) rapid shipment in response to sales orders. If the firm has large production facilities, they can produce and ship with relatively small inventories at the firm's location. The trucks are carrying the inventory!

INVENTORY MODEL

```
INPUT_X(t) = INPUT_X(t - dt) + (DEL_X) * dt
INIT INPUT_X = 40 {Units of the Input per Time Period}
INFLOWS:
DEL_X = GRAPH(DEL_INVENTORY {Units of the Input Per
Time Period^2})
(-2, -1) (-1.6, -0.55) (-1.2, -0.26) (-0.8, -0.12)
(-0.4, -0.04) (0, 0) (0.4, 0.04) (0.8, 0.12) (1.2,
0.27) (1.6, 0.55) (2, 1)


INVENTORY(t) = INVENTORY(t - dt) + (PRODUCTION -
SALES) * dt
INIT INVENTORY = 600 {Units of the Product}
INFLOWS:
PRODUCTION = 2.5 * INPUT_X ^ 0.4 {Units of the Product
per Time Period}
OUTFLOWS:
SALES = SALES_LEVEL {Units of the Product per Time
Period}


SALES_LEVEL(t) = SALES_LEVEL(t - dt) + (DEL_SALES) *
dt
INIT SALES_LEVEL = 10 {Units of the Product}
INFLOWS:


DEL_SALES = GRAPH(DEL_PROFIT)
(-0.5, -1) (-0.4, -0.51) (-0.3, -0.23) (-0.2, -0.1)
(-0.1, -0.02) (0, 0) (0.1, 0.02) (0.2, 0.11) (0.3,
0.23) (0.4, 0.48) (0.5, 1)
DAMPING_FACTOR = IF ((DESIRED_INVENTORY -
INVENTORY)^2  >
  DELAY((DESIRED_INVENTORY - INVENTORY)^2,DT,0.1)) THEN 8
  ELSE 1 {The square removes the sign effect of the
difference. 1/Time Period}
```

```
DEL_INVENTORY = DAMPING_FACTOR * (DESIRED_INVENTORY -
INVENTORY) / INVENTORY
{Units of the Product per Time Period}
DEL_PROFIT = (PROFIT - DELAY (PROFIT,DT,0.1)) / SQRT
(PROFIT ^ 2) {$ per Time Period}
DESIRED_INVENTORY = 250 {Units of the Product}
INTEREST_RATE = 0.002
{$ Interest per $ Principal per Time Period}
INVENTORY_COST = INTEREST_RATE * PRICE * INVENTORY {$
per Time Period}
PRICE = 10 - 0.00338 * SALES {$ per Unit of the
Product}
PRODUCTION_COST = UNIT_COST * INPUT_X {$ per Time
Period}
PROFIT = PRICE * SALES - PRODUCTION_COST -
INVENTORY_COST {$ per Time Period}
UNIT_COST = 1 {$ per Unit of the Input}
```

30
Optimal Tree Cutting

> Then I say the earth belongs to each generation during its course, fully and in its own right. Then, no generation can contract debts greater than may be paid during the course of its own existence.
>
> —Thomas Jefferson, *Letters*

In the previous chapters on economic problems we dealt with assumptions on the optimal behavior of firms in competitive or monopolistic markets, and illustrated the concepts of substitution, time preference, and opportunity cost. In this chapter, we turn our attention to the optimal harvest of a renewable resource by an individual firm who owns that resource but operates in a competitive market. In the next chapter, we model extraction from a common property resource, before we turn our attention to the optimal extraction of a nonrenewable resource.

Let us model the optimal use of a renewable resource. For this purpose, assume you are an owner of a forestry operation that grows walnut trees. Your operation is small in comparison with the rest of the walnut tree growers in the country, i.e., the price of trees is given and cannot be influenced by your production. Assume also for simplicity that the price of trees, measured in dollars per board-feet, is constant and equal to 1. You must make an initial investment of $100 per hectare in the plantation. This investment has to be made after each harvest. There are no other costs during the growth phase of your walnut trees and no costs of harvesting the trees. It is the goal of your operation to maximize cumulative discounted profits. The discount rate is given and fixed at 5%. As discussed in the previous chapter, we use the market interest rate as the discount rate in order to represent the opportunity cost of capital investment.

Your operation is faced with the problem of determining the optimal time for the harvest of walnut trees. In order to find the optimal time period, you need to know how the harvestable amount of walnut trees changes over time. For our model, the biomass (measured in board-feet) of walnut trees at different age classes, starting at year 21 after planting, is given in the Table 30.1 and used in the STELLA model (Fig. 30.1) in the form of a graphical relationship. From this graphically expressed relationship it is readily possible to calculate an approximation for the underlying growth rate of walnut trees, BIO GROWTH RATE.

The even-aged stand is cut and replanted over and over again and the profit formula is set to represent this infinitely repeating cycle, all discounted to the present. As the walnut tree manager you face the initial investment of I_0. After t periods, the revenues from selling Q board-feet of walnut tree biomass at a price P is PQ. At that period you have to replant

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TABLE 30.1

Age	Biomass	Age	Biomass	Age	Biomass
21	847.00	42	4770.00	63	10032.00
24	1238.00	45	5555.00	66	10760.00
27	1730.00	48	6241.00	69	11478.00
30	2253.00	51	6955.00	72	12414.00
33	2848.00	54	7685.00	75	13154.00
36	3442.00	57	8418.00	78	13885.00
39	4087.00	60	9293.00	81	14460.00

again at the cost of I_0, and so on into the distant definite future. Consequently, the present value of your profits is

$$
\begin{aligned}
\text{PVP} &= -I_0 + (PQ - I_0)e^{-rt} + (PQ - I_0)e^{-r2t} \\
&\quad + (PQ - I_0)e^{-r3t} + \ldots + (PQ - I_0)e^{-rnt} \\
&= PQ(e^{-rt} + e^{-r2t} + e^{-r3t} + \ldots + e^{-rnt}) \\
&\quad - I_0 (1 + e^{-rt} + e^{-r2t} + e^{-r3t} + \ldots + e^{-rnt})
\end{aligned} \tag{1}
$$

For simplicity of this analytical expression, let us define a new variable a $= e^{-rt}$, so we have

$$
\begin{aligned}
\text{PVP} &= PQ (a + a^2 + a^3 + \ldots + a^n) \\
&\quad - I_0 (1 + a + a^2 + a^3 + \ldots + a^n)
\end{aligned} \tag{2}
$$

Our calculation of the cumulative present value of profits depends on the choice of the planning horizon. The longer our planning horizon, the more profits are accumulated. However, the further in the distance these profits occur, the lower their contribution to the cumulative present value.

Ideally, we would like to determine the optimal harvest decision over an infinite time horizon. This requires infinite summation in the previous equa-
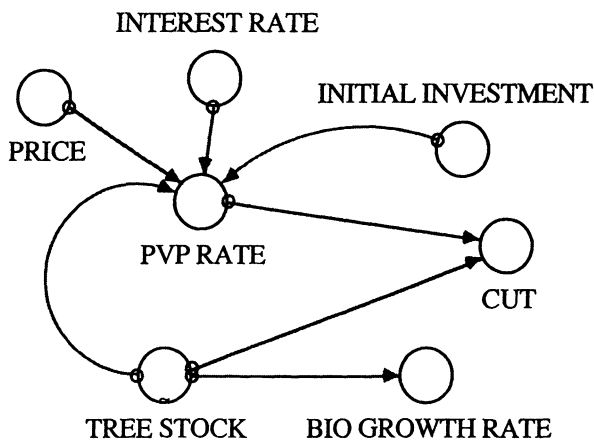
FIGURE 30.1

tion. With a little mathematical trick, we can get rid of this infinite sum of the a's. For that purpose, let us define that sum as

$$s = 1 + a + a^2 + a^3 + \ldots + a^n \tag{3}$$

$$\Rightarrow \quad as = a + a^2 + a^3 + \ldots + a^n + a^{n+1} \tag{4}$$

$$\Rightarrow \quad (s - as) = 1 - a^{n+1} \tag{5}$$

$$\Rightarrow \quad s(1 - a) = 1 - a^{n+1} \tag{6}$$

and therefore

$$s = \frac{1 - a^{n+1}}{1 - a} \tag{7}$$

Since $a = e^{-rt}$, the numerator will get very close to 1 for long time horizons, i.e., for large n. We can thus, for all intents and purposes, write

$$s = \frac{1}{1 - a} \tag{8}$$

which we insert into the present value profit function to eliminate the sum of the a's, leading to

$$PVP = (PQ - I_0)/(1 - e^{-rt}) - PQ \tag{9}$$

This is the expression for the present value of profit rate function used in the STELLA model (see Fig. 30.1) for the optimal harvest of walnut trees. Figure 30.2 shows that the present value of the profit rate rises to a maximum as the age and size of the stand increase, and when that profit rate is at a maximum, the trees are cut. From the table in the model we see that the maximum of the discounted profit is approximately \$561 and that the
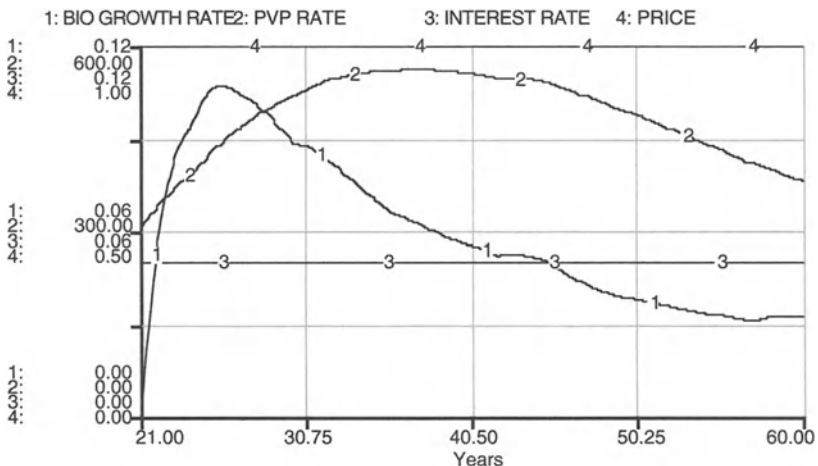
FIGURE 30.2

corresponding cut period is 36 years. Try varying the interest rate to see how the cut time changes.

We are using real tree growth data here. What is causing the sudden changes in the growth rate curve? Can you modify these data to give a more meaningful result? In the mode below, we used a smoothing procedure to accomplish that task.

OPTIMAL TREE CUTTING

```
BIO_GROWTH_RATE =
SMTH1(DERIVN(TREE_STOCK,1)/TREE_STOCK,2) {Board ft. per
Time Period}
CUT = IF ((PVP_RATE - DELAY(PVP_RATE,DT,.1)) < 0) AND
(TIME > 22) THEN TREE_STOCK/DT ELSE 0 {Board ft.}
INITIAL_INVESTMENT = 100
INTEREST_RATE = .05 {$ per $ Invested per Time Period}
PRICE = 1 {$ per board ft.}
PVP_RATE = (PRICE*TREE_STOCK-INITIAL_INVESTMENT)/(1-
EXP(-INTEREST_RATE*TIME)) - PRICE*TREE_STOCK      {$ Per
Time Period}
TREE_STOCK = GRAPH(TIME)
(21.0, 847), (24.0, 1238), (27.0, 1730), (30.0, 2253),
(33.0, 2848), (36.0, 3442), (39.0, 4087), (42.0, 4770),
(45.0, 5555), (48.0, 6241), (51.0, 6955), (54.0, 7685),
(57.0, 8418), (60.0, 9293), (63.0, 10032), (66.0,
10760), (69.0, 11478), (72.0, 12414), (75.0, 13154),
(78.0, 13885), (81.0, 14460)
```

31
Fisheries Reserve Model[1]

> In reality, marine creatures are the *only* wild animals still hunted on a large scale. The language used in fisheries amounts to a forced attempt to induce amnesia on this point. Fisheries people talk incessantly of "harvesting" fishes, and even of "harvesting" whales—trying to impart a sanitized and agricultural tone, as though hunting the largest creatures ever to live on earth by firing bombs into their bodies is analogous to picking watermelons that have been planted and cultivated. Fish populations are referred to as "stock," like shoes in a warehouse. And when sea turtles that have drowned in nets wash up on beaches, they are referred to as—of all things—"stranded". (. . .) They are no more "stranded" than a person killed in a plane crash is "grounded."
>
> —Carl Safina, *Song for the Blue Ocean*

In Chapter 22 we addressed the fate of a common property resource, and in the previous chapter we dealt with the optimal use of a renewable resource. Let us now discuss the management of common property, renewable resources. A typical example for such a resource are fisheries. Virtually anybody who has access to oceans and shorelines may engage in fishing activities. As long as profits can be made from fisheries there is the incentive to increase fishing effort. As a result, pressure on the fish population increases, potentially leading to the collapse of the resource. Standard approaches to dealing with the tragedy of the commons include licensing of fishing boats and gear, restrictions of fishing seasons, and limitations on the catchable size of fish. Regulations and control measures affecting the fishing industries are frequently difficult to enforce, particularly on the oceans, and are therefore likely to be not very effective.

Given the lack of effectiveness of standard measures of controlling and regulating fisheries, we may choose a quite different approach to enable fish populations to be sustained. One such approach is the establishment of reserves in which no fishing is allowed. Fishing effort outside the reserve is treated as a common property resource to which some of the standard regulations and control measures apply, such as legal size of catchable fish. How must such a reserve be established to enable sustainability of a fish population?

The model developed in this chapter applies to an inshore reef fishery with a reserve closed to fishing. However, fishing proceeds unregulated in the remainder of the fishery. The model is based on red snapper data from

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[1]Our model is based on a more elaborate version of marine fishery management worked out by Holland, D.S. 1993. *Managing Fisheries Without Restricting Catch or Effort: The Use of Marine Reserves for Inshore Fisheries*, MS Thesis, Agricultural Economics, University of Illinois, Urbana.

the Gulf of Mexico, however the red snapper is meant to be representative of several major target species. Thus, the model represents a multiple species fishery without explicitly modeling the interaction between species. This assumes that the various target species will be affected similarly by the management strategy of reserves. Fishing effort and biological parameters can be easily adjusted for application of the model to other fish species and for purposes of sensitivity analyses.

The purpose of the model is to examine the economic and ecological effects of various sizes of reserves and fishery production. Thus, our model should allow both reserve size and fishing effort to be adjusted before a run begins. The primary control variable in the model, however, is the size of the reserve.

Most important for the dynamics of the model, and the fate of the fishery, are the spatial range of fish, reproduction strategies, growth rates, and weight-fecundity relationships. A basic assumption of the model is that eggs are produced on the reef but undergo an extended larval stage out on the open ocean before returning to be spread evenly around the reef.

In the model, fish are divided into age classes and their physical position, either inside or outside the reserve. Thus, this model combines features of the age cohort models and spatial models discussed earlier. For simplicity, we distinguish here only three age classes. This simplified model works somewhat differently from a more complex model with 10 age classes. The reserve quickly becomes very dense and recruitment is cut off. Recruitment for the fishery remains high and the fishery exports young fish to the reserve while the reserve ships back older fish to the fishery. The reserve benefits the fishery in two ways. It increases egg production, and thus recruits for the fishery, and provides a refuge for adults to grow and then move back to the fishery to be caught.

Each period, the fish in each age class are subject to fishing and are caught according to an annual fishing mortality rate multiplied by the stock of that age class. Fish in the reserve are subject to fishing also, but at a much lower rate. These fish are assumed to be caught while foraging outside the reserve. Fish from each age class may also die from other causes besides fishing. An age-specific annual natural mortality rate is multiplied by the stock to determine the number that die. Both fishing mortality rates and natural mortality rates are originally put into the model as instantaneous rates. These rates are summed to get a total mortality rate and integrated over time to calculate annual rates. The rest of the fish grow and move to the next age class. Most fish go to the next age class in the same location. However, some fish emigrate from the fishery to the reserve or vise versa. The age classes for fisheries (F) and reserve (R) are calculated separately in the model. The main structure of the model is shown in Figure 31.1.

Each period, a number of eggs are produced based on the number and age of fish in both the reserve and the nonreserve fishery. There is an ex-
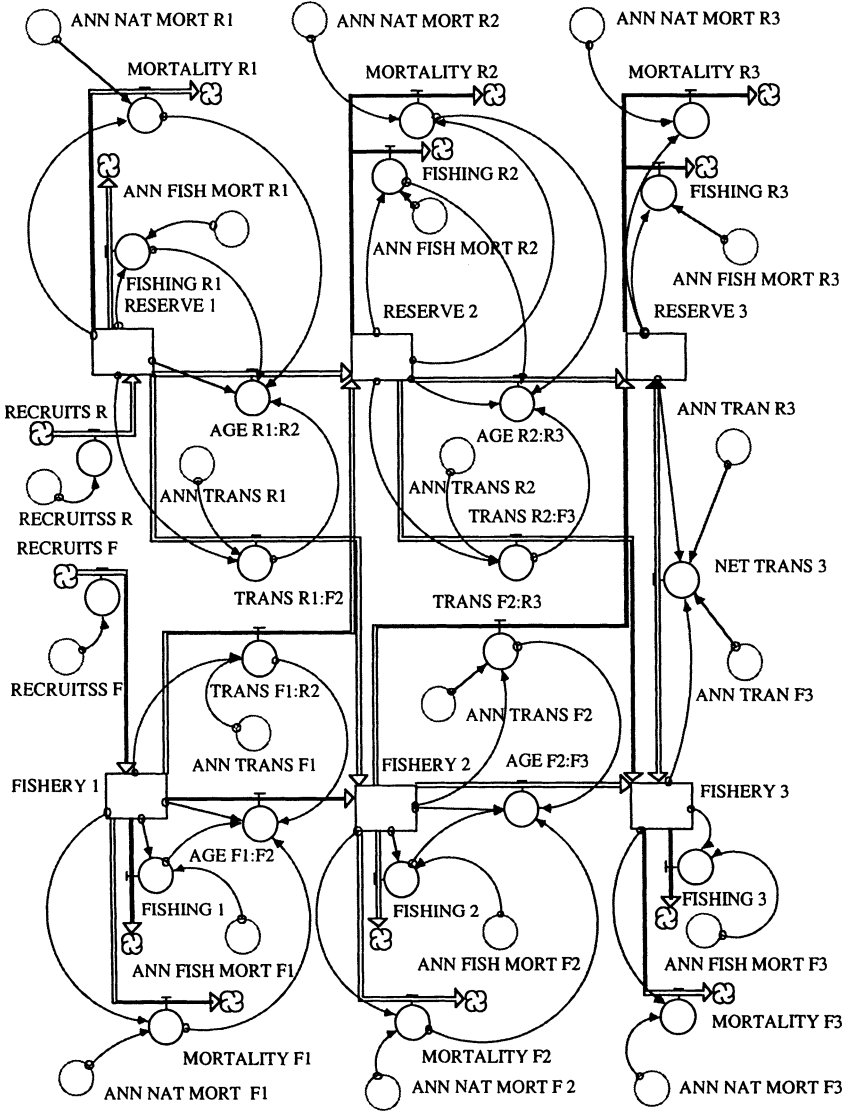
FIGURE 31.1

ponential relationship between fish age/size and fecundity. In the model, fish weights and fecundity conform to data from the Gulf of Mexico for the Red Snapper. The eggs produced leave the system (undergoing a pelagic larval stage) and return to be dispersed evenly over the fishery and reserve, i.e., if the reserve is 10% of total area, it receives 10% of all eggs produced. The number of recruits that enter the reserve and fishery each period is a function of the number of eggs received, a base recruitment rate (representing a survival rate for larvae and juveniles) and a density-dependent
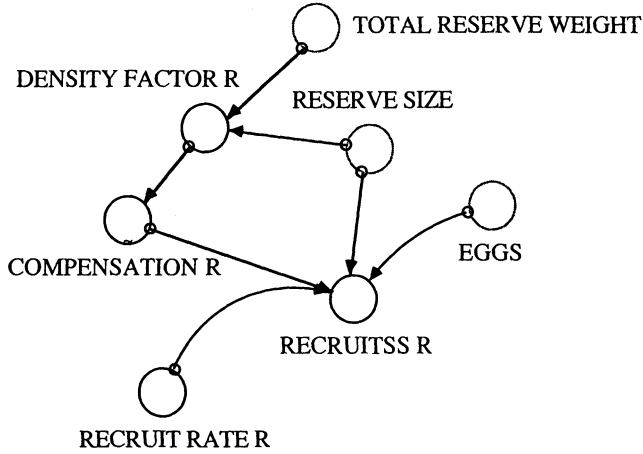
FIGURE 31.2

compensation mechanism. The compensation mechanism reduces the re-cruitment rate as biomass density rises or increases it as density falls. This mechanism keeps the system stable. Without this mechanism, disturbance would cause the fishery to either increase to infinity or collapse. The compensation is relatively slight when the fishery/reserve is at moderate densities, but increases its effect when the density gets very low or high.

Compensation and recruitment for the reserve are calculated in the module of Figure 31.2. Compensation is modeled with a graphical function as plotted in Figure 31.3.

FIGURE 31.3

COMPENSATION F

TOTAL FISHERY WEIGHT

RECRUITSS F

EGGS

DENSITY FACTOR F

RECRUIT RATE F

RESERVE SIZE
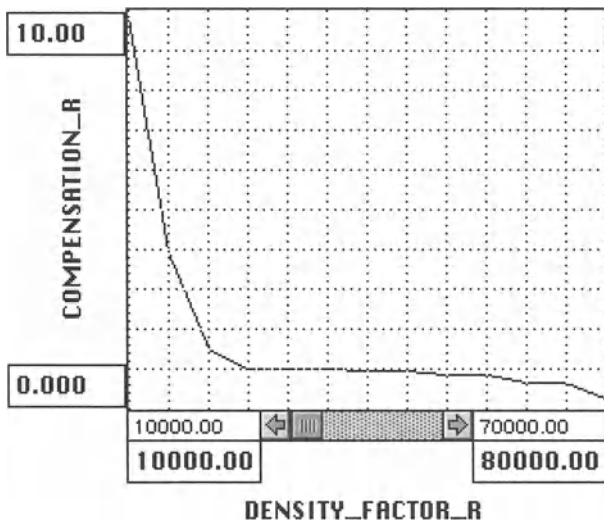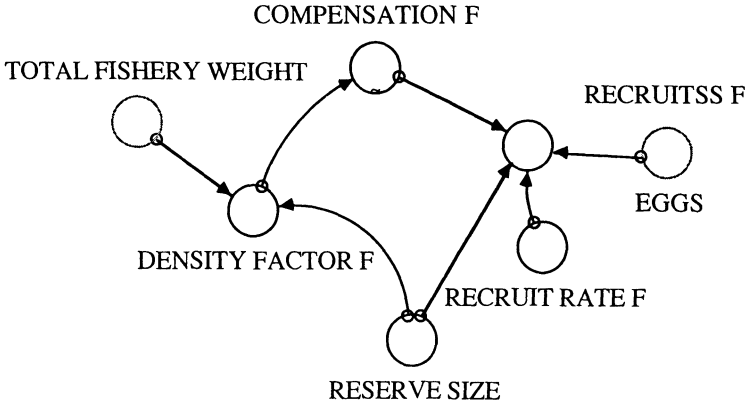
FIGURE 31.4

The calculation of compensation and recruits for the fishery is done anal-
ogously to that for the reserve. The corresponding module is shown in Fig-
ure 31.4, and compensation in the fishery is plotted in Figure 31.5.

The production of eggs, in turn, is a function of the female ratio in the
population, the fecundity factor, and the mean gonad weight in the respec-
tive fishery. These relationships are captured in the module of Figure 31.6.

The instantaneous transfer of fish between reserve and fishery, TRANS
R:F, and between fishery and reserve, TRANS F:R, are a function of the in-
stantaneous transfer rates that are characteristic for the particular fish spe-
cies. The relationships are shown in Figure 31.7.
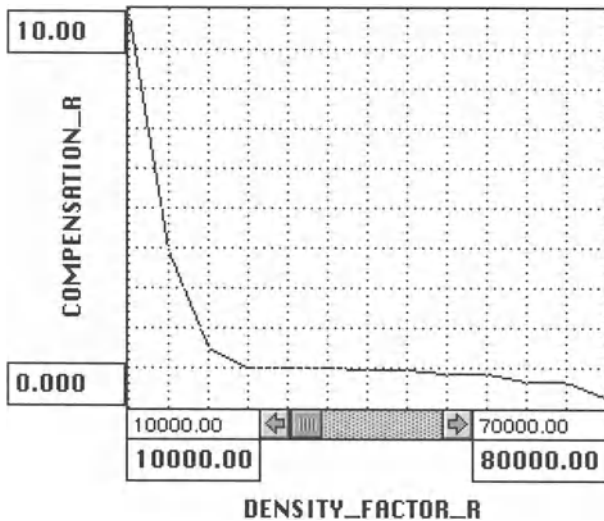
FIGURE 31.5
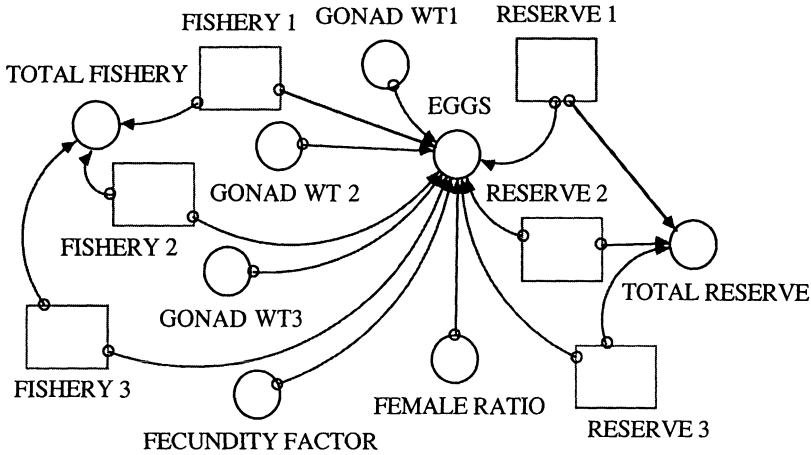
FIGURE 31.6

From these instantaneous transfer rates we can calculate the annual transfer of fish among the reserves and fisheries. Together with the coefficients that determine fishing effort of fish that reside in the fisheries and those living in the reserves but temporarily leaving the reserves, we can calculate the mortalities for the fish population (Figs. 31.8 to 31.13). The effort coefficients are exogenously specified.

In order to determine the amount of fish that can be caught, total fish biomass must be divided among the fishery and the reserve. Additionally, we need to specify the mass of fish at different age classes. This is done exogenously by the translation variables WEIGHT F1, WEIGHT F2, WEIGHT F3 (Fig. 31.14).

Each period the total value (price*weight) of fish caught is discounted according to the length of time that has passed from the institution of the reserve and added to the cumulative net present value of harvests (Fig. 31.15). Price is assumed to remain constant at one unit per pound of

FIGURE 31.7

EFFORT COEFF F   EFFORT COEFF R

EFFORT COEFF R

BASE INST FISH MORT R1

INST NAT MORT R1

INST TRANS R:F

INST FISH MORT R1

INST TOTAL MORT R1

ANN FISH MORT R1

ANN TRANS R1

ANN NAT MORT R1

FIGURE 31.8

EFFORT COEFF R   INST TRANS R:F

BASE INST FISH MORT R 2

INST NAT MORT R 2

INST FISH MORT R2

INST TOTAL MORT R2

ANN TRANS R2

ANN FISH MORT R2

ANN NAT MORT R2

FIGURE 31.9

EFFORT COEFF R   INST TRANS R:F

BASE INST FISH MORT R3

INST NAT MORT R3

INST FISH MORT R3

INST TOTAL MORT R3

ANN TRAN R3

ANN FISH MORT R3

ANN NAT MORT R3

FIGURE 31.10

BASE INST FISH MORT F1   EFFORT COEFF F

INST FISH MORT F F   INST NAT FISH MORT F1

INST TRANS F:R

INST TOTAL MORT F1

ANN FISH MORT F1

ANN NAT MORT F1   ANN TRANS F1

FIGURE 31.11

EFFORT COEFF F INST TRANS F:R
BASE INST FISH MORT F2

INST FISH MORT F2   INST NAT MORT F2

INST TOTAL MORT F2

ANN FISH MORT F2   ANN TRANS F2

ANN NAT MORT F 2

FIGURE 31.12

EFFORT COEFF F

BASE INST FISH MORT F3   INST TRANS F:R

INST NAT MORT F3

INST FISH MORT F3

INST TOTAL MORT F3

ANN FISH MORT F3   ANN TRAN F3

ANN NAT MORT F3

FIGURE 31.13

FIGURE 31.14

fish caught. When the reserve is first instituted, fish production drops roughly proportional to the reserve size relative to the original fishery. It takes several years for production to respond to the reserve and recover to prior levels. Thus the discount rate is very important.

The model is initialized with a population structure that results from a set of age-specific natural mortality rates and fishing mortality rates in the absence of a reserve. A reserve size is then chosen and the model is run for several years until a steady state is reached. Steady state of the fish population almost always occurs within 20 years, but always within 60 years if one is reached at all.

The results indicate that recruits to the fishery and egg production increase over time (Figs. 31.16 and 31.17). Not only does the total biomass in

NET PRESENT VALUE OF HARVEST

NET PRESENT VALUE OF TOTAL FISHING WEIGHT

TOTAL FISHING WEIGHT
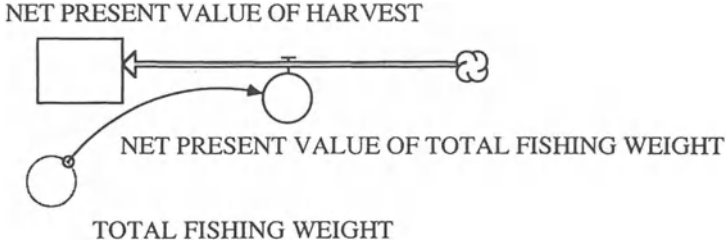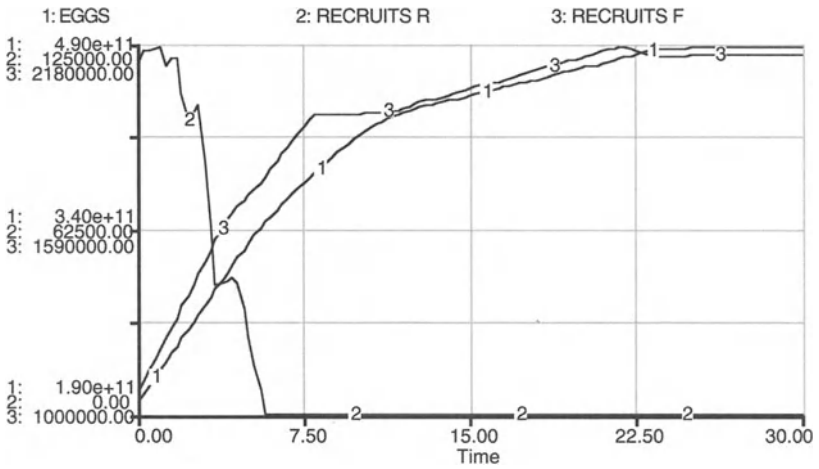
FIGURE 31.15
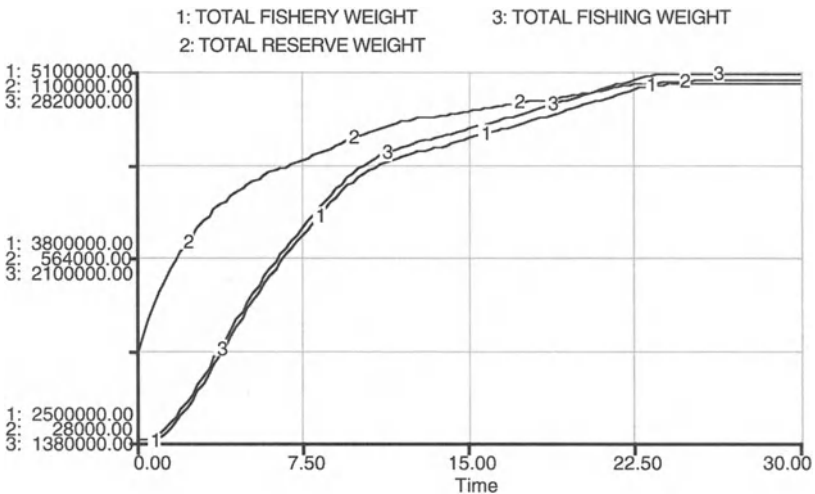
FIGURE 31.16

FIGURE 31.17

the reserve increase but also the total biomass in the fishery. Thus, the overall effect of the introduction of the reserve on the fish population is positive, effectively protecting biodiversity and habitat as well as fishery production.

Since conventional methods of common property management that depend on reducing fishing are frequently expensive and difficult or impossible to apply, marine reserves provide a feasible alternative management strategy. The analytical model developed here not only indicates that marine reserves can be effective in sustaining or increasing fishery yields, it also demonstrates the advantages of age-class-specific modeling of fishery dynamics over previous models that did not differentiate the different spatio-temporal dynamics of different age classes.

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                    FISHERY RESERVE MODEL

 FISHERY_1(t) = FISHERY_1(t - dt) + (RECRUITS_F -
 FISHING_1 - AGE_F1:F2 - MORTALITY_F1 - TRANS_F1:R2) *
 dt
 INIT FISHERY_1 = .5475*INITIAL_FISH_TOTAL*(1-
 RESERVE_SIZE)
 INFLOWS:
 RECRUITS_F = RECRUITSS_F
 OUTFLOWS:
 FISHING_1 = FISHERY_1*ANN_FISH_MORT_F1
 AGE_F1:F2 = FISHERY_1 - MORTALITY_F1 - TRANS_F1:R2 -
 FISHING_1
 MORTALITY_F1 = FISHERY_1*ANN_NAT_MORT__F1
 TRANS_F1:R2 = FISHERY_1*ANN_TRANS_F1

 FISHERY_2(t) = FISHERY_2(t - dt) + (AGE_F1:F2 +
 TRANS_R1:F2 - FISHING_2 - AGE_F2:F3 - MORTALITY_F2 -
 TRANS_F2:R3) * dt
 INIT FISHERY_2 = .3255*INITIAL_FISH_TOTAL*(1-
 RESERVE_SIZE)
 INFLOWS:
 AGE_F1:F2 = FISHERY_1 - MORTALITY_F1 - TRANS_F1:R2 -
 FISHING_1
 TRANS_R1:F2 = RESERVE_1*ANN_TRANS_R1
 OUTFLOWS:
 FISHING_2 = FISHERY_2*ANN_FISH_MORT_F2
 AGE_F2:F3 = FISHERY_2-MORTALITY_F2-TRANS_F2:R3-
 FISHING_2
 MORTALITY_F2 = FISHERY_2*ANN_NAT_MORT_F_2
 TRANS_F2:R3 = FISHERY_2*ANN_TRANS_F2
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FISHERY_3(t) = FISHERY_3(t - dt) + (AGE_F2:F3 +
TRANS_R2:F3 + NET_TRANS_3 - FISHING_3 - MORTALITY_F3)
* dt
INIT FISHERY_3 = INITIAL_FISH_TOTAL*.1270*(1-
RESERVE_SIZE)
INFLOWS:
AGE_F2:F3 = FISHERY_2-MORTALITY_F2-TRANS_F2:R3-
FISHING_2
TRANS_R2:F3 = RESERVE_2*ANN_TRANS_R2
NET_TRANS_3 = (RESERVE_3*ANN_TRAN_R3)-
(ANN_TRAN_F3*FISHERY_3)
OUTFLOWS:
FISHING_3 = FISHERY_3*ANN_FISH_MORT_F3
MORTALITY_F3 = FISHERY_3*ANN_NAT_MORT_F3

NET_PRESENT_VALUE_OF_HARVEST(t) =
NET_PRESENT_VALUE_OF_HARVEST(t - dt) +
(NET_PRESENT_VALUE_OF_TOTAL_FISHING_WEIGHT) * dt
INIT NET_PRESENT_VALUE_OF_HARVEST = 0
INFLOWS:
NET_PRESENT_VALUE_OF_TOTAL_FISHING_WEIGHT =
TOTAL_FISHING_WEIGHT/(1.1^TIME)

RESERVE_1(t) = RESERVE_1(t - dt) + (RECRUITS_R -
AGE_R1:R2 - MORTALITY_R1 - TRANS_R1:F2 - FISHING_R1) *
dt
INIT RESERVE_1 = .5475*INITIAL_FISH_TOTAL*RESERVE_SIZE
INFLOWS:
RECRUITS_R = RECRUITSS_R
OUTFLOWS:
AGE_R1:R2 = RESERVE_1-MORTALITY_R1-TRANS_R1:F2-
FISHING_R1
MORTALITY_R1 = RESERVE_1*ANN_NAT_MORT_R1
TRANS_R1:F2 = RESERVE_1*ANN_TRANS_R1
FISHING_R1 = RESERVE_1*ANN_FISH_MORT_R1

RESERVE_2(t) = RESERVE_2(t - dt) + (AGE_R1:R2 +
TRANS_F1:R2 - AGE_R2:R3 - MORTALITY_R2 - TRANS_R2:F3 -
FISHING_R2) * dt
INIT RESERVE_2 = .3255*INITIAL_FISH_TOTAL*RESERVE_SIZE
INFLOWS:
AGE_R1:R2 = RESERVE_1-MORTALITY_R1-TRANS_R1:F2-
FISHING_R1
```

```
TRANS_F1:R2 = FISHERY_1*ANN_TRANS_F1
OUTFLOWS:
AGE_R2:R3 = RESERVE_2-MORTALITY_R2-TRANS_R2:F3-
FISHING_R2
MORTALITY_R2 = RESERVE_2*ANN_NAT_MORT_R2
TRANS_R2:F3 = RESERVE_2*ANN_TRANS_R2
FISHING_R2 = RESERVE_2*ANN_FISH_MORT_R2

RESERVE_3(t) = RESERVE_3(t - dt) + (AGE_R2:R3 +
TRANS_F2:R3 - MORTALITY_R3 - FISHING_R3 - NET_TRANS_3)
* dt
INIT RESERVE_3 = INITIAL_FISH_TOTAL*RESERVE_SIZE*.1270
INFLOWS:
AGE_R2:R3 = RESERVE_2-MORTALITY_R2-TRANS_R2:F3-
FISHING_R2
TRANS_F2:R3 = FISHERY_2*ANN_TRANS_F2
OUTFLOWS:
MORTALITY_R3 = RESERVE_3*ANN_NAT_MORT_R3
FISHING_R3 = RESERVE_3*ANN_FISH_MORT_R3
NET_TRANS_3 = (RESERVE_3*ANN_TRAN_R3)-
(ANN_TRAN_F3*FISHERY_3)

ANN_FISH_MORT_F1 =
INST_FISH_MORT_F_F1/INST_TOTAL_MORT_F1*(1-EXP(-
INST_TOTAL_MORT_F1))
ANN_FISH_MORT_F2 =
INST_FISH_MORT_F2/INST_TOTAL_MORT_F2*(1-EXP(-
INST_TOTAL_MORT_F2))
ANN_FISH_MORT_F3 =
INST_FISH_MORT_F3/INST_TOTAL_MORT_F3*(1-EXP(-
INST_TOTAL_MORT_F3))
ANN_FISH_MORT_R1 =
INST_FISH_MORT_R1/INST_TOTAL_MORT_R1*(1-EXP(-
INST_TOTAL_MORT_R1))
ANN_FISH_MORT_R2 =
INST_FISH_MORT_R2/INST_TOTAL_MORT_R2*(1-EXP(-
INST_TOTAL_MORT_R2))
ANN_FISH_MORT_R3 =
INST_FISH_MORT_R3/INST_TOTAL_MORT_R3*(1-EXP(-
INST_TOTAL_MORT_R3))
ANN_NAT_MORT_F3 =
INST_NAT_MORT_F3/INST_TOTAL_MORT_F3*(1-EXP(-
INST_TOTAL_MORT_F3))
```

```
ANN_NAT_MORT_F_2 =
INST_NAT_MORT_F2/INST_TOTAL_MORT_F2*(1-EXP(-
INST_TOTAL_MORT_F2))
ANN_NAT_MORT_R1 =
INST_NAT_MORT_R1/INST_TOTAL_MORT_R1*(1-EXP(-
INST_TOTAL_MORT_R1))
ANN_NAT_MORT_R2 =
INST_NAT_MORT_R_2/INST_TOTAL_MORT_R2*(1-EXP(-
INST_TOTAL_MORT_R2))
ANN_NAT_MORT_R3 =
INST_NAT_MORT_R3/INST_TOTAL_MORT_R3*(1-EXP(-
INST_TOTAL_MORT_R3))
ANN_NAT_MORT__F1 =
INST_NAT_FISH_MORT_F1/INST_TOTAL_MORT_F1*(1-EXP(-
INST_TOTAL_MORT_F1))
ANN_TRANS_F1 = INST_TRANS_F:R
ANN_TRANS_F2 = INST_TRANS_F:R
ANN_TRANS_R1 = INST_TRANS_R:F
ANN_TRANS_R2 = INST_TRANS_R:F
ANN_TRAN_F3 = INST_TRANS_F:R
ANN_TRAN_R3 = INST_TRANS_R:F
BASE_INST_FISH_MORT_F1 = .01
BASE_INST_FISH_MORT_F2 = .5
BASE_INST_FISH_MORT_F3 = .5
BASE_INST_FISH_MORT_R1 = .01
BASE_INST_FISH_MORT_R3 = .5
BASE_INST_FISH_MORT_R_2 = .5
BASE_INST_TRANS_RATE = .4
DENSITY_FACTOR_F = TOTAL_FISHERY_WEIGHT/((1-
RESERVE_SIZE)*100)
DENSITY_FACTOR_R =
TOTAL_RESERVE_WEIGHT/(RESERVE_SIZE*100)
EFFORT_COEFF_F = 2
EFFORT_COEFF_R = .02
EGGS =
(((FISHERY_1+RESERVE_1)*GONAD_WT1)+((FISHERY_2+RESERVE
_2)*GONAD_WT_2)+((FISHERY_3+RESERVE_3)*GONAD_WT3))*FEC
UNDITY_FACTOR*FEMALE_RATIO
FECUNDITY_FACTOR = 12000
FEMALE_RATIO = .5
GONAD_WT1 = 0
GONAD_WT3 = 100
GONAD_WT_2 = 10
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INITIAL_FISH_TOTAL = 2085556
INST_FISH_MORT_F2 =
BASE_INST_FISH_MORT_F2*EFFORT_COEFF_F
INST_FISH_MORT_F3 =
BASE_INST_FISH_MORT_F3*EFFORT_COEFF_F
INST_FISH_MORT_F_F1 =
BASE_INST_FISH_MORT_F1*EFFORT_COEFF_F
INST_FISH_MORT_R1 =
BASE_INST_FISH_MORT_R1*EFFORT_COEFF_R
INST_FISH_MORT_R2 =
BASE_INST_FISH_MORT_R_2*EFFORT_COEFF_R
INST_FISH_MORT_R3 =
BASE_INST_FISH_MORT_R3*EFFORT_COEFF_R
INST_NAT_FISH_MORT_F1 = .5
INST_NAT_MORT_F2 = .3
INST_NAT_MORT_F3 = .2
INST_NAT_MORT_R1 = .5
INST_NAT_MORT_R3 = .2
INST_NAT_MORT_R_2 = .3
INST_TOTAL_MORT_F1 =
INST_FISH_MORT_F_F1+INST_NAT_FISH_MORT_F1
INST_TOTAL_MORT_F2 =
INST_FISH_MORT_F2+INST_NAT_MORT_F2
INST_TOTAL_MORT_F3 =
INST_FISH_MORT_F3+INST_NAT_MORT_F3
INST_TOTAL_MORT_R1 =
INST_FISH_MORT_R1+INST_NAT_MORT_R1
INST_TOTAL_MORT_R2 =
INST_FISH_MORT_R2+INST_NAT_MORT_R_2
INST_TOTAL_MORT_R3 =
INST_FISH_MORT_R3+INST_NAT_MORT_R3
INST_TRANS_F:R = BASE_INST_TRANS_RATE*RESERVE_SIZE
INST_TRANS_R:F = BASE_INST_TRANS_RATE*(1-RESERVE_SIZE)
RECRUITSS_F = EGGS*RECRUIT_RATE_F*(1-
RESERVE_SIZE)*COMPENSATION_F
RECRUITSS_R =
RECRUIT_RATE_R*EGGS*RESERVE_SIZE*COMPENSATION_R
RECRUIT_RATE_F = .00000594
RECRUIT_RATE_R = .00000594
RESERVE_SIZE = .1
TOTAL_BIOMASS =
TOTAL_FISHERY_WEIGHT+TOTAL_RESERVE_WEIGHT
TOTAL_FISHERY = SUM(FISHERY_1,FISHERY_2,FISHERY_3)
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TOTAL_FISHERY_WEIGHT =
(FISHERY_1*WEIGHT_F1)+(FISHERY_2*WEIGHT_F2)+(FISHERY_3
*WEIGHT_F3)
TOTAL_FISHING_WEIGHT =
((FISHING_1+FISHING_R1)*WEIGHT_F1)+((FISHING_2+FISHING
_R2)*WEIGHT_F2)+((FISHING_3+FISHING_R3)*WEIGHT_F3)
TOTAL_RESERVE = SUM(RESERVE_1,RESERVE_2,RESERVE_3)
TOTAL_RESERVE_WEIGHT =
(RESERVE_1*WEIGHT_F1)+(RESERVE_2*WEIGHT_F2)+(RESERVE_3
*WEIGHT_F3)
WEIGHT_F1 = .05
WEIGHT_F2 = 2
WEIGHT_F3 = 5
COMPENSATION_F = GRAPH(DENSITY_FACTOR_F)
(10000, 9.80), (15000, 3.90), (20000, 1.50), (25000,
1.00), (30000, 1.00), (35000, 1.00), (40000, 0.95),
(45000, 0.95), (50000, 0.85), (55000, 0.85), (60000,
0.65), (65000, 0.65), (70000, 0.25), (75000, 0.25),
(80000, 0.00)
COMPENSATION_R = GRAPH(DENSITY_FACTOR_R)
(10000, 9.80), (15000, 3.90), (20000, 1.50), (25000,
1.00), (30000, 1.00), (35000, 1.00), (40000, 0.95),
(45000, 0.95), (50000, 0.85), (55000, 0.85), (60000,
0.65), (65000, 0.65), (70000, 0.25), (75000, 0.25),
(80000, 0.00)
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32
Dynamic Scarcity

> Preposterous as it may seem at first blush, it is probably true that, even if all the timber in the U.S., or all the oil, or gas or anthracite, were owned by absolute monopoly, entirely free of public control, prices to consumers would be fixed lower than the long-run interests of the public would justify.
>
> —John Ise, *The United States Oil Policy*

32.1 Competitive Scarcity

In the previous chapter we identified the optimal harvest of a renewable resource. Let us assume now that we are the owners of a nonrenewable resource such as a mineral in a mine. As the mine owners, we attempt to maximize the present value of profits that accrue from selling the mining output. Again, we assume perfect competition, i.e., at each period of time the price of the mineral is a given. Let us also assume that there are no substitutes for the mineral and that it cannot be recycled. Furthermore, for the simplicity of the model, we assume that the resource stock is known and that there are no new discoveries.

Since the mine is exhaustible, we are faced with the following choice: We can either extract all the minerals today, make profits, invest these profits and earn interest from this investment, or, we may wait until other mining operations exhausted their mines, thereby decreasing the amount that can be supplied, and thus, driving up the mineral's price. Once the price is high, we start extracting from our mine. How should we proceed? We would expect that the optimal behavior for the mining operation falls between these two extremes of total extraction today and waiting with the extraction process.

It is optimal for each of the mining operations to purposely develop a production rate that continuously forces up the price. Initially, we extract a large amount that yields a high present value of profits. The amount of minerals in the mine is reduced by that amount, and thus, less can be supplied in the following periods. A decreased supply means that prices increase along the demand curve. With a higher price and less of the minerals left in the ground the rate of extraction is lower in subsequent periods.

The problem of maximizing the present value of profits, PV PROFIT, from the sale of a nonrenewable resource can be formalized as follows. The objective of our mining operation is to

$$\text{maximize PV profit} = \text{maximize} \int_0^T [P(t)*Q(t) - C(t)]e^{-rt}dt \qquad (1)$$

over the time horizon starting today (t = 0) up to the future period T in which the mine is entirely exhausted. This terminal time, T, will be determined when we run our model. Output, Q(t), is the variable that can be controlled by the mining operation in order to achieve a maximum. Here, the price at time period t is P(t), and cost of extraction is C(t). The rate of discount, r, is known and constant over time. The cost of extraction is a function of the mining output,

$$C(t) = B^*Q(t)^{\wedge}DELTA \qquad (2)$$

where B and DELTA are constants. Production cost increases with an increasing rate of extraction.

Since the mine contains only a finite amount of minerals, the maximization is constrained by the condition that the initial resource stock must be greater than or equal to the total amount that is being extracted over the T periods, i.e.,

$$X(t = 0) \geq \int_0^T Q(t)dt \qquad (3)$$

An additional constraint that is placed on our optimizing mining operation by the condition that refilling of the mine is not possible, i.e.,

$$Q(t) \geq 0 \qquad (4)$$

A final condition is given by the fact that the change of the amount of minerals in the mine, $\dot{X}(t) = \partial X/\partial t$, is solely brought about by the extraction process, i.e., no discoveries or adjustments in our estimates of the reserve size take place. The formal statement of this condition is

$$\dot{X}(t) = -Q(t) \qquad (5)$$

The negative sign indicates that X decreases over time.

Analytically, the optimal rate of extraction at each time period t, Q(t), can be calculated by maximizing the following function with respect to Q(t), our control variable:

$$H = [P(t)^*Q(t) - C(t)] - \lambda(t)^*Q(t) \qquad (6)$$

The term in square brackets is the *present* value of profits. The last term of this equation subtracts an amount $\lambda(t)^*Q(t)$ from this present value of profits. This amount is a "penalty" for reducing the resource stock by an amount Q(t). The parameter $\lambda(t)$ reflects this "penalty" and can be interpreted alternatively as the amount that a mining operation is willing to pay to have an additional unit of the mineral in the ground. It is referred to as the scarcity rent.

This approach to intertemporal optimization is known as optimal control theory.[1] The function H is referred to as a Hamiltonian. The conditions for

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[1] Dorfman, R. 1969. An Economic Interpretation of Optimal Control Theory, *The American Economic Review*, Vol. 59, pp. 817–831.

the optimal extraction path are given by the first partial derivative of the Hamiltonian with respect to the control variable Q(t),

$$\frac{\partial H}{\partial Q(t)} = P(t) - \frac{\partial C(t)}{\partial Q(t)} - \lambda(t) = 0 \qquad (7)$$

and the condition that describes the way in which H is influenced by a change in the resource stock X. The latter condition is

$$-\frac{\partial H}{\partial X(t)} = \lambda(t) - r^*\lambda(t) \qquad (8)$$

Since H is not a function of X(t), we know

$$\dot{\lambda}(t) - r^*\lambda(t) = 0 \qquad (9)$$

and thus,

$$\frac{\dot{\lambda}(t)}{\lambda(t)} = r \qquad (10)$$

This last condition is known as the Hotelling Rule[2] and can be interpreted easily: In the optimum of the mining operation (i.e., the maximum of present value of profits) the scarcity rent of the finite resource must rise at the rate of interest. This rule summarizes our discussion above: as the owners of the mining operation we extract just that much from the mine at each period of time that the resulting percentage change in the scarcity rent, $\dot{\lambda}(t)/\lambda(t)$, is exactly the rate of interest that would be applied for an investment of our profits. Even though this rule is rather intuitive, we still need to know the value for the scarcity rent, since this is not directly observable by us—unlike the price of the mineral or the extraction cost. However, a little bit more calculus gives us the necessary information.

The first partial derivative of H with respect to Q(t) can be rewritten as

$$\lambda(t) = P(t) - \frac{\partial C(t)}{\partial Q(t)} = P(t) - MC(t) \qquad (11)$$

Recall that in the profit maximum of a firm that produces an ordinary good, the price is equal to marginal cost. For the mining operation that extracts a nonrenewable resource, price equals marginal cost plus a rent, $\lambda(t)$, to the mine owners that is due to the nonrenewability of the good supplied by the mine. This is the reason why $\lambda(t)$ is called the scarcity rent.

From the condition above we can also calculate the change of the scarcity rent over time,

$$\frac{\partial \lambda(t)}{\partial t} = \dot{\lambda}(t) = \dot{P}(t) - \dot{MC}(t) \qquad (12)$$

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[2]Hotelling, H. C. 1931. The Economics of Exhaustible Resources, *Journal of Political Economy*, Vol. 39, pp. 137–175.

and together, these two equations yield

$$\dot{P} = r^*(\dot{P} - \dot{MC}) + MC \tag{13}$$

We have now expressed the optimal time path for the extraction of the minerals from the mine in observable quantities. As the owners of the mine, we must now translate these conditions into a STELLA program to find exactly how much we can extract from our mine, with an initial amount of 60,000 units of the mineral, at each period. In our use of equation 13, we will call the time derivative of price, DEL P and the time derivative of MC, DEL MC.

   Note when you are running this model, there are some terminal conditions you must meet. These are called the sufficiency conditions and are needed to insure optimality, in addition to the necessary conditions that lead to equation 13. The sufficiency conditions for this problem are: $\lambda(T) \geq 0$, $\lambda(T)^*X(T) = 0$, and because we have no prespecified terminal time T in this model, we have a third sufficiency condition: $H(T) = 0$. The last of these three conditions is met rather naturally in this model. We can readily see from equation 6 that if $Q(T) = 0$, then $H(T) = 0$. So at the terminal time, the production rate must be 0. From equation 7 we can see that $\lambda \geq 0$ since $P - MC \geq 0$ for all time. Therefore X(T) must be 0 to satisfy the second condition. So we pick an initial price that will bring X and Q to zero simultaneously and we will have achieved the maximum associated CUMULATIVE PV PROFIT.

   To solve such a problem in STELLA, we use a trial and error procedure to constantly adjust the initial price and rerun the model so that the cumulative present value of profits, CUMULATIVE PV PROFIT, is maximized (Fig. 32.1). Use the Table Window to do this. Adjust the starting price such that the production rate is 0 just as the resource stock reaches 0 (i.e., the sufficiency conditions for the optimum). This will always be the case when the cost does not depend on the stock size or when there are no independent additions or reductions in the stock. A trial and error process is not the only way to solve this problem. We can set up the terminal conditions and run the problem backwards in time. The result is a curve of initial prices as a function of initial reserve size. We can pick the correct starting price directly from the curve for the 60,000 unit reserve.

   The graph of Figure 32.2 shows a smooth decline in the production rate, Q(t), a steady rise in the price, and a fall in marginal cost, MC. Repeated solution of the problem with different initial prices finally produced the graph shown in Figure 32.2. Can you find the proper starting price if the initial reserve is 100,000 units? What is the optimum path if we change the DT to 0.1 instead of 1.0? How is the result altered if we stop the problem at t = 20 and reoptimize on the remaining reserve?

   Run the model to assure yourself that the scarcity rent rises at the rate of interest as the analytic theory indicates it should in the absence of any stock effects. Also demonstrate that CUMULATIVE PV PROFIT is maximized

FIGURE 32.1

FIGURE 32.2

when the resource is exhausted by a production rate that smoothly goes to 0 at the same time.

This model does not handle all cases. For example, if the cost of extracting the resource is related to the size of the reserve (the known quantity of that resource in the ground that is economically extractable), then the model no longer holds. The scarcity rent will rise more slowly than the rate of interest. This means that not all of the resource might be used, i.e., that the production rate reaches 0 before the stock is exhausted. In cases of biological resources, the interest rate in the above equation is changed by the rate of change of the growth of the resource with respect to a change in resource stock size. We will augment our little model from the theory and apply it in the following section to the case of full monopoly.

```
                    COMPETITIVE SCARCITY

CUMULATIVE__PV_PROFIT(t)  = CUMULATIVE__PV_PROFIT(t -
dt) + (PV_PROFIT) * dt
INIT CUMULATIVE__PV_PROFIT = 0 {The cumulative present
value of the profit; $}
INFLOWS:
PV_PROFIT = CV_PROFIT/(1+r)^TIME {The present value of
the rate of profit; $ per Year}
P(t) = P(t - dt) + (DEL_P) * dt
INIT P = 1.95 {The resource exhaustion time depends on
this value as it indicates where the problem is started
wrt the shutoff price of 10. $ per Units of X}
INFLOWS:
DEL_P = r*(P - MC) + DEL_MC {$ per Units of X per Time
Period}


X(t) = X(t - dt) + (- Q) * dt
INIT X = 60000 {Units of X}
OUTFLOWS:
Q = (10 - P)/.00338   {q goes to zero simultaneously
with stock exhaustion. This is the optimal
(profit-maximizing) path and the simultaneity is
achieved by adjusting the initial price. Units of X per
Time Period}


B = .1
C = B*Q^DELTA {This is the total cost and it is
independent of any stock size or interest rate. $ per
Year}
CV_PROFIT = P*Q-C {$ per Year}
```

```
DELTA = 1.2   {This value must be > one so that the
cost vs q curve is convex.}
DEL_C = C - DELAY(C,DT,C+1.23) {$ per Unit of X per
Year}
DEL_EXTRACTION = Q - DELAY(Q,DT,Q+2.17) {Units of X
per Time Period}
DEL_MC = (MC- DELAY(MC,DT))/DT   {$ per Unit of X}
LAMBDA = P - MC   {$ per Unit X}
MC = DEL_C/DEL_EXTRACTION   {$ per Unit X}
r = .05 {$ per $ Invested per Time Period}
```

32.2 Monopoly Scarcity

Let us return to the basic model of section 32.1 but now assume that our firm has a monopoly for the resource it extracts. As a monopolist, we could influence the price by our control over the rate of extraction. The extraction rate will be chosen such that we maximize the cumulative present value of profits, CPVP. The Hamiltonian in this model is

$$H = [P(Q(t)) * Q(t) - C(Q(t))]e^{-r*t} - \lambda(t) * Q(t) \tag{14}$$

with

$$MR = 10 - .00676 * Q \frac{\partial H}{\partial Q} = \left[MR - \frac{\partial C}{\partial Q}\right] * e^{-r*t} - \lambda$$

$$\text{where MR = Marginal Revenue} = \frac{\partial[P(Q)*Q(t)]}{\partial Q} \tag{15}$$

$$-\frac{\partial H}{\partial X} = \dot{\lambda} \tag{16}$$

as the optimality and adjoint equations. These equations are virtually the same as those for the perfect competitors, with the exception that price is now a function of Q, and therefore, marginal revenues are no longer equal to the price. When you set up the model, remember to correct for the size of DT to calculate the change in MR over time.

For the demand curve

$$P = 10 - .00338 * Q \tag{17}$$

and therefore, marginal revenues are

$$MR = 10 - .00676 * Q. \tag{18}$$

Solving the demand curve for Q allows the extraction rate to be determined from MR, which is a state variable in our model. Using the demand curve

lets us find the price once Q has been determined. Price in the monopoly case is MR plus a monopoly rent rate.

The marginal revenue curve for a straight-line demand curve is a straight line at twice the slope of the demand curve, as shown in equation (18) and the preceding sketch. Can you prove this? In Figure 32.3 we show the four possibilities for market equilibrium for the competitive/monopoly, finite/nonfinite resource conditions. We have now covered all four of these conditions with our models:

1. P = MC for the pure competition condition with no finiteness in the inputs.
2. P = MC + SRR (the scarcity rent rate) for the competition/finiteness condition.
3. MR = MC and P = MC + MRR (the monopoly rent rate) for the monopoly/nonfiniteness condition.
4. MR = MC + SRR and P = MC + SRR + MRR for the monopoly/finiteness condition.

Note well that the monopoly extracts both types of rent rates if it realizes that its input is finite. The rent rate is a possible scarcity indicator for the input resource. The problem lies in distinguishing between MRR and SRR, that is, in distinguishing the monopolist from the competitive firms that are using an increasingly scarce resource.

Before trying to find the rent rate and how it may have changed for a particular industry, one must first determine the degree of monopoly in the industry. One can assume away monopolistic possibility and then look for differences between P and MC for the industry. This approach, however, is
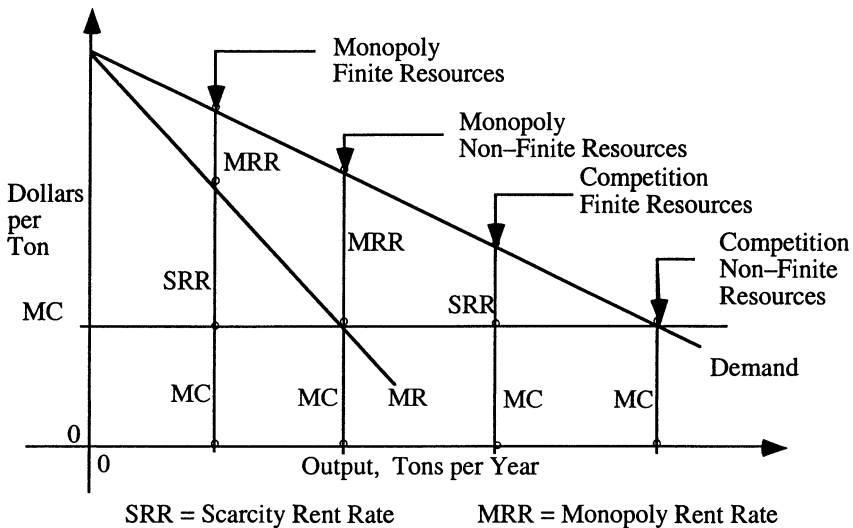
FIGURE 32.3

hardly satisfying! As we have seen earlier, there exists an approximate sur-
rogate for the SRR, the marginal discovery cost, the cost of finding the next
unit of a resource. If that is not 0, then there probably exists some SRR and
the resource is probably becoming scarce.

The model of a monopoy is solved here for a 60,000 unit finite resource
(the same as section 32.1) using trial and error values for the initial MR
(Fig. 32.4). The MR is adjusted until the resource stock and the production
rate are simultaneously 0 (to meet the sufficiency conditions noted in sec-
tion 32.1) and at this point, the cumulative present value of profits, CPVP, is
maximized. Use the Table Window for this effort. We used a DT = 1 here
(Figs. 32.5 to 32.7). Would a smaller DT bring more accuracy to this model?

We can now compare the competitive and the monopoly under scarcity.
Begin the scarcity model of the collection of firms in section 32.1 and the
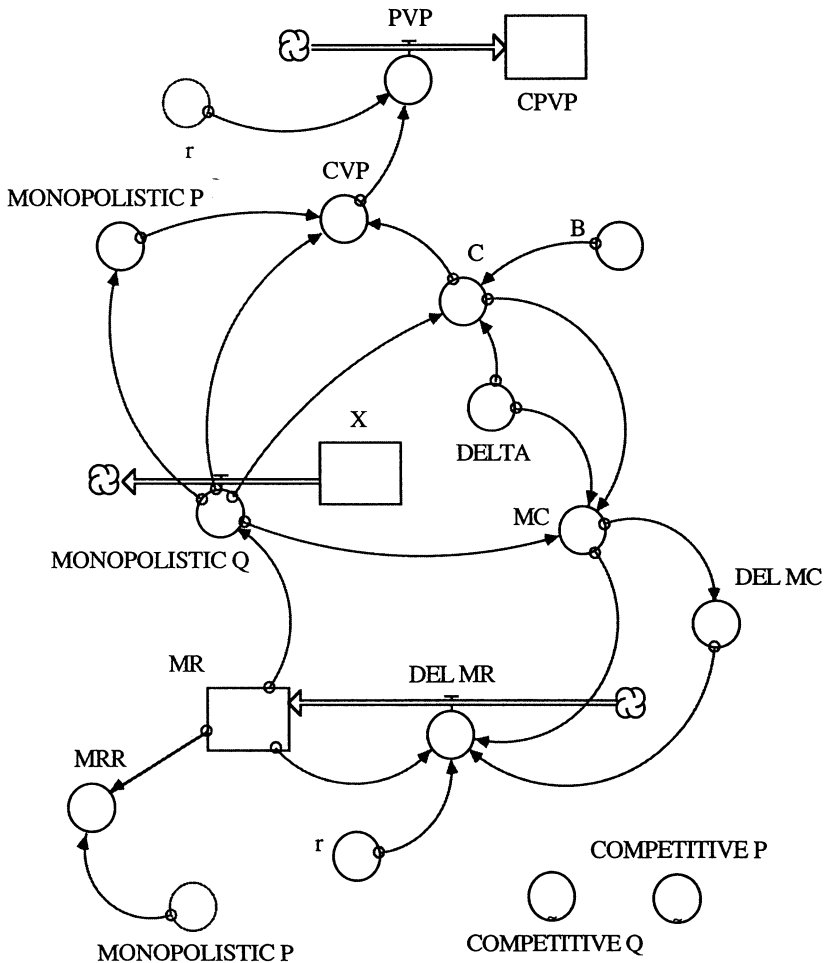
FIGURE 32.4
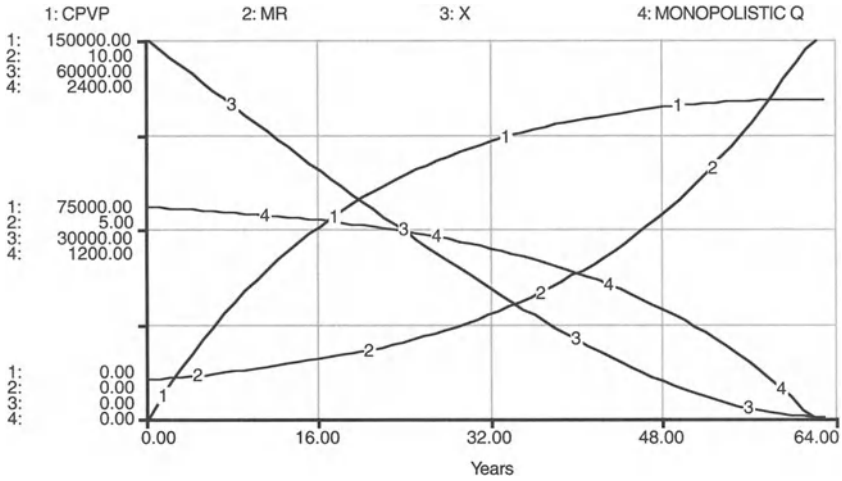
FIGURE 32.5

monopoly model of this section with the same amount, 60,000 units. Figures 32.8 and 32.9 depict the differences.

Note how much higher the price is here, in the initial and intermediate times, when compared with the competitive case (section 32.1). Interestingly, the competitive price rises above the monopoly price in the 23rd year. Note also how much lower cumulative present value of profits is and how much longer the resource lasts: approximately 63 years for the monopoly versus approximately 33 years with perfect competition. The total
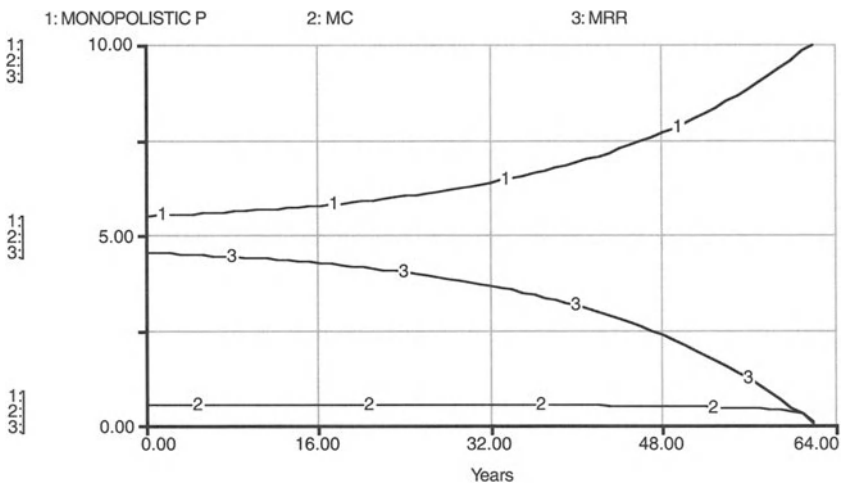
FIGURE 32.6
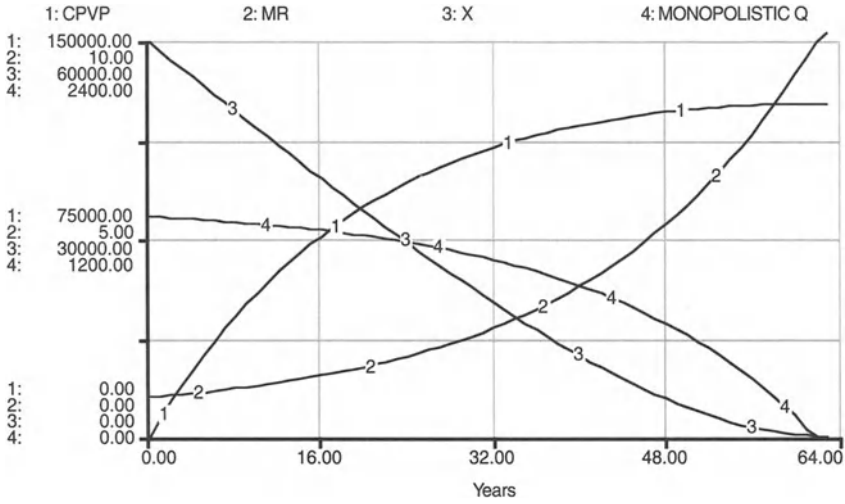
FIGURE 32.7

cumulative present value of the monopoly profit is about one-third higher than the competitive counterpart.

Is it always the case that monopolies will slow the rate of resource use? For the linear demand curve this is always true. But for a demand curve with a rising price elasticity of demand, the monopolist will use the resource faster than the competitors! The price elasticity of demand is

$$\frac{\partial Q}{\partial P} \frac{P}{Q} \tag{19}$$
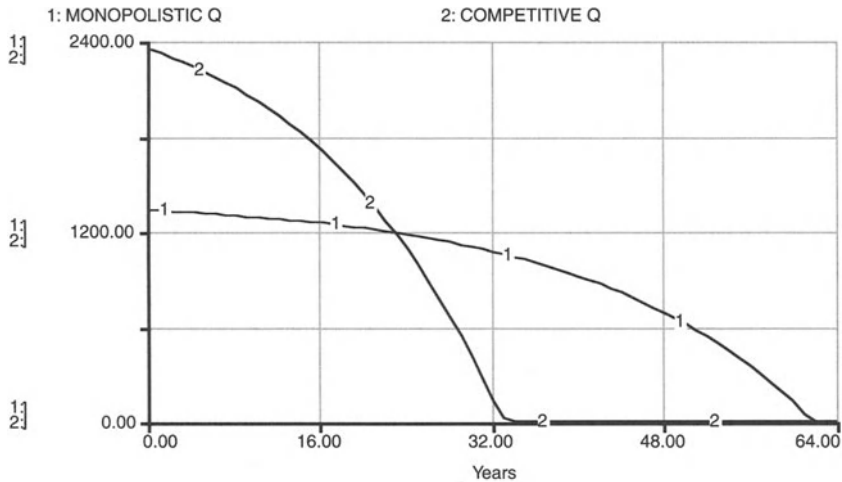
FIGURE 32.8
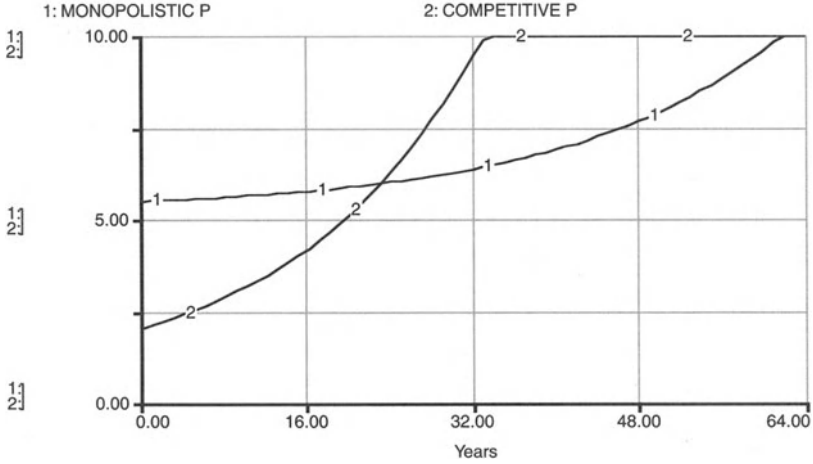
FIGURE 32.9

and an example of a demand curve with increasing price elasticity is

$$P = \frac{120}{Q^{0.5}} \tag{20}$$

Solve for the optimal extraction path under perfect competition and under monopolistic behavior, using this demand curve to show that the monopolist is now more "shortsighted" than the competitors. Because the monopoly profit is lower here than in the total competitive case, will monopolies be less likely to form in the finite resource markets than in regular markets? If a monopoly extends the life of a finite resource and accrues smaller cumulative present value profit, would you prefer that a monopoly controlled your society's finite resources? Before you jump to conclusions, recall also that discount rates, which play a primary role in determining the rate of extraction, are not usually constant over time.[3]

```
                    MONOPOLISTIC SCARCITY

  CPVP(t) = CPVP(t - dt) + (PVP) * dt
  INIT CPVP = 0 {Dollars}
  INFLOWS:
  PVP = CVP*EXP(-r*TIME) {Dollars per Year}


  MR(t) = MR(t - dt) + (DEL MR) * dt
```

[3]For optimal depletion models with varying discount rates and with many other modifications, such as discovery of new reserves or recycling, see Ruth, M. and B. Hannon (1997) *Modeling Dynamic Economic Systems*, Springer-Verlag, New York.

```
INIT MR = .978
INFLOWS:
DEL MR = r*(MR - MC) + DEL MC/DT {Dollars per Year}

X(t) = X(t - dt) + (- MONOPOLISTIC_Q) * dt
INIT X = 60000 {Tons}
OUTFLOWS:
MONOPOLISTIC_Q = (10 - MR)/.00676 {Tons per Year}

B = .1 {Dollars per Ton}
C = B*MONOPOLISTIC_Q^DELTA {Dollars per Year}
CVP = MONOPOLISTIC_P*MONOPOLISTIC_Q-C {Dollars per
Year}
DELTA = 1.2  {This value must be > one so that the C
versus Q curve is convex.}
MC = IF MONOPOLISTIC_Q > 0 then DELTA*C/MONOPOLISTIC_Q
else 0 {Dollars per Ton}
MONOPOLISTIC_P = 10 - .00338*MONOPOLISTIC_Q {Dollars
per Ton}
MRR = MONOPOLISTIC_P-MR {Dollars per Ton}
r = .05 {Dollars per Dollar per Year}
DEL MC = (MC-DELAY(MC,DT))/DT  {Dollars per Ton}
COMPETITIVE_P = GRAPH(TIME) See data from Chapter
32.1.
COMPETITIVE_Q = GRAPH(TIME) See data from Chapter
32.1.
```

33
Market Game

> At the bottom of the uncertainty problem in economics is the forward-looking character of the economic process itself.
> —Frank Knight, *Risk, Uncertainty and Profit*

Economic game theory coupled with computer modeling holds much promise in the development of an economic subdiscipline.[1] The idea that theories of behavior can be coupled with numerical experimentation on the computer means that economists will have a way to bench-test their ideas. Some theories may lead to results that are clearly absurd at the computer modeling level. Other theories could lead to results that would suggest testing for validity by field observation.

This model[2] is an example of an economic game that can be partially verified by the reasonableness of its results. It combines the behavioral purpose of two firms deciding anew each year whether to enter either of two markets whose sizes are randomly determined each year. The model combines the two forces that also drive evolutionary behavior in biology, a directedness (sometimes called fitness), and randomness. This means that over the long term, the average behavior of a firm is predictable. It acts on the average according to an expected norm. But in any given period, the behavior is unpredictable, with the resulting action at that time perhaps being harmful to a firm.

The model is developed in pieces. The first step is for each firm to make the decision to enter or refrain from the market in the upcoming period. Those decisions are based on an expected profit function. Once those two decisions are made, the market size is found and a chain of events is set into motion that leads to the revenues, costs, and payoffs for the period.

Each firm has an expected profit function based on its history, on its strategy with respect to the other firm, and on knowledge gained from any research on possible market size. The possession of this function allows the firm to actually decide whether to join or refrain from the market for each period. The decisions (informed or directed randomness) and the resulting market size (randomness with a probability distribution) combine to determine the expected costs. These costs and the expected revenues, determined by actual market size and market price (a function of the number of

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[1]Ruth, M. and B. Hannon. 1997. *Modeling Dynamic Economic Systems*, Springer-Verlag, New York.
[2]This model is based on Kreps, D.M., 1990, *A Course in Microeconomic Theory*, Princeton University Press, pp. 356–360. We thank Darla Anderson for her help in its construction.

firms in the market in this period), are used to find the expected net reve-
nues or expected payoffs for each firm.

For our model we assume that two firms, J and B, must make a decision
whether they enter the market based on their expectations of making prof-
its. The market for their goods will be either large (total demand equaling
20,000 units) or small (total demand equaling 6,000 units):

$$\text{LARGE M DEMAND} = \text{IF RAND} < .4 \text{ THEN } 20000 \text{ ELSE } 0 \qquad (1)$$

$$\text{SMALL M DEMAND} = \text{IF RAND} \leq .4 \text{ THEN } 6000 \text{ ELSE } 0 \qquad (2)$$

The owner of B conducts market research to improve her chances of
knowing the market size, but the owner of J has takes no such action. The
critical decision for each of the two firms is whether to enter the market. B's
decision to enter depends on its expected annual profit and whether it ex-
pects the market will be large or small. B enters the market if its normally
distributed expected profit is positive:

$$\text{B ENTER} = \text{IF EXP PROFIT B} > 0 \text{ THEN } 1 \text{ ELSE } 0 \qquad (3)$$

J also enters the market if its normally distributed profit is positive, re-
gardless of any anticipated market size:

$$\text{J ENTER} = \text{IF EXP PROFIT J} > 0 \text{ THEN } 1 \text{ ELSE } 0 \qquad (4)$$

Since the expected profits are normally distributed, their standard deviation
must also be specified. The mean and standard deviation in these normally
distributed expected profit functions are derived in part from prior expe-
rience in this market and in part on research-derived information. B's market
research provides her with some information on the size of the market in the
next period. She knows that if the market is small, her actual profits are likely
to be small or negative. But if the market is large, her profits are likely to turn
out to be large. J knows that B is acting in this way and likely abstaining from
the expected small market. In this case, J is likely to have the small market all
to herself. J also knows that if the market is large, both firms will do well.
These possibilities tell J to favor entering the market regardless of size, but
her mean profit is lower than is that of B should the market be large. Here, J
actually knows what B is doing, which may not always be the case. In some
sense, J is piggybacking on the research done by B.

Once the market-entering decision has been made, we know how many
firms (0, 1, or 2) actually joined the market (Fig. 33.1). Knowing the
number of firms in the market, N, allows the price to be computed (either
10 or 12, depending on the number of active firms). The expected revenues
of each firm for this period can then be calculated if the quantity sold, Q
PER N, can be found. Q PER N is just the market demand (small or large)
divided by N, which in turns determines the expected quantities sold by
each firm (Q J and Q B).

The companies have different cost structures. J has a fixed cost of
$40,000, and a unit production cost of $5. B has a fixed cost of $60,000 and

LARGE M DEMAND

B ENTER

QB

B COST

N

Q PER N

RAND

J COST

Q J

J ENTER

SMALL M DEMAND
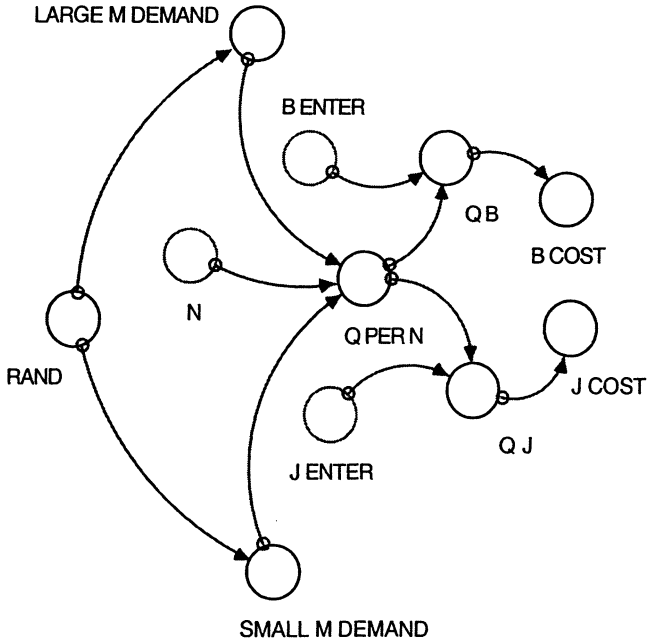
FIGURE 33.1

a unit cost of $3. Given this simple cost function, made up of both fixed and variable costs, the expected cost for J and B, J COST and B COST, can be determined based on the expected quantities to be produced and a fixed cost (see Fig. 33.1).

Information on expected profits is needed to make the market-entering decision. Now that one or both firms are in the market, they can make a better estimate of their expected payoffs (Fig. 33.2). They know what their revenues will be once the market size is known, given expected costs. This enables them to *calculate* their expected payoff for the period. Expected payoff is discounted over time by an interest rate to correct for the future period in which it is expected—the ensuing stream of discounted payoffs can be calculated and summed, resulting in a cumulative present valued profit, CPVP, for each firm (Fig. 33.3). Long-term values of CPVP determine the most successful firm. The strategies that ultimately lead to these long-term CPVP values are the issue here. Is there a superior strategy that can consistently win under these marketing rules?

We expect B to enter the market when the market is large and not to enter when it is small—independent of J's actions. J, given her rational expectation of B, will know this fact and nearly always choose to enter. B will not enter when the market is small, so J can count on usually making at least a small profit. Therefore, we would predict that J will nearly always enter, and B will tend to enter when the market is large.
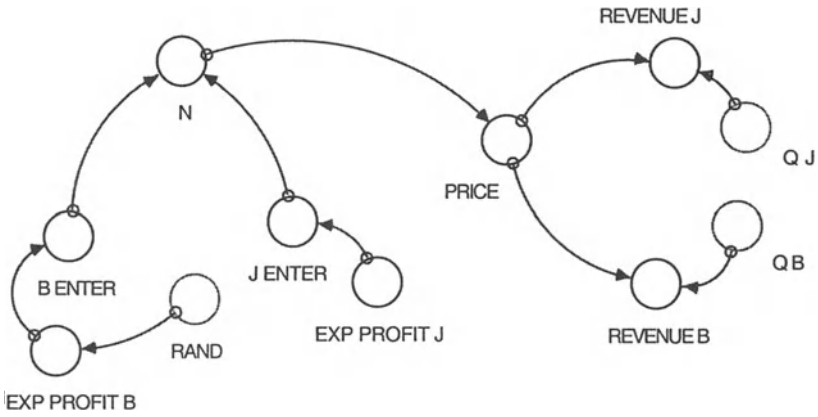
FIGURE 33.2

We can count the number of times each of these firms actually enters the market, realizing that it is possible that neither will enter in any given period (Fig. 33.4). We simply count each of the resulting possible outcomes based on the payoffs for each firm. There are four combinations for the pairing of the firms, and two market sizes, giving eight possible outcome conditions for the period. We show that indeed B enters the market less often than J and that each firm usually makes some profit (Figs. 33.5 and 33.6). But the "winning" strategy, the firm with the largest CPVP, is likely to be different in ensuing model runs.

One set of model results are shown in Figures 33.5 and 33.6. As we imagined, in repeated play, these firms, regardless of their expected payoffs
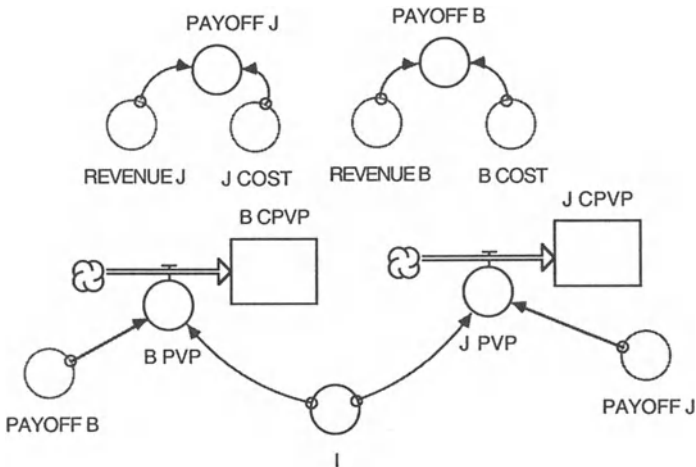
FIGURE 33.3
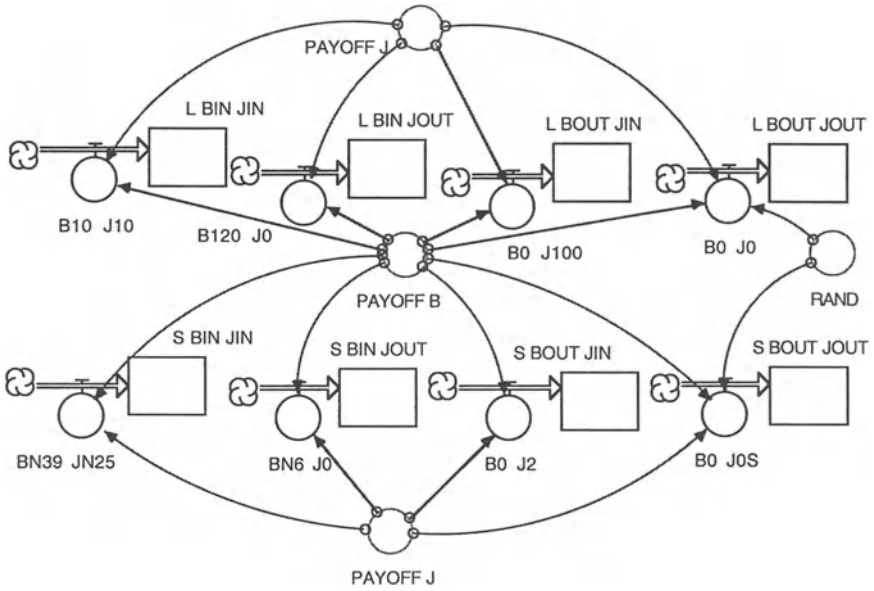
FIGURE 33.4

use all strategies. We find that some strictly dominated strategies, e.g., B entering when the market is small, were still played occasionally.

Run this model to show these conclusions. Over a large number of runs, does it pay for B to spend to fix an estimate of market size for the next period and allow this knowledge to influence its market entering possibilities?
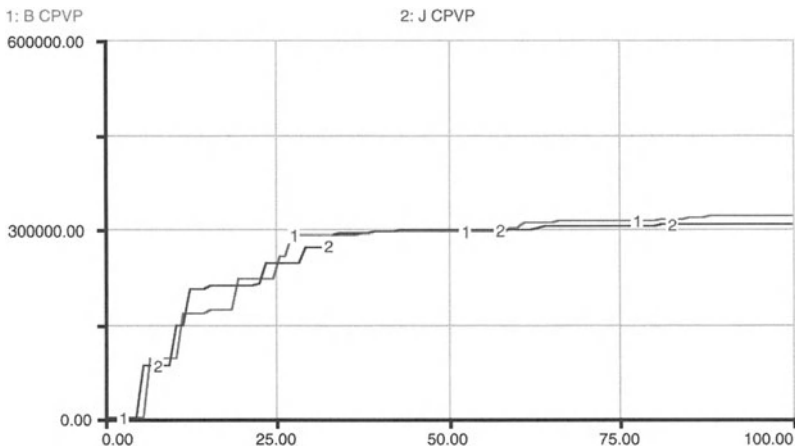
FIGURE 33.5
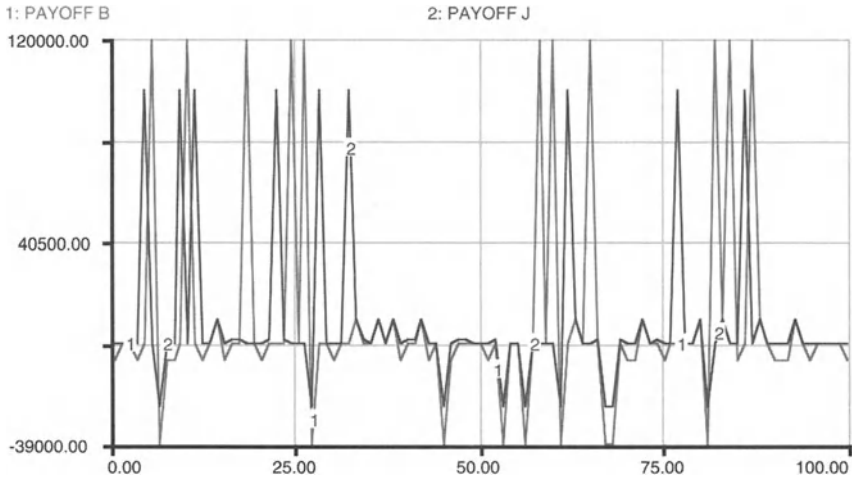
FIGURE 33.6

That is does B "win" in this larger CPVP sense more often than J? Can you refine the expected profit functions to change over time, based on what was earned in the previous period? Can you make the firms calculate their expected profits so that the losing strategies are played less often?

```
                        MARKET GAME

Expected Costs
B_COST = IF Q_B > 0 THEN 60000 + 3*Q_B ELSE 0 {TOTAL
COST FOR BELJEAU = FIXED COST PLUS VARIABLE COSTS}
J_COST = IF Q_J>0 THEN 40000+ 5*Q_J ELSE 0
LARGE_M_DEMAND = IF RAND < .4 THEN 20000 ELSE 0
{PROBABILITY OF A LARGE MARKET AND THE UNITS DEMANDED
IN THAT CASE}
Q_B = Q_PER_N*B_ENTER {IF B ENTERS, THIS NUMBER IS
STRICTLY POSITIVE}
Q_J = Q_PER_N*J_ENTER {IF J ENTERS, THIS NUMBER IS
POSITIVE}
Q_PER_N = IF N>0 THEN (LARGE_M_DEMAND +
SMALL_M_DEMAND)/N ELSE 0 {THIS DETERMINES WHETHER THE
MARKET IS SPLIT OR CAPTURED BY ONE FIRM}
RAND = RANDOM(0,1)
SMALL_M_DEMAND = IF RAND ≤ .4 THEN 6000 ELSE 0
{PROBABILITY OF A SMALL MARKET AND THE UNITS DEMANDED
IN THAT CASE}
```

```
Expected Profits
B_CPVP(t) = B_CPVP(t - dt) + (B_PVP) * dt
INIT B_CPVP = 0
INFLOWS:
B_PVP = PAYOFF_B/EXP(I*TIME)


J_CPVP(t) = J_CPVP(t - dt) + (J_PVP) * dt
INIT J_CPVP = 0
INFLOWS:
J_PVP = PAYOFF_J/EXP(I*TIME)


I = .05
PAYOFF_B = REVENUE_B-B_COST
PAYOFF_J = REVENUE_J-J_COST


Expected Revenues
B_ENTER = IF EXP_PROFIT_B>0 THEN 1 ELSE 0
EXP_PROFIT_B = IF RAND>=.4 THEN NORMAL(-11.25,350.25)
ELSE NORMAL (32.5,3425) {THE MEAN AND STD DEV OF
BELJEAU'S PROFIT GIVEN LARGE OR SMALL MARKET}
EXP_PROFIT_J = NORMAL(10.875,1397.554) {THE MEAN AND
STANDARD DEVIATION OF JOKYX'S POSSIBLE PROFITS IN
000'S}
J_ENTER = IF EXP_PROFIT_J>0 THEN 1 ELSE 0
N = B_ENTER+J_ENTER {Number of Firms in the Market}
PRICE = IF N=2 THEN 10 ELSE 12 {price per unit for a
monopolist or duopolist}
REVENUE_B = PRICE*Q_B
REVENUE_J = PRICE * Q_J


Strategy Count
L_BIN_JIN(t) = L_BIN_JIN(t - dt) + (B10_J10) * dt
INIT L_BIN_JIN = 0
INFLOWS:
B10_J10 = IF PAYOFF_B=10000 AND PAYOFF_J=10000 THEN 1
ELSE 0


L_BIN_JOUT(t) = L_BIN_JOUT(t - dt) + (B120_J0) * dt
INIT L_BIN_JOUT = 0
INFLOWS:
B120_J0 = IF PAYOFF_B=120000 AND PAYOFF_J=0 THEN 1
ELSE 0
```

```
L_BOUT_JIN(t) = L_BOUT_JIN(t - dt) + (B0_J100) * dt
INIT L_BOUT_JIN = 0
INFLOWS:
B0_J100 = IF PAYOFF_B=0 AND PAYOFF_J=100000 THEN 1
ELSE 0


L_BOUT_JOUT(t) = L_BOUT_JOUT(t - dt) + (B0_J0) * dt
INIT L_BOUT_JOUT = 0
INFLOWS:
B0_J0 = IF PAYOFF_B=0 AND PAYOFF_J=0 AND RAND<.4 THEN
1 ELSE 0


S_BIN_JIN(t) = S_BIN_JIN(t - dt) + (BN39_JN25) * dt
INIT S_BIN_JIN = 0
INFLOWS:
BN39_JN25 = IF PAYOFF_B=-39000 AND PAYOFF_J=-25000
THEN 1 ELSE 0


S_BIN_JOUT(t) = S_BIN_JOUT(t - dt) + (BN6_J0) * dt
INIT S_BIN_JOUT = 0
INFLOWS:
BN6_J0 = IF PAYOFF_B=-6000 AND PAYOFF_J=0 THEN 1 ELSE
0


S_BOUT_JIN(t) = S_BOUT_JIN(t - dt) + (B0_J2) * dt
INIT S_BOUT_JIN = 0
INFLOWS:
B0_J2 = IF PAYOFF_B=0 AND PAYOFF_J=2000 THEN 1 ELSE 0


S_BOUT_JOUT(t) = S_BOUT_JOUT(t - dt) + (B0_J0S) * dt
INIT S_BOUT_JOUT = 0
INFLOWS:
B0_J0S = IF PAYOFF_B=0 AND PAYOFF_J=0 AND RAND>=.4
THEN 1 ELSE 0
```

Part 7

Engineering Models

34

Pig Cycle[1]

> Life is understood backwards, but must be lived forwards.
> —Sören Kirkegaard, *Concept of Irony*

The model of this chapter has a long history, dating back to the 1920s. It captures mechanisms behind adjustments in prices and quantities of pigs in England. The identification and interpretation of mechanisms were the source of debate between the Reorganisation Commission for Pigs and Pig Products on the one hand, and R. H. Coase and R. F. Fowler on the other hand. Summarizing its observations of market fluctuations, a report by the Reorganisation Commission stated that

the chief factor influencing the volume of supplies of pigs at any time is the anticipation of profit or loss from pig raising by the mass of farmers. Although the decision to increase or decrease production does not affect the market supplies until many months later, owing to the time required for breeding and feeding, most farmers appear to assume that the existing relation of pig prices and feeding costs will continue. Thus, farmers take steps to increase or decrease production when food is cheap or dear relatively to pig prices. But the full results of such decisions are not felt until many months later, when there comes a change in the relation between feed costs and pig prices that has not been expected by the mass of farmers. The latter then begin to reverse their previous policy, and so the pig cycle is perpetuated.

  (Report of the Reorganisation Commission for Pigs and Pig Products, 1932)

The report describes in some length the biology and economics of breeding and selling pigs and comes to the conclusion that the underlying mechanisms are cause for the observed pig cycle that repeats itself roughly every 4 years. The cyclic behavior is assumed to be consistent with the cobweb theorem,[2] which posits that if producers assume that present prices and costs will continue unchanged and if the supply curve is more elastic than the demand curve, continuous fluctuations in prices and output will occur if there is a change in demand or supply conditions.

In a debate that extended into the 1940s, Coase and Fowler[3] argued that the necessary conditions for a cobweb cycle do not hold in the case of pig

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[1]This chapter is based on M. Cloutier and M. Ruth, 1997. The "Pig-cycle" out of the "Cobweb": A Revisit of R. H. Coase's Contribution to Economic Dynamics, paper presented at the History of Economics Society 1997 Conference in Charleston, South Carolina, June 1997.
[2]Cohen, R. and J.D. Barker. 1935. The Pig Cycle: A Reply, *Economica*, Vol. 2, pp. 408–421.
[3]Coase, R.H. and R.F. Fowler. 1935. Bacon Production and the Pig-cycle in Great Britain, *Economica*, Vol. 2, pp. 142–167; Coase, R.H. and R.F. Fowler. 1935. The Pig-

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farming, and that if one uses the Commission's assumptions, a 1- to 2-year pig cycle would result instead of the 4-year cycle actually observed. In an effort to shed light on the debate, we made use of the Commission's assumptions and data as much as possible, and modeled the dynamics of the pig cycle.[4] Where necessary, we used our own judgments to make up for missing information, and then explored the sensitivity of the model results to alternative specifications.

The model captures the bacon pig market based on the assumption that the whole market consists of breeder and slaughter pigs (Fig. 34.1). We assume an initial pig population of 10 breeders.
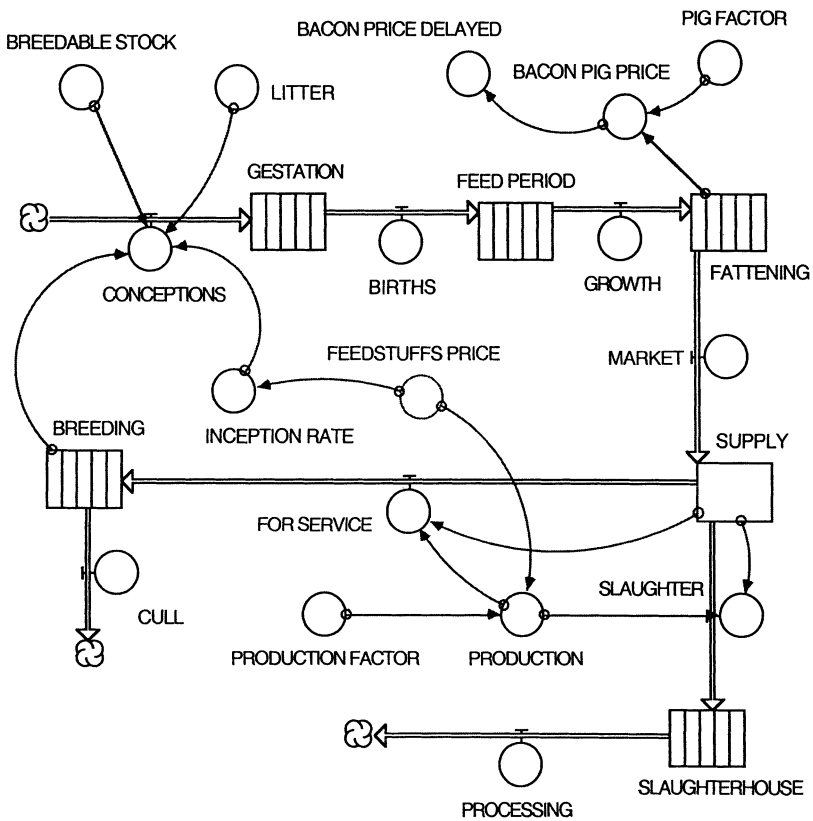
FIGURE 34.1

Cycle: A Rejoinder, *Economica*, Vol. 2, pp. 423–428; Coase, R.H. and R.F. Fowler. 1937. The Pig Cycle in Great Britain: An Explanation, *Economica*, Vol. 4, pp. 55–81; Coase, R.H. and R.F. Fowler. 1940. The Analysis of Producer's Expectations, *Economica*, Vol. 7, pp. 280–292.

The variable LITTER is the number of piglets produced by a pregnant female pig each month. With LITTER = 3.5 piglets per breeder per month and a gestation period of 4 months, the size of each litter will be 14 piglets.

Two-thirds of the breeders are assumed to be female. The report states that each female has on average 4 litters which equates to 16 months worth of breeding throughout the life span of a breeder, assuming a GESTATION of 4 months. With an average life span of 28 months for a breeder, this leaves 12 months for the females in the breeder population to be receptive. Twelve months divided by 28 months is the percentage of time of receptivity. This percentage times two-thirds (the female percentage population of breeders) gives the BREEDABLE STOCK figure of two-sevenths.

Subsequent to gestation, a FEED PERIOD and FATTENING period of, respectively, 3 and 5 months raise piglets to pigs that can either be sold for bacon or used as breeders. The PIG FACTOR used in our model is simply the total of feed and fattening period, i.e., 8 months. Separating the FEED PERIOD from the FATTENING period enables the model to be easily expanded to include variations in the age at which pigs can be slaughtered and diverted to the bacon market from the porker market, which in reality are somewhat separate.

The decision which route a pig will go—to breeding or slaughter—depends on the price of feedstuffs. As the price of feed increases, farmers do not want to produce more pigs, so they slow the rate of reproduction. This price responsiveness is modeled as

$$\text{INCEPTION RATE} = .5 - (\text{FEED PRICE}/10) \tag{1}$$

with

$$\text{FEEDSTUFFS PRICE} = 6 - (\text{BACON PIG PRICE}/\text{FEED FACTOR}) \tag{2}$$

and

$$\begin{aligned} \text{BACON PIG PRICE} = \ &\text{IF } 10 - (\text{FATTENING/PIGFACTOR}) < 2 \\ &\text{THEN 2 ELSE } 10 - (\text{FATTENING/PIG FACTOR}) \end{aligned} \tag{3}$$

based on a simple extrapolation of data plotted in Coase and Fowler.[5] The prices for feedstuffs and bacon pigs are measured in £ per unit of feedstuffs and £ per pig, respectively, and each of them depends on some factor that we have chosen to properly scale the relationships. Each of these factors, in turn, is discussed in more detail below.

Production cost, revenues, and profits are calculated in the module of Figure 34.2. Virtually no information was available to estimate feed prices and per capita feedstuffs consumption by pigs. As a result, the profit function is unlikely to have the correct values but the shape of the curve is correct.

Run the model and you will find that it begins cycling at around 100 months, the time that the conveyors run through a full cycle and the pig

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[5]Figure I, p. 151 in Coase, R.H. and R.F. Fowler. 1935. Bacon Production and the Pig-cycle in Great Britain, *Economica*, Vol. 2, pp. 142–167.
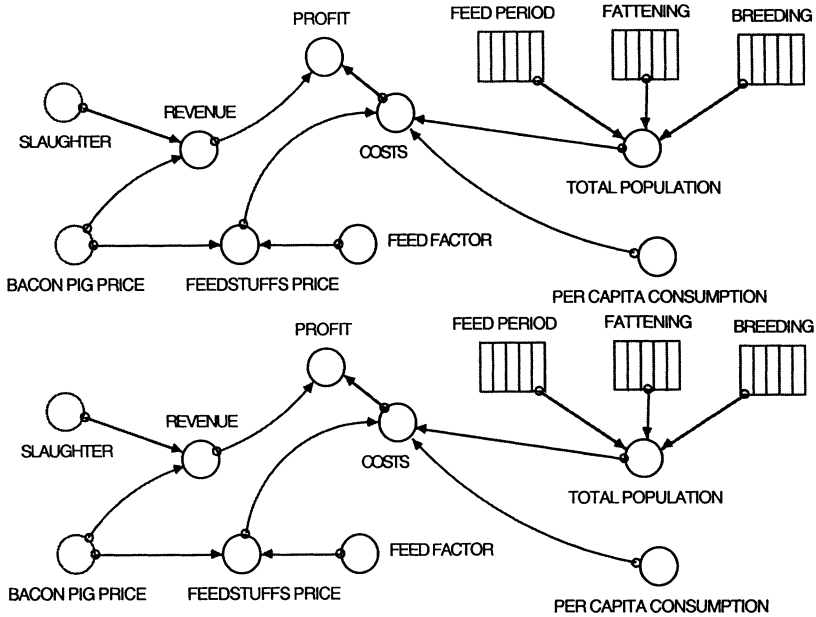
FIGURE 34.2

population has time to adjust to its fluctuations. The sizes of the stocks are shown in Figure 34.3. Corresponding relationships among fattening, gestation, and feedstuffs prices are plotted in Figures 34.4 and 34.5. As Figure 34.6 shows, there is an inverse relationship between the behavior of feedstuffs and bacon pig prices—when bacon pig prices are up, demand for
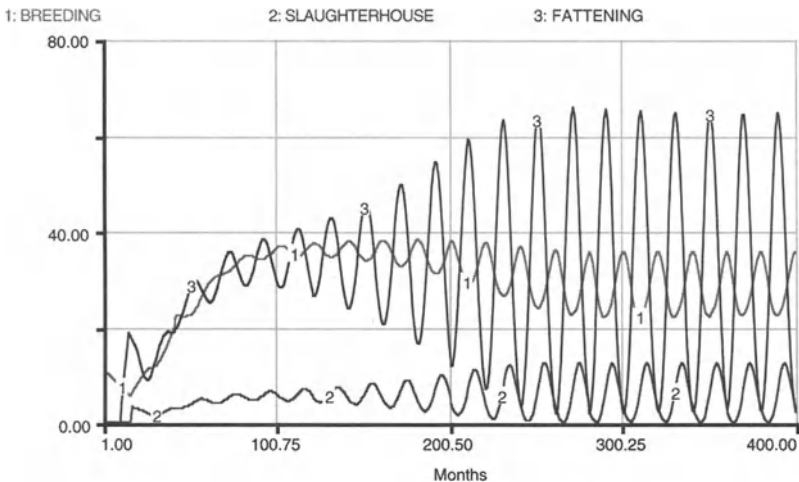
FIGURE 34.3
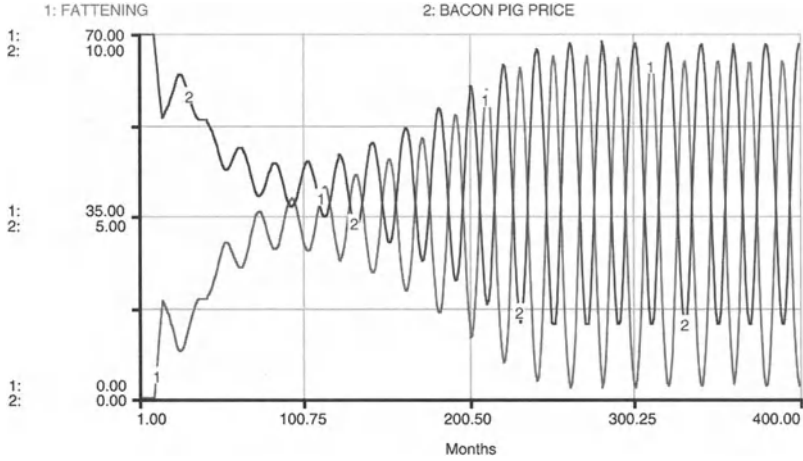
FIGURE 34.4

feedstuffs is low, and vice versa. The resulting implications for farmers' costs of production and profit are displayed in Figure 34.7.

Figures 34.3 to 34.7 clearly illustrate that once the pig cycle is in full swing, the length of each cycle is very nearly 2 years, as Coase and Fowler argued. However, our model is based on several assumptions that are not all directly derived from actual data. Along with assumptions about feedstuffs prices and consumption of feedstuffs by pigs, a number of other model specifications had to be either inferred from data plotted in graphs of the original reports and papers, or assumed. Sensitivity analyses can shed light on the implications of our assumptions.
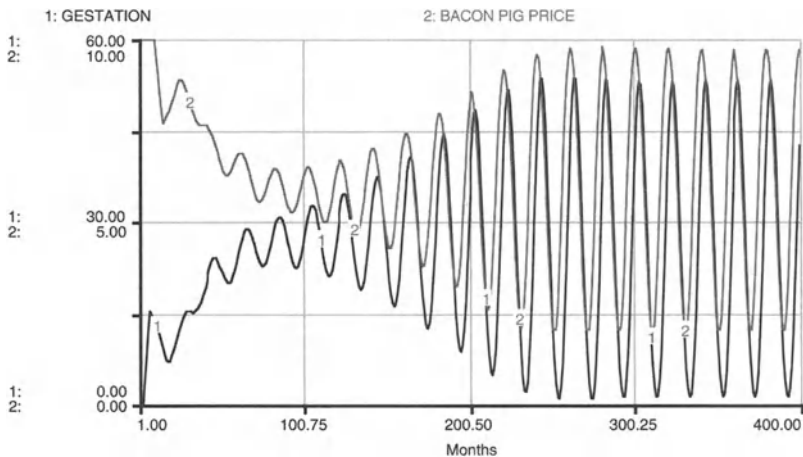
FIGURE 34.5
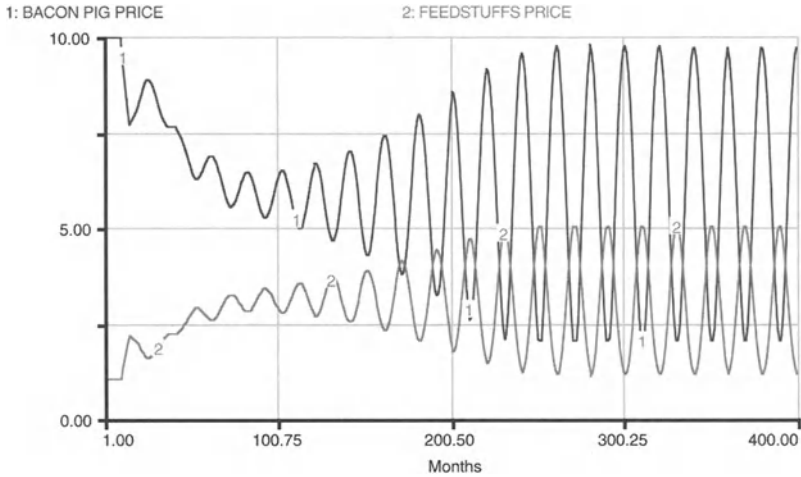
FIGURE 34.6

Explore the implications of alternative assumptions about litter sizes, and you should find that increasing litter size means equally frequent waves with greater amplitude. A greater total population is realized with a greater LITTER value. Sensitivity analysis for the breedable stock and for the inception rate show similar results to those of larger litter sizes. This is as expected since breedable stock size, litter size, and inception rate all determine the rate at which piglets are generated.

Perhaps the least supported assumptions of the model are with respect to the PRODUCTION FACTOR and PIG FACTOR. However, as Figure 34.8
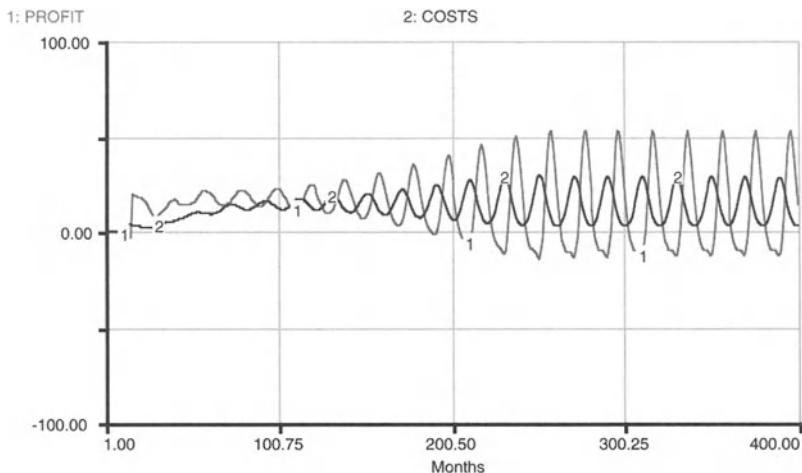
FIGURE 34.7
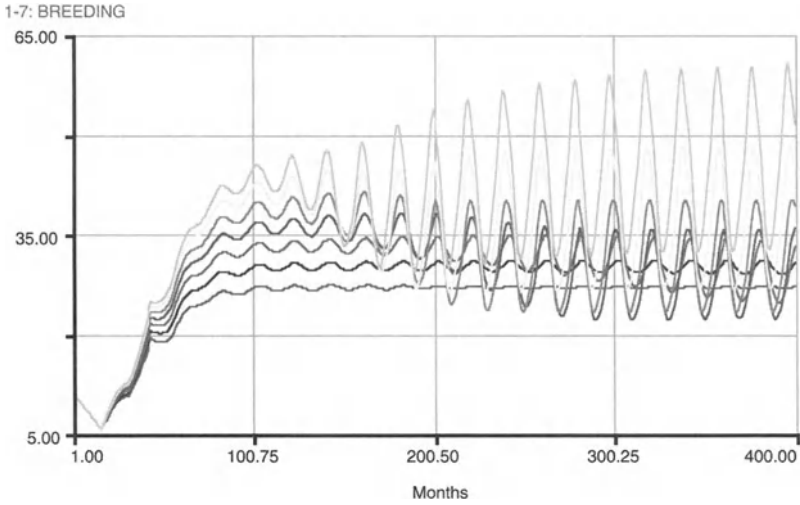
FIGURE 34.8

shows, varying the PRODUCTION FACTOR between 12 and 18 simply alters the amplitude of the pig cycle, not its periodicity. Similarly, Figure 34.9 shows for a PIG FACTOR ranging for subsequent runs from 6 to 12 that only the amplitude of the cycle is affected. So, even though these numbers are purely speculative, the results are fairly robust to our assumptions, and provide ample confidence to side with Coase and Fowler and help us (finally) put to rest their debate with the Reorganisation Commission for Pigs and Pig Products.
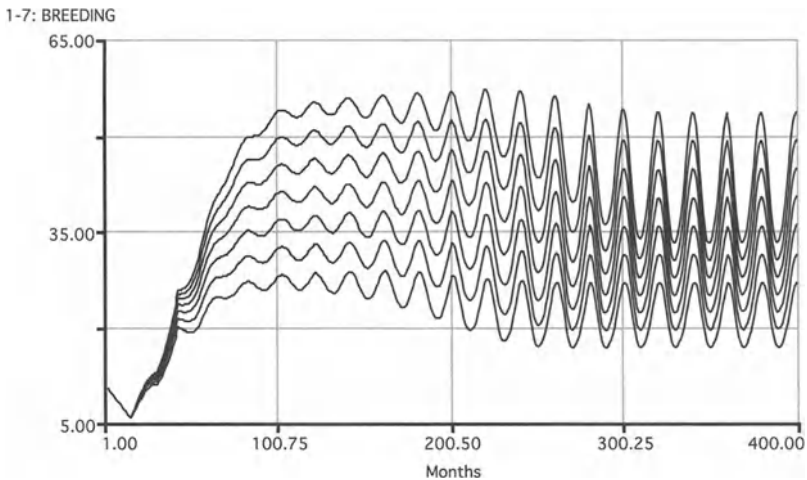
FIGURE 34.9

PIG CYCLE

```
BREEDING(t) = BREEDING(t - dt) + (FOR_SERVICE - CULL)
* dt
INIT BREEDING = 10
        TRANSIT TIME = 28
        INFLOW LIMIT = ∞
        CAPACITY = ∞
INFLOWS:
FOR_SERVICE = PRODUCTION*SUPPLY {breeders per month}
OUTFLOWS:
CULL = CONVEYOR OUTFLOW


FATTENING(t) = FATTENING(t - dt) + (GROWTH - MARKET) *
dt
INIT FATTENING = 0
        TRANSIT TIME = 5
        INFLOW LIMIT = ∞
        CAPACITY = ∞
INFLOWS:
GROWTH = CONVEYOR OUTFLOW
OUTFLOWS:
MARKET = CONVEYOR OUTFLOW


FEED_PERIOD(t) = FEED_PERIOD(t - dt) + (BIRTHS -
GROWTH) * dt
INIT FEED_PERIOD = 0
        TRANSIT TIME = 3
        INFLOW LIMIT = ∞
        CAPACITY = ∞
INFLOWS:
BIRTHS = CONVEYOR OUTFLOW
OUTFLOWS:
GROWTH = CONVEYOR OUTFLOW


GESTATION(t) = GESTATION(t - dt) + (CONCEPTIONS -
BIRTHS) * dt
INIT GESTATION = 0
        TRANSIT TIME = 4
        INFLOW LIMIT = ∞
        CAPACITY = ∞
INFLOWS:
CONCEPTIONS =
BREEDING*INCEPTION_RATE*LITTER*BREEDABLE_STOCK
{inceptions per month}
```

```
OUTFLOWS:
BIRTHS = CONVEYOR OUTFLOW

SLAUGHTERHOUSE(t) = SLAUGHTERHOUSE(t - dt) +
(SLAUGHTER - PROCESSING) * dt
INIT SLAUGHTERHOUSE = 0
     TRANSIT TIME = 1
     INFLOW LIMIT = ∞
     CAPACITY = ∞
INFLOWS:
SLAUGHTER = (1-PRODUCTION)*SUPPLY {pigs to be
butchered per month}
OUTFLOWS:
PROCESSING = CONVEYOR OUTFLOW

SUPPLY(t) = SUPPLY(t - dt) + (MARKET - FOR_SERVICE -
SLAUGHTER) * dt
INIT SUPPLY = 0 {pigs}
INFLOWS:
MARKET = CONVEYOR OUTFLOW
OUTFLOWS:
FOR_SERVICE = PRODUCTION*SUPPLY {breeders per month}
SLAUGHTER = (1-PRODUCTION)*SUPPLY {pigs to be
butchered per month}

BACON_PIG_PRICE = IF 10-(FATTENING/PIG_FACTOR) < 2
THEN 2 ELSE 10-(FATTENING/PIG_FACTOR) {pounds per pig}
BACON_PRICE_DELAYED = DELAY(BACON_PIG_PRICE,12)
BREEDABLE_STOCK = 2/7 {pigs per breeding per time
period that are able to become pregnant}
COSTS = TOTAL_POPULATION*FEEDSTUFFS_PRICE*
PER_CAPITA_CONSUMPTION {pounds}
FEEDSTUFFS_PRICE = 6-(BACON_PIG_PRICE/FEED_FACTOR)
{pounds per unit of feed}
FEED_FACTOR = 2
INCEPTION_RATE = .5-(FEEDSTUFFS_PRICE/10) {inceptions
per breeders}
LITTER = 3.5 {pigs per month per breeders}
PER_CAPITA_CONSUMPTION = .05 {feed units per pig per
month}
PIG_FACTOR = 8
PRODUCTION = .4-(FEEDSTUFFS_PRICE/PRODUCTION_FACTOR)
{breeders per pigs}
```

```
PRODUCTION_FACTOR = 15
PROFIT = REVENUE - COSTS {pounds}
REVENUE = SLAUGHTER*BACON_PIG_PRICE {pounds}
TOTAL_POPULATION = BREEDING+FATTENING+FEED_PERIOD
{pigs}
```

35

The Assembly Line

The assembly-belt is like an army on the move.
> —W.H. Auden and Christopher Isherwood, *On the Frontier*

35.1 Basic Model

The production line process is the secret to low-cost products. It is the ultimate, in the almost microscopic division of labor. It is also monotonous and eventually debilitating for workers unless variety in the work week assignments can be found.

In the first section of this chapter we model the allocation of workers among different tasks. In the following section we develop a model of car manufacturing that adds detail and variety to the basic insights generated here. Each of the models makes use of STELLA's ovens, queues, and conveyors. We have already encountered conveyors in Chapter 17. Conveyors are able to take up some value or values of state variables, keep these values for a fixed length of time and then release the values describing the state variable. Conveyors can be compared to conveyor belts on which some object moves before it gets off again. Whatever goes into the conveyor first also leaves the conveyor first. The same holds for queues. These are lines of objects that await entry into some process. Objects first in line, i.e., arriving first in the queue, will enter the process first after waiting for a specified length of time. Lines of people waiting at a grocery store checkout are an example. Ovens are processors of discrete batches of objects, very much like ovens used in a bakery. Ovens receive a number of objects over some period of time, retain these objects for some time, and then unload them all in an instant.

Conveyors, queues, and ovens are forms of stocks that were designed for use in modeling discrete processes. One such use of conveyors and ovens is shown in the following example. We do not make use of queues since we are here not interested in keeping track of the order of some of the objects moving inside the system that is being modeled.

Imagine a manufacturing process that has three assembly points with holding stages between each process and a transport time between the first and the second. The raw product must be delivered from the warehouse to the first holding stage by the person performing the third process. A second person performs the first and second process. People cannot be in two places at once, of course, and the program must recognize this fact.

The three stages of production take 4, 5, and 14 minutes respectively. The transit time between processes 2 and 3 is 2 minutes.

The program developed in this chapter determines how many items should be performed at each station before the workers switch to their other task. Such modeling is extremely valuable to designers of assembly lines and for "just in time" production. Incidentally, "just in time" production comes from a desire to minimize the capital of the inventory (the parts waiting for assembly). Since an improperly made part arriving just in time would stop the whole production process, quality control at the sub-assembly plant is an absolute necessity. So, production quality seems to be a product of cost-minimizing efforts rather than an ephemeral goal or an inherent characteristic of a producer.

To model the assembly line as described above, first lay out its general structure as shown in Figure 35.1 with only the stocks and flows. Next, specify STATION 1, STATION 2, and STATION 3 as ovens. Do so by double-clicking on the reservoir symbols and checking the Oven button. Start with STATION 1. Set its cook time to 4 and the capacity of the oven, its fill time, and initial value each to 1. This station can now take on one unit each time period it is empty and a non-zero inflow occurs. It will then process that entry for 4 minutes. Note that when you click OK, the question mark in the stock for STATION 1 should disappear, and with it the question mark in the outflow should disappear as well. The outflow from STATION 1 (and similarly of the other ovens and the conveyor for TRANSFER 1 2) is automatically determined by the logic internal to the definition of the queue and conveyor. Proceed to specify the other stocks and flows of the model as discussed above and shown in Figure 35.2.
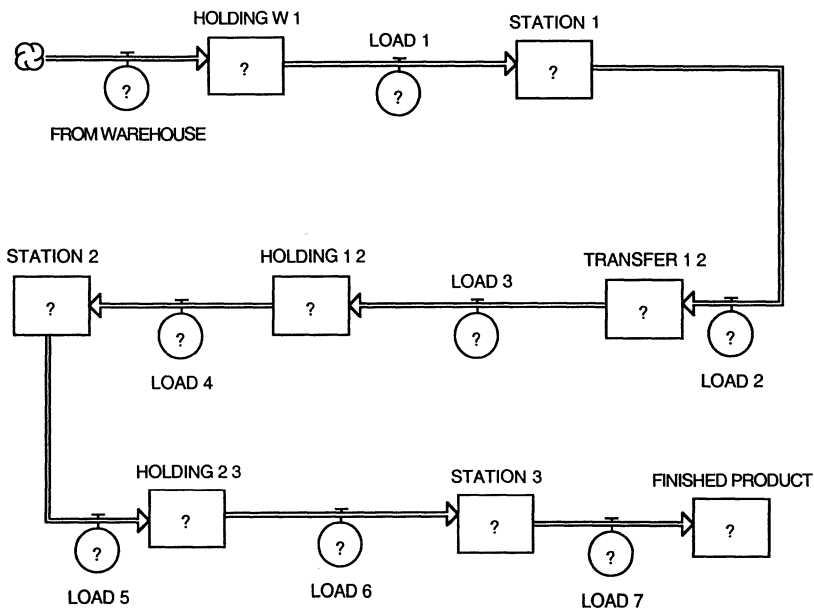
FIGURE 35.1
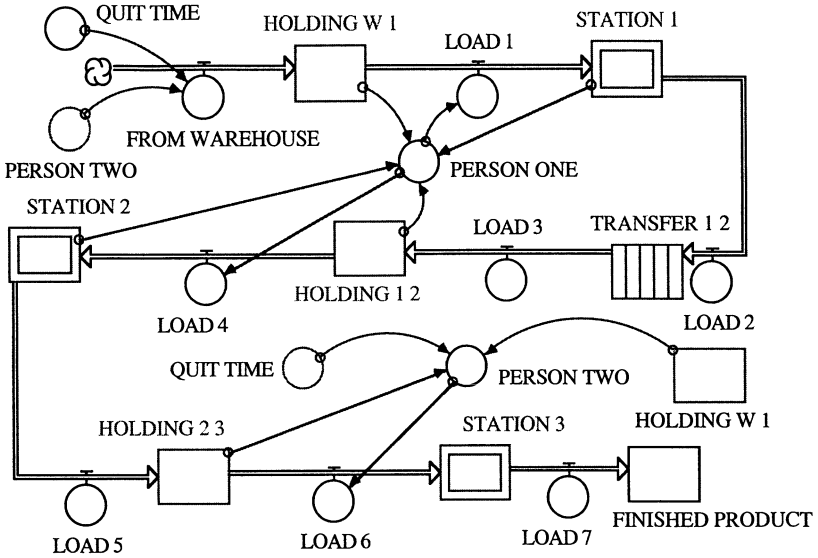
FIGURE 35.2

The results of our model, presented in Figure 35.3, show a stepwise increase in finished products, and the changes in delivery and holding stages. The model assumes that the workers get no break and they can't screw up in their assembly plant. They are robotic. Add humanity to this model. Let them take preplanned breaks and let them have a finite error rate. You may also need an inspector who supervises the production process. If you wish to keep track of the order in which goods are produced, replace the HOLDING . . . stocks by Queues.
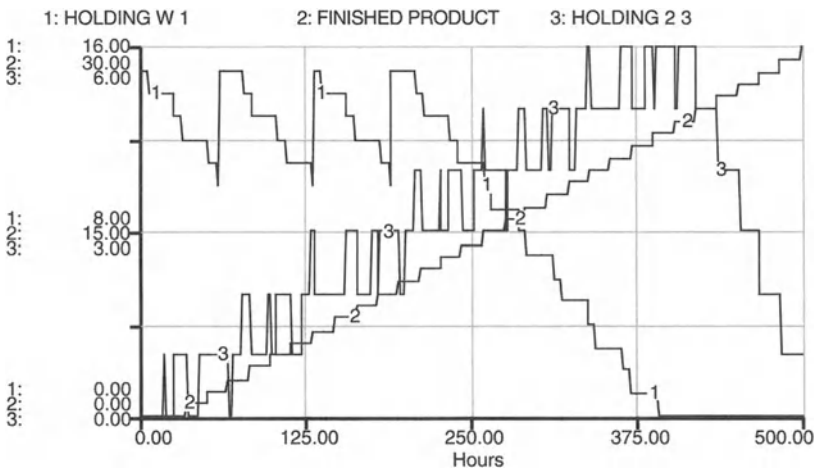
FIGURE 35.3

ASSEMBLY LINE

```
FINISHED_PRODUCT(t) = FINISHED_PRODUCT(t - dt) +
(LOAD_7) * dt
INIT FINISHED_PRODUCT = 0
INFLOWS:
LOAD_7 = CONTENTS OF OVEN AFTER COOK TIME, ZERO
OTHERWISE

HOLDING_1_2(t) = HOLDING_1_2(t - dt) + (LOAD_3 -
LOAD_4) * dt
INIT HOLDING_1_2 = 0
INFLOWS:
LOAD_3 = CONVEYOR OUTFLOW
OUTFLOWS:
LOAD_4 = IF PERSON_ONE = 2 THEN 1 ELSE 0

HOLDING_2_3(t) = HOLDING_2_3(t - dt) + (LOAD_5 -
LOAD_6) * dt
INIT HOLDING_2_3 = 0
INFLOWS:
LOAD_5 = CONTENTS OF OVEN AFTER COOK TIME, ZERO
OTHERWISE
OUTFLOWS:
LOAD_6 = IF (PERSON_TWO = 1) THEN 1 ELSE 0

HOLDING_W_1(t) = HOLDING_W_1(t - dt) + (FROM_WAREHOUSE
- LOAD_1) * dt
INIT HOLDING_W_1 = 5
INFLOWS:
FROM_WAREHOUSE = IF (PERSON_TWO =2) AND (TIME <
QUIT_TIME)  THEN 5 ELSE 0
OUTFLOWS:
LOAD_1 = IF (PERSON_ONE = 1) THEN 1 ELSE 0

STATION_1(t) = STATION_1(t - dt) + (LOAD_1 - LOAD_2) *
dt
INIT STATION_1 = 1
      COOK TIME = 4
      CAPACITY = 1
      FILL TIME = 1
INFLOWS:
LOAD_1 = IF (PERSON_ONE = 1) THEN 1 ELSE 0
```

```
OUTFLOWS:
LOAD_2 = CONTENTS OF OVEN AFTER COOK TIME, ZERO
OTHERWISE

STATION_2(t) = STATION_2(t - dt) + (LOAD_4 - LOAD_5) *
dt
INIT STATION_2 = 0
       COOK TIME = 5
       CAPACITY = 1
       FILL TIME = 1
INFLOWS:
LOAD_4 = IF PERSON_ONE = 2 THEN 1 ELSE 0
OUTFLOWS:
LOAD_5 = CONTENTS OF OVEN AFTER COOK TIME, ZERO
OTHERWISE

STATION_3(t) = STATION_3(t - dt) + (LOAD_6 - LOAD_7) *
dt
INIT STATION_3 = 0
       COOK TIME = 14
       CAPACITY = 1
       FILL TIME = 1
INFLOWS:
LOAD_6 = IF (PERSON_TWO = 1) THEN 1 ELSE 0
OUTFLOWS:
LOAD_7 = CONTENTS OF OVEN AFTER COOK TIME, ZERO
OTHERWISE

TRANSFER_1_2(t) = TRANSFER_1_2(t - dt) + (LOAD_2 -
LOAD_3) * dt
INIT TRANSFER_1_2 = 0
       TRANSIT TIME = 5
       INFLOW LIMIT = ∞
       CAPACITY = ∞
INFLOWS:
LOAD_2 = CONTENTS OF OVEN AFTER COOK TIME, ZERO
OTHERWISE
OUTFLOWS:
LOAD_3 = CONVEYOR OUTFLOW

PERSON_ONE = IF (STATION_1 = 0) AND (STATION_2 = 0)
AND (HOLDING_1_2 = 0) AND (HOLDING_W_1 ≠ 0) THEN 1 ELSE
```

```
IF (STATION_2 = 0) AND (STATION_1 = 0) AND
(HOLDING_1_2 ≠ 0) THEN 2 ELSE 0
PERSON_TWO = IF ((HOLDING_2_3 ≠ 0) AND (HOLDING_W_1 >
10)) OR ((HOLDING_2_3 ≠ 0) AND (TIME > QUIT_TIME)) THEN
1 ELSE IF HOLDING_W_1 ≤ 10 THEN 2 ELSE 0
QUIT_TIME = 190
```

35.2 Car Assembly Line[1]

Here we model an auto assembly line where two labor teams must accomplish the following five tasks: engine manufacuring (10 minutes), chassis manufacturing (20 minutes), engine parts shipping (2 minutes), chassis parts shipping (2 minutes), and finally, auto assembly (15 minutes), where time is the time per product unit. The labor teams are given three breaks per shift: two 30-minute morning and afternoon breaks and one 60-minute meal break. Each labor team has two tasks. The distribution of these tasks between the teams will affect the production rate. For any run of the model, workers have a particular set of production priorities, switching to fill the three parts inventories when inventory levels become critically low. Worker scheduling must be consistent, not requiring them to do two things at once.

Naturally, the first question is: How many cars can this process produce under differing labor team assignments? And second: Can the assembly line be made more efficient if the distribution of tasks between the two labor teams is changed?

The use of the STELLA ovens is prominent in this model. Ovens can be set to take in any number of units at once, with an adjustable maximum capacity, and then process (hold) the item(s) for a specified time, whereupon the item(s) are released to an inventory stock for the next stage.

The labor team controls are set such that they work only when Working Hours equals 1, and this variable is set to 0 over the lunch hour and for two breaks. If a task is in progress when lunch or a break time is reached, the workers complete the current task before stopping their work.

The data for this model are not actual data but made up for the purposes of illustrating the use of parallel and series "production chain" models in industry. Here the engine and chassis are produced in parallel, followed by their assembly into a single unit, whose assembly rate depends on the availability of engines, chassis, and labor (Fig. 35.4).

Figure 35.5 illustrates the output of the model's first run for the course of one work day. Here, the model is set up so that labor team 1 is taking care of engine and chassis manufacture while labor team 2 completes the other tasks. More specifically, team 1 would build engines as long as the parts are

---

[1]We thank Mark Hersam for his work on this model.
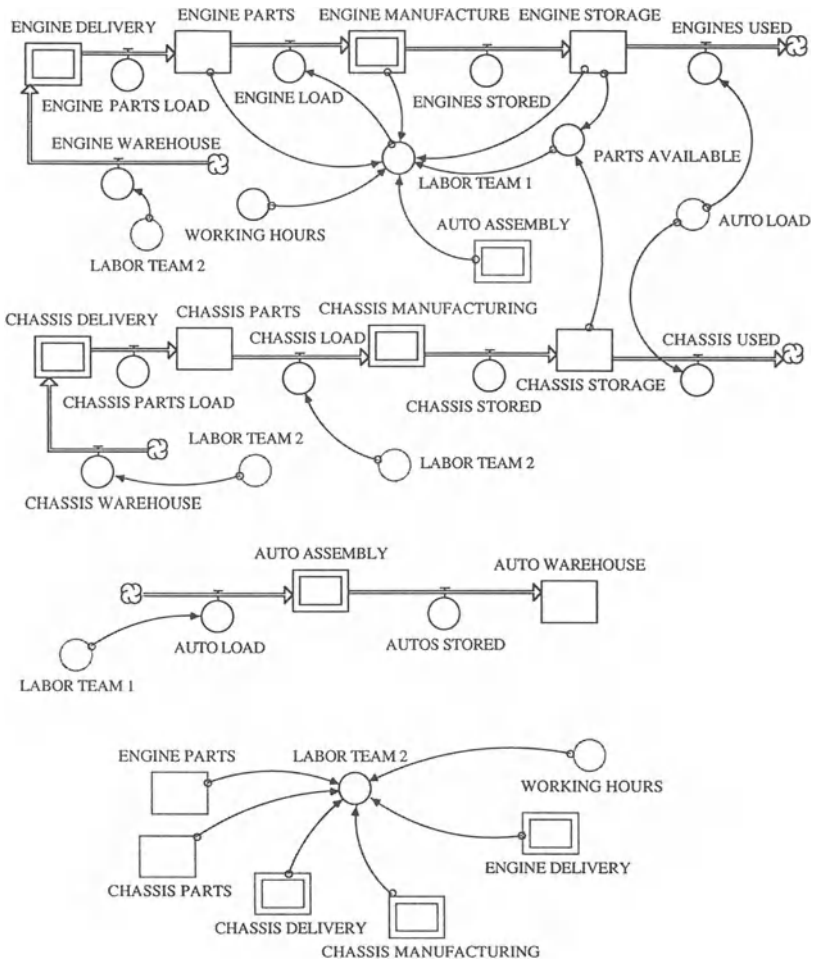
FIGURE 35.4

available and no engines are in storage; otherwise, team 1 would manufacture chassis, assuming that chassis parts are available. On the other hand, team 2 would fill the engine parts or chassis parts stocks if either is empty; if the engine and chassis parts stocks are not empty, team 2 would assemble cars if both the chassis and engine were ready. At the end of one day, the total number of cars produced equals 13 automobiles.

Would a change in the the distribution of labor allow for more cars to be built in one day? Figure 35.6 shows the results if labor team 1 is in charge of engine manufacturing and auto assembly, while labor team 2 takes care of the rest of the work. Now, 15 cars are built in the course of one day. Altering the distribution of labor, automobile production was increased by 15.4 percent. Through careful positioning of labor, assembly lines can be made more efficient at no additional cost.
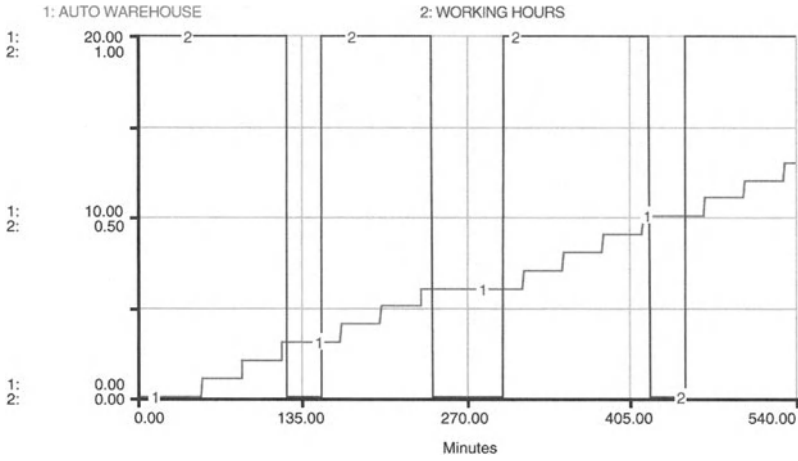
FIGURE 35.5

Examine the chassis and engine stocks in this model. The chassis stocks seem to grow indefinitely, while the engine stocks oscillate from 0 to 1! Check this statement with a graph of these variables. This is not good inventory control. Apparently there is an engine shortage in the current model. Can you improve production by further changing the allocation of labor? Don't use any more labor-minutes than you do now . . . just try to rallocate them. If engine production should be increased, in which stage is it most efficient to do so? Team 2 does the delivery of engines when it is not delivering parts and building chassis. Should we consider a thrid labor team? What data are mssing here for a complete optimization? Could the breaks be scheduled at different times to improve production efficiency?
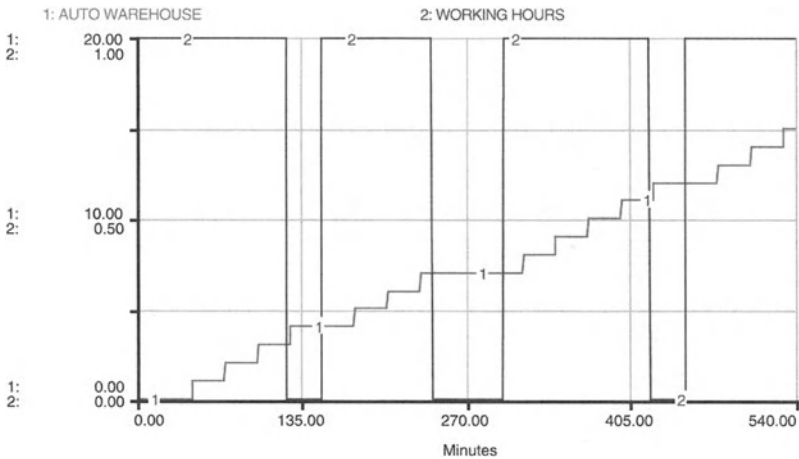
FIGURE 35.6

CAR ASSEMBLY

```
AUTO_ASSEMBLY(t) = AUTO_ASSEMBLY(t - dt) + (AUTO_LOAD
- AUTOS_STORED) * dt
INIT AUTO_ASSEMBLY = 0
      COOK TIME = 15
      CAPACITY = 1
      FILL TIME = 1
INFLOWS:
AUTO_LOAD = IF LABOR_TEAM_1 = 2 THEN 1 ELSE 0
OUTFLOWS:
AUTOS_STORED = CONTENTS OF OVEN AFTER COOK TIME, ZERO
OTHERWISE

AUTO_WAREHOUSE(t) = AUTO_WAREHOUSE(t - dt) +
(AUTOS_STORED) * dt
INIT AUTO_WAREHOUSE = 0
INFLOWS:
AUTOS_STORED = CONTENTS OF OVEN AFTER COOK TIME, ZERO
OTHERWISE

CHASSIS_DELIVERY(t) = CHASSIS_DELIVERY(t - dt) +
(CHASSIS_WAREHOUSE - CHASSIS_PARTS_LOAD) * dt
INIT CHASSIS_DELIVERY = 0
      COOK TIME = 2
      CAPACITY = 10
      FILL TIME = 1
INFLOWS:
CHASSIS_WAREHOUSE = IF LABOR_TEAM_2 = 3 THEN 10 ELSE 0
OUTFLOWS:
CHASSIS_PARTS_LOAD = CONTENTS OF OVEN AFTER COOK TIME,
ZERO OTHERWISE

CHASSIS_MANUFACTURING(t) = CHASSIS_MANUFACTURING(t -
dt) + (CHASSIS_LOAD - CHASSIS_STORED) * dt
INIT CHASSIS_MANUFACTURING = 0
      COOK TIME = 20
      CAPACITY = 1
      FILL TIME = 1
INFLOWS:
CHASSIS_LOAD = IF LABOR_TEAM_2 = 1 THEN 1 ELSE 0
OUTFLOWS:
CHASSIS_STORED = CONTENTS OF OVEN AFTER COOK TIME,
ZERO OTHERWISE
```

```
CHASSIS_PARTS(t) = CHASSIS_PARTS(t - dt) +
(CHASSIS_PARTS_LOAD - CHASSIS_LOAD) * dt
INIT CHASSIS_PARTS = 0
INFLOWS:
CHASSIS_PARTS_LOAD = CONTENTS OF OVEN AFTER COOK TIME,
ZERO OTHERWISE
OUTFLOWS:
CHASSIS_LOAD = IF LABOR_TEAM_2 = 1 THEN 1 ELSE 0

CHASSIS_STORAGE(t) = CHASSIS_STORAGE(t - dt) +
(CHASSIS_STORED - CHASSIS_USED) * dt
INIT CHASSIS_STORAGE = 0
INFLOWS:
CHASSIS_STORED = CONTENTS OF OVEN AFTER COOK TIME,
ZERO OTHERWISE
OUTFLOWS:
CHASSIS_USED = IF AUTO_LOAD=1 THEN 1 ELSE 0

ENGINE_DELIVERY(t) = ENGINE_DELIVERY(t - dt) +
(ENGINE_WAREHOUSE - ENGINE_PARTS_LOAD) * dt
INIT ENGINE_DELIVERY = 0
        COOK TIME = 2
        CAPACITY = 10
        FILL TIME = 1
INFLOWS:
ENGINE_WAREHOUSE = IF LABOR_TEAM_2 = 2 THEN 10 ELSE 0
OUTFLOWS:
ENGINE_PARTS_LOAD = CONTENTS OF OVEN AFTER COOK TIME,
ZERO OTHERWISE

ENGINE_MANUFACTURE(t) = ENGINE_MANUFACTURE(t - dt) +
(ENGINE_LOAD - ENGINES_STORED) * dt
INIT ENGINE_MANUFACTURE = 0
        COOK TIME = 10
        CAPACITY = 1
        FILL TIME = 1
INFLOWS:
ENGINE_LOAD = IF LABOR_TEAM_1 = 1 THEN 1 ELSE 0
OUTFLOWS:
ENGINES_STORED = CONTENTS OF OVEN AFTER COOK TIME,
ZERO OTHERWISE
```

```
ENGINE_PARTS(t) = ENGINE_PARTS(t - dt) +
(ENGINE_PARTS_LOAD - ENGINE_LOAD) * dt
INIT ENGINE_PARTS = 0
INFLOWS:
ENGINE_PPARTS_LOAD = CONTENTS OF OVEN AFTER COOK TIME,
ZERO OTHERWISE
OUTFLOWS:
ENGINE_LOAD = IF LABOR_TEAM_1 = 1 THEN 1 ELSE 0

ENGINE_STORAGE(t) = ENGINE_STORAGE(t - dt) +
(ENGINES_STORED - ENGINES_USED) * dt
INIT ENGINE_STORAGE = 0
INFLOWS:
ENGINES_STORED = CONTENTS OF OVEN AFTER COOK TIME,
ZERO OTHERWISE
OUTFLOWS:
ENGINES_USED = IF AUTO_LOAD=1 THEN 1 ELSE 0

LABOR_TEAM_1 = IF ((ENGINE_MANUFACTURE = 0) AND
(AUTO_ASSEMBLY = 0) AND (ENGINE_PARTS > 0) AND
(ENGINE_STORAGE = 0) AND (WORKING_HOURS = 1)) THEN 1
ELSE IF ((AUTO_ASSEMBLY = 0) AND (ENGINE_MANUFACTURE =
0) AND (WORKING_HOURS = 1) AND (PARTS_AVAILABLE = 1))
THEN 2 ELSE 0
LABOR_TEAM_2 = IF ((CHASSIS_PARTS = 0) AND
(WORKING_HOURS = 1) AND (CHASSIS_MANUFACTURING = 0) AND
(CHASSIS_DELIVERY = 0) AND (ENGINE_DELIVERY = 0)) THEN
3 ELSE IF ((ENGINE_PARTS = 0) AND (WORKING_HOURS = 1)
AND (CHASSIS_MANUFACTURING = 0) AND (CHASSIS_DELIVERY =
0) AND (ENGINE_DELIVERY = 0)) THEN 2 ELSE IF
((CHASSIS_PARTS>0) AND (CHASSIS_MANUFACTURING = 0) AND
(CHASSIS_DELIVERY = 0) AND (ENGINE_DELIVERY = 0) AND
(WORKING_HOURS = 1)) THEN 1 ELSE 0
PARTS_AVAILABLE = If(CHASSIS_STORAGE > 0 AND
ENGINE_STORAGE > 0)THEN(1)ELSE(0)
WORKING_HOURS = IF ((TIME > 120 and TIME < 150) OR
(TIME > 240 and TIME < 300) OR (TIME > 420 and TIME <
450)) THEN 0 ELSE 1
```

36
Models of Gravity and Acceleration

The shrill bell rings, the censer swings.
—Alfred, Lord Tennyson, *Sir Galahad*

36.1 Falling Rock

Imagine that we are standing on top of a tall building. We release a lead weight off the side and watch it fall. It seems to fall faster and faster before it hits the ground. We try pieces of the same shape and different weights and they hit the ground at the same time when released at the same time. This is due to the acceleration of gravity in the absence of air resistance.

To build a model that describes the change in position of a falling object, we must enter the world of the engineer and physicist. They have a more advanced understanding of their (simpler) world than biologists and economists do of theirs. By resorting to the concept of force and the idea of conservation of force (Newton) and the brilliant idea of inertial force (Huygens, Newton), they can describe a system without talking about its history at all! This is very different from our previous models, in which the system's history is a very important component.

The flight of a projectile can be very accurately predicted knowing only its initial direction and speed, regardless of where it has been. The history of the projectile is not important in terms of predicting its future states. Thus, the idea of force equilibrium with the addition of the inertial force frees the engineer and physicist from concern for the locational and speed history of the stone. This discovery is one of the unique principles that these two disciplines bring to our understanding of the physical world. The forces acting upon the projectile are illustrated in Figure 36.1.

The inertial force arises from the rate of change of the "quantity of motion" as Newton called it. Momentum is the modern name for "quantity of motion." Momentum is defined as the mass of the moving object times its velocity, V. The mass, M, of an object depends on its weight and the gravity field in which it is embedded. If the object is a rocket, then it is losing mass as it travels (fuel burnup) and Newton's inertial force is complicated. But for our experiment of a rock falling from a tall building, the mass is constant and since velocity is dY/dt, the inertial force is

$$M*dV/dt = M*d^2Y/d^2t \qquad (1)$$

(mass times the second derivative of distance, Y, with respect to time). The weight force of the dropped object acts downward and the inertial force acts opposite to the direction of motion (up) and so the balance of force is

INERTIAL
FORCE

DRAG
FORCE

FALLING
WEIGHT
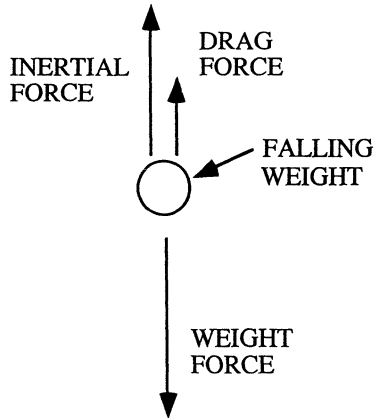
WEIGHT
FORCE

FIGURE 36.1

$$M*d^2Y/d^2t = M*g = weight \qquad (2)$$

Here, g is the acceleration of gravity. It is the one-way attraction by which the Earth pulls on the weight. The weight at rest has an inertial force

$$M*d^2Y/d^2t = M*g = weight \qquad (3)$$

so the alternate expression for any weight (e.g., yours) is its mass times g.

To solve this second order differential equation in STELLA, we use the substitution

$$q = dY/dt \qquad (4)$$

Then

$$dq/dt = q \, dot = d^2Y/d^2t \qquad (5)$$

The diagram (Fig. 36.2) and model equations show how to arrange such a description. It can be done for any order differential equation.

Check out the falling rock model. Set A = 0. Run the model and note how the velocity continues to grow (Fig. 36.3). Run the model again with different initial velocities. The velocity keeps growing (linearly) because there is no air resistance.

Abstracting away from air resistance may be a convenient procedure to develop and test our model. However, we know from many real-world experiments that air resistance may be significant. Therefore, we include some air resistance in our model. The simplest way to do that is to add in a force that is proportional to the velocity of the weight: the faster it falls, the greater the force of air resistance. Now the balance of forces shows that the mass of the weight matters, where before it did not (Fig. 36.4) Try adding in the air resistance (A = 10) and what do you find?
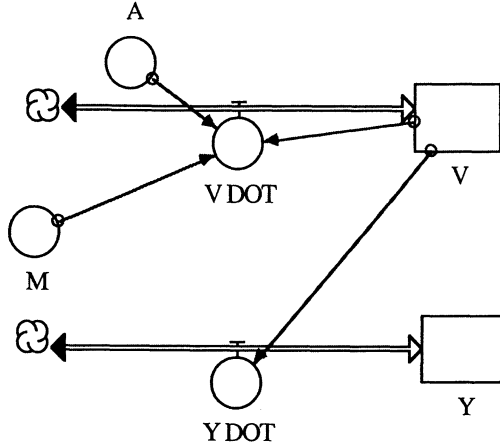
FIGURE 36.2

The object reaches a terminal velocity, its acceleration goes to 0 if given enough falling time. Actually, it isn't so much the mass of the object alone that causes our model to be sensitive to the mass dropped, but shape plays a significant role, too. Obviously, a feather dropped from the building would reach terminal velocity almost immediately. A feather and a rock of the same weight would not fall at the same rate. These differences in the way different objects fall or glide are reflected in the value of A.

Where does the A come from? Its value was determined from carefully measured actual experiments such as the one we have been talking about. So the engineer "forces" his or her theory to work by finding the experi-
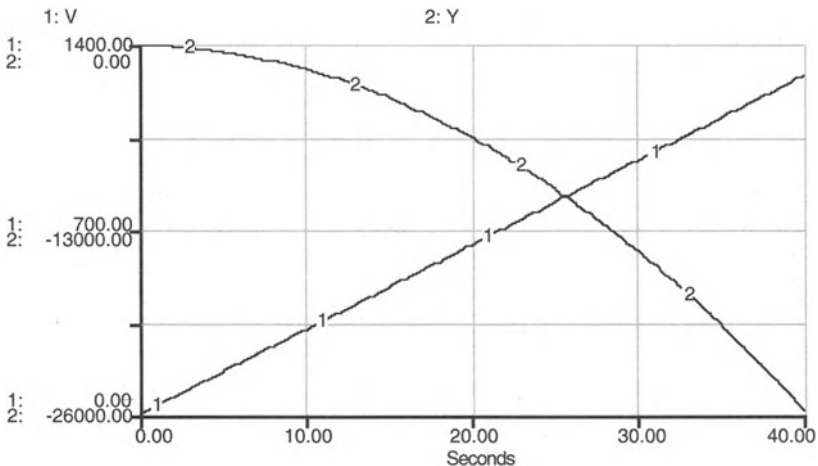
FIGURE 36.3
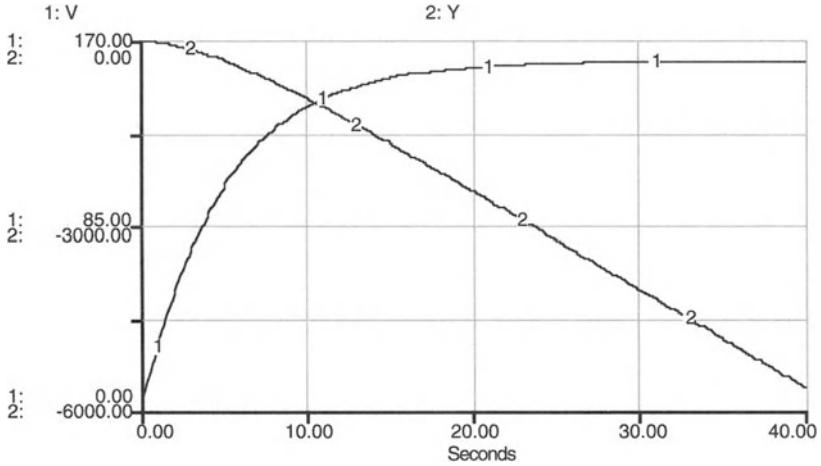
FIGURE 36.4

mental values of the theoretical constants. Measurements of these constants are frequently affected by a myriad of influences on the respective objects or systems. Thus, it may very well happen that measurements of these theoretical constants are incorrect.

In order to better understand the model, try turning the falling stone into a falling rocket with a specific force or thrust and that linearly loses half its weight (the fuel) in some specified time. And then try to make the coefficient A depend on the velocity.

Try throwing the rock straight up from the surface of a lake and then project its path for 10 seconds after it hits the water. Let the water have 10 times the velocity-resisting effect as the air.

```
                    FALLING ROCK

V(t) = V(t - dt) + (V_DOT) * dt
INIT V = 0 {Integral of the acceleration dv/dt =
v_dot. v_dot becomes v, the velocity of the falling
weight. Note that also v = = y_dot.}
INFLOWS:
V_DOT = 32.2 - A*V/M {This equation comes from the
conservation of forces acting on the body: the inertial
force
(up), the air resistance force (up) and the weight
(down).}


Y(t) = Y(t - dt) + (Y_DOT) * dt
```

```
INIT Y = 0 {Integral of dy/dt = y_dot into y, the
distance of fall.}
INFLOWS:
Y_DOT = -V
DOCUMENT: The negative sign means that the rock is
falling downward.

A = 10*1 {The nonzero constant (pounds-seconds^2/foot)
is the coefficient which converts velocity into a drag
force due to air resistance.}
M = 50 {The mass of the falling object = weight/g.}
```

36.2 Projectile Motion

In the previous section we modified the model of a falling rock into a model of a falling rocket, losing mass as it falls. Why let rocks fall when one could direct them and send them off with an initial velocity? The problem is really the same as in the one-dimensional case, complicated by our need to keep track now of both the height and distance reached by the projectile. The relevant forces are illustrated in Figure 36.5.

Now we need two state variables, one for the horizontal distance, X, and one for the vertical distance, Y. The initial velocity (100 feet per second) and the firing angle must be specified by the rocket scientist launching the object. The model for this problem is shown in Figure 36.6.

Set A = 0 to find the path with no air resistance. The result is shown Figure 36.7. Vary the firing angle to find the maximum height and maximum horizontal distance possible with this initial velocity. Can you verify this angle using the force conservation equation and a little calculus?
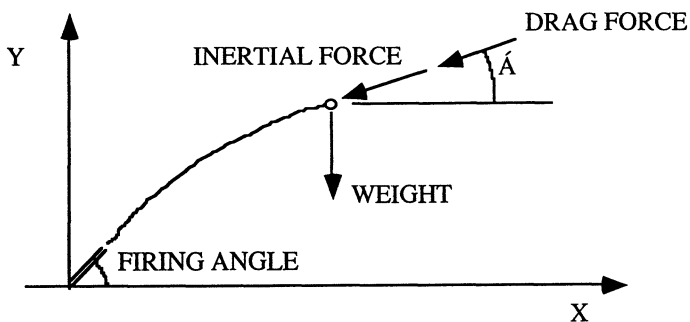
FIGURE 36.5
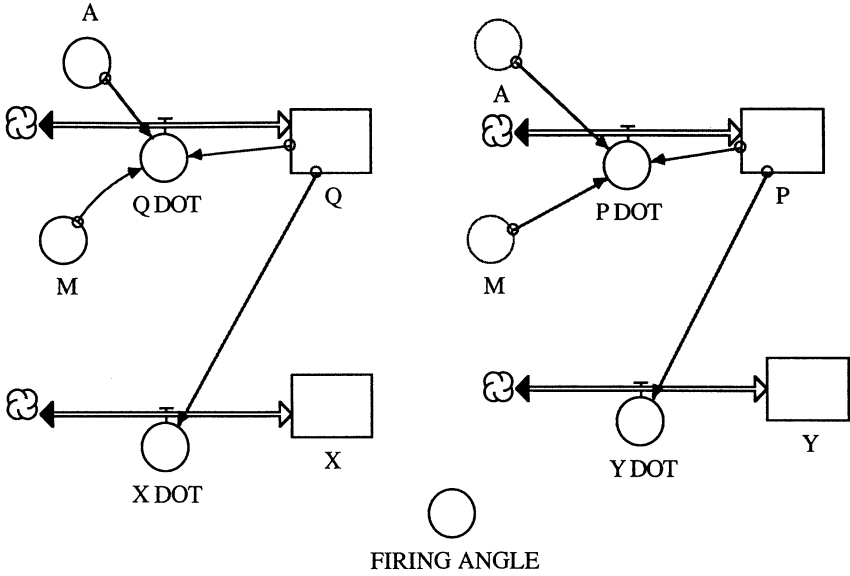
FIGURE 36.6

Now set A = 15 and note the difference in the path. We plotted our re-
sults in Figure 36.8. It is no longer symmetrical! But it does fit our intuition
a little better. Now find the firing angle for maximum horizontal reach. Can
you verify the result analytically?

Finally, suppose that half the projectile weight is propellant with a thrust
of 1 pound per pound of propellant burned per second and it burns at a

FIGURE 36.7

1: X v. Y
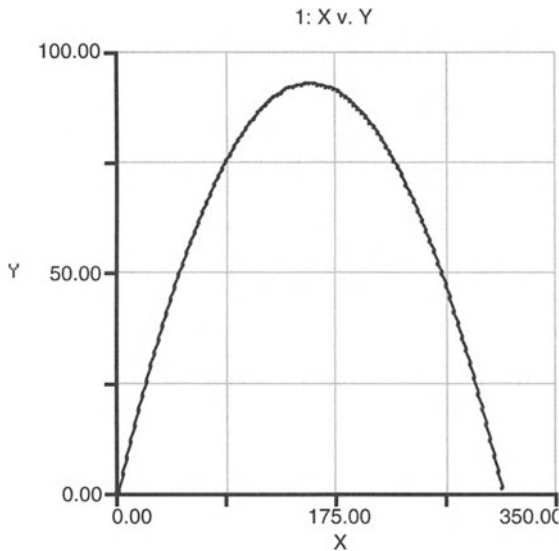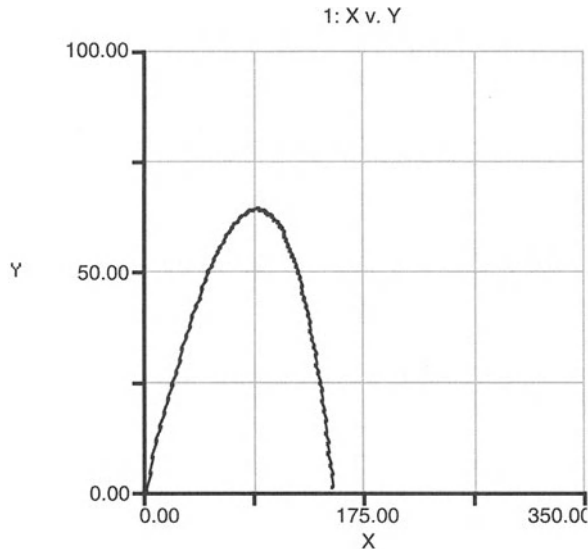
FIGURE 36.8

rate of 10 pounds per second. Firing the projectile with the given initial velocity ignites the propellant. Now what is the resulting path? This problem is more difficult than it generally seems. Be sure that you remember how Newton specified the inertial force: "The rate of alteration of motion is ever proportional to the motive force impressed."

Newton wrote in Latin but thought in English. His Latin writings were then translated back into English by someone else. Given this bizarre linguistic history and realizing that he wrote in the mid 1600s at the beginning of the mechanical age, it is no wonder that he is hard for us to understand. He is saying that the inertial force is equal to the time rate of change of the momentum (mass times velocity) or d(M*V)/dt. If the mass is constant, we have the familiar

$$F = M*A \tag{6}$$

but in the rocket problem we do not have a constant mass. This is the problem that NASA faces when it fires a shuttle, except that the shuttle does not have an initial velocity. At least, we can abstract away from the initial velocity given by the Earth's rotation provided we are planning an interplanetary trip. In our model, if we include the Earth's rotation, our projectile would not fly. You can fix that problem by shuting off the initial velocity and increasing the thrust force.

Imagine a wind is blowing directly against the x direction and that the wind velocity drops off to 0 at the ground. Make use if a logarithmic function to model the change in velocity of the wind. How would such a wind affect the path of the projectile?

```
                    PROJECTILE MOTION

P(t) = P(t - dt) + (P_DOT) * dt
INIT P = 100*SIN(FIRING_ANGLE) {initial vertical
velocity.}
INFLOWS:
P_DOT = -A*P/M - 32.2


Q(t) = Q(t - dt) + (Q_DOT) * dt
INIT Q = 100*COS(FIRING_ANGLE) {initial velocity in
the horizontal direction.}
INFLOWS:
Q_DOT = -A*Q/M


X(t) = X(t - dt) + (X_DOT) * dt
INIT X = 0
INFLOWS:
X_DOT = Q


Y(t) = Y(t - dt) + (Y_DOT) * dt
INIT Y = 0
INFLOWS:
Y_DOT = P


A = 15
FIRING_ANGLE = 50*PI/180{radians, angle with the
horizontal.}
M = 50
```

36.3 Mass-Damper-Spring

Let us assume a weight that hangs on a spring that exerts a vertical force. That force is proportional to, and oriented in, the same direction as the stretch. We can model this system of mass-damper-spring by resorting again to Newton and the conservation of forces. We make the distance traveled by the mass equal to Y (positive downward) and realize that the weight force is just cancelled by the force of the initial extension of the spring. So the 0 for the displacement Y is where the weight is at rest when hanging on the spring. Therefore, when a positive Y displacement is seen, the spring force (K*Y), the damping force (q*dY/dt = q*V), and the inertial force (M*dV/dt) all act upward. Thus,

$$dV/dt = -q*V/M - K*Y/M + P/M, \qquad (7)$$

INERTIAL
FORCE

SPRING
FORCE

DAMPING
FORCE

WEIGHT

Y
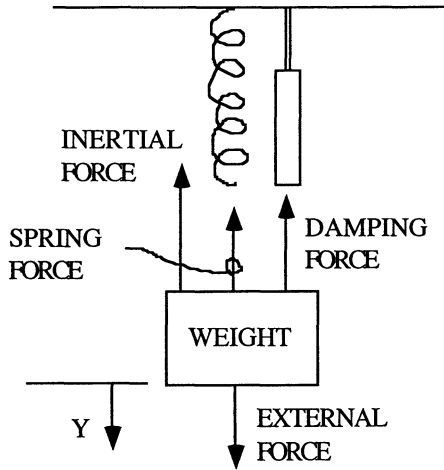
EXTERNAL
FORCE

FIGURE 36.9

from the conservation equation. P is an external force that can be set up any way you wish. Here we choose to make it a harmonic force, varying as the cosine of an angle that is a function of time. The relevant forces are illustrated in Figure 36.9 and the corresponding STELLA model is shown in Figure 36.10.

The choice of DT in this problem is nontrivial; use as the rule for the choice of DT: DT = 1/2 the smallest time constant in the problem. Here this turns out to be DT = 0.02. Try running the model with a DT = 1, and observe the results.

It is hard to tell when setting up a new and complex problem whether the weird results are from the wrong choice of DT or some programming

Y

Y DOT

V DOT

V

M

Q

P

K

BUMP

FIGURE 36.10

1: Y
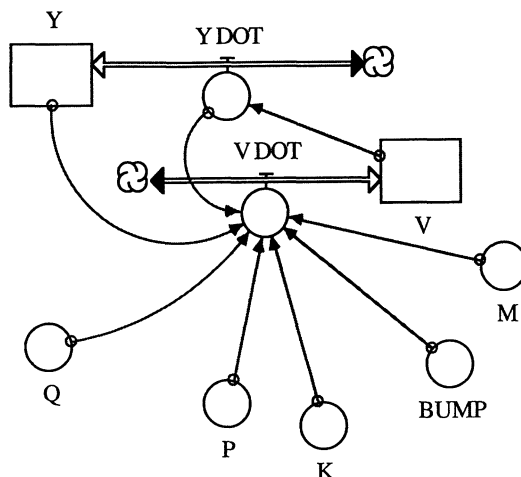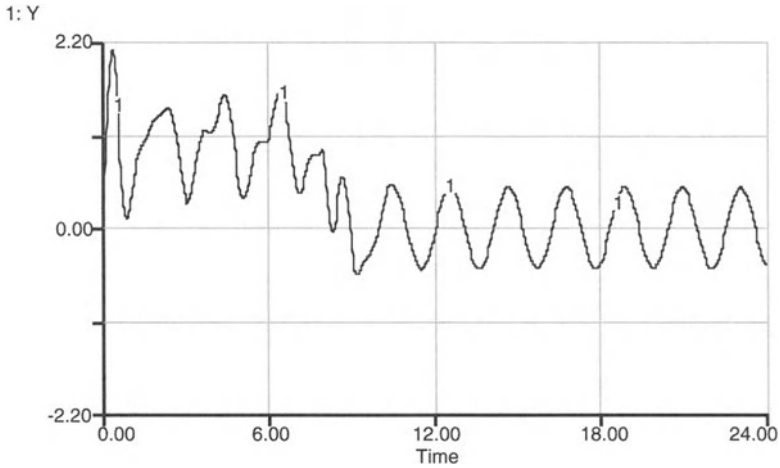
FIGURE 36.11

error. Most often this question is easily resolved by trial and error: keep cutting the DT in half and run the model again. If this improves the appearance of the result, then you are probably doing the correct thing.

Figure 36.11 shows how the mass swings over time, moving from the initially almost erratic behavior into a constant bounce up and down. The same process can be shown graphically by plotting velocity versus distance. At early stages the mass-damper-spring system follows a seemingly unsystematic pattern that corresponds to the excentric curves in Figure 36.12. After
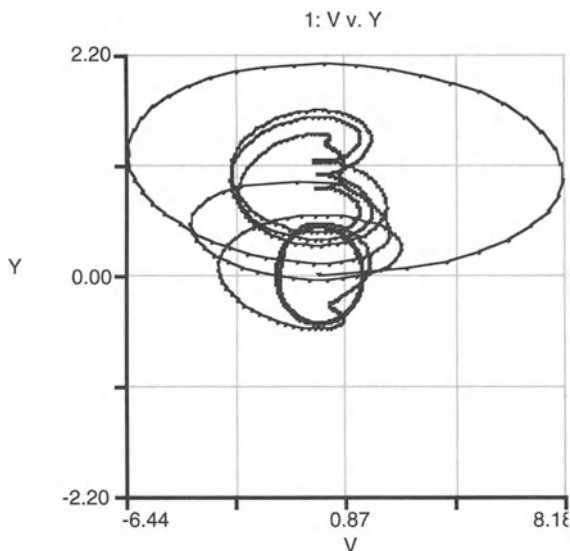
FIGURE 36.12

some time, a constant pattern of velocity-distance relationships establishes it-self, leading to the almost uniform circle in the lower part of the graph.

Run the model that you have constructed with the external force and the damping force turned off. Then turn each on and note in turn the effects. Any feature of this model can overwhelm the others: the spring can be so large that it dominates the external and damping force, for example.

```
                    MASS-DAMPER-SPRING

V(t) = V(t - dt) + (V_DOT) * dt
INIT V = 0 {v = velocity of the mass. This problem has
a weight of mass m hanging on a spring with spring
constant k. It is displaced a distance y with velocity
v. The force of the weight is cancelled by the intial
spring displacement.}
INFLOWS:
V_DOT = -K/M*(Y-BUMP) + 1*P/M - Q/M*(Y_DOT-
DERIVN(BUMP,1)) {Remove zeros to get the effects of the
exteranl forcing and the damping. The inertial force,
m*v_dot, opposes the direction of motion.}

Y(t) = Y(t - dt) + (Y_DOT) * dt
INIT Y = 0 {Initial velocity. This problem will run
only on Runga-Kutta. The time constant is 1/sqrt(k/m)
or about 1/8 sec. So try DT = about 1/16 as a start.}
INFLOWS:
Y_DOT = V  {The basic differential equation is:
d2(x)/d2(t) =
-k/m*x + P - q/m*v.}

BUMP = IF TIME ≤ 8 THEN (1 - COSWAVE(.1,1)) ELSE 0
K = 12 {Spring constant, weight/unit extension of the
spring.}
M = .2  {mass, weight/gravity acceleration.}
P = 5*COS(3*TIME) {This is the exciting force. The
natural frequency of the system is sqrt(k/m).}
Q = .50  {Damping coefficient. Damping slows the
velocity as would air resistance, for example.}
```

36.4 Mechanical Amplifier

Perhaps you can recall the childhood experience on the playground swing. Without any understanding of the physics behind the process, every pre-

schooler learns quickly to "pump" the swing in order to increase and sustain the swinging motion. The pumping process requires that the swingers quickly pull themselves up to a full sitting position as near the bottom arc of the swing as possible. To do this, the swinger lays out flat at the extreme positions of the arc. The swinger pulls on the swing ropes to very quickly raise themselves to the full sitting position just as they cross the bottom of the arc, and then quickly lay out flat again as the swing begins to rise. It takes energy and arm strength to do this as the swinger is pulling against not only their body weight but also the centifugal force, which reaches a maximum at the bottom of the arc. Through this pulse, the swinger inserts energy into the process that allows them to rise higher on the backswing than previously, until such a time as the air resistance and mechanical friction energy dissipate the input pulse.

The principle can be demonstrated in the classroom with a small board fixed to the desk at one end. Through a small hole in the free end a string is passed to a small iron weight (the swing seat and the swinger). The instructor holds the upper end of the string in one hand and gives the weight a small push with the other. The instructor pulls on the string in rhythm with the weight's cycle; the weight is pulled up slightly and quickly just as it passes directly under the board, and it is let out fully and quickly when the weight has just passed the bottom of the arc. The swinging amplitude of the weight will then increase.

To model this process, we develop the equation of motion for the pendulum swing. From engineering mechanics, the transverse inertial force is $M*[L*d^2(PHI)/d^2t + 2*dL/dt*d(PHI)/dt]$, where PHI is the angle of rotation from the vertical, Q is the damping coefficient due to air resistance, M is the mass of the seat and rider, and L is the (varying) length of the string from the hinge to the center of mass. The transverse air resistance force is $Q*L*d(PHI)/dt$ and the transverse component of the weight is $M*g*SIN(PHI)$, with g as the gravity constant. The balance of these forces (in the same direction when PHI is increasing) gives a second-order differential equation that can be decomposed into two first-order differential equations:

$$d2(PHI)/d2t = dY/dt = -[\,2*d(L)/dt/L + Q/M]*Y - g/L*SIN(PHI) \qquad (8)$$

and

$$d(PHI)/dt = Y \qquad (9)$$

These equations are solved simultaneously in STELLA (Fig. 36.13). The principles of their derivation can be found in many mechanics textbooks.[1]

If L is not a constant but instead varies cyclically with the cycle of the swing such that it is the shortest when passing the low point and longest at the extremes of PHI, then the swing is pumped. It is pumped by the energy

---

[1]For example: Beer, F. and E. Johnson, *Statics and Dynamics*, McGraw-Hill, New York, 1962, eq. 11.46, p. 432.
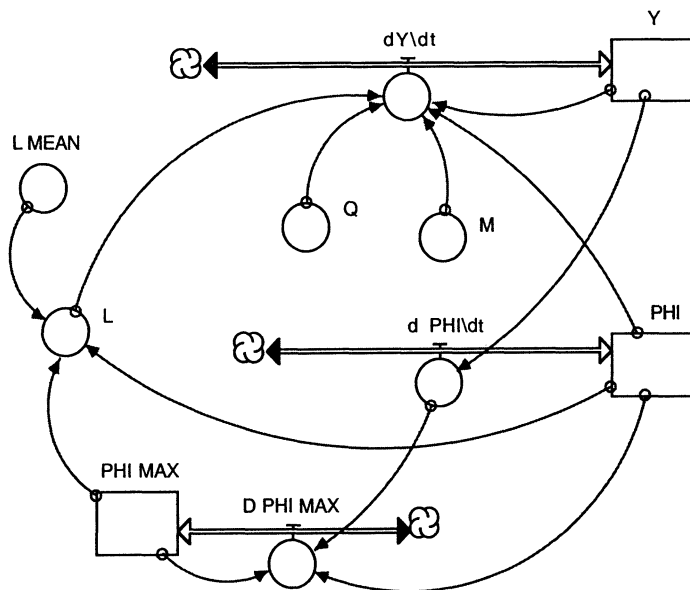
FIGURE 36.13

needed to shorten the swing when the centifugal force is greatest. Lengthening the pendulum extracts energy from the system. Therefore the pumping should be done in a very short period near the lowest position of the swing. The swing responds to this energy input by increasing its arc until the air resistance (damping) energy loss overcomes the pumping energy input. The swing retains (approximately) its arc as long as the rider can continue the energy input. Note how it was necessary to determine the maximum angle of the swing so that the swinger has a goal. If the maximum swing angle is greater than the goal, pumping is suspended.

Figure 36.14 shows the pulsed shortening of the effective swing length and the resulting cycles of the swing. The pulsing is intermittent and irregular (due to the limits set in L) but they keep the swing in motion.

Systems with this sort of a pulse input of energy are sometimes called "kicked" systems. Certain electronic and electrical systems can exhibit such oscillating behavior when kicked repeatedly by a voltage or current pulse.[2]

This is a complex form of the simple pendulum problem. There are many other interesting variations on the simple pendulum. Do you know that you can make a pendulum vibrate steadily back and forth, in an inverted position if you shake its suspension point vertically in the right frequency and amplitude range? Try to write the equations of motion of this problem. It's tough and the solution in STELLA is trial and error. But the phenomenon is real.

[2]Yurke, B. 1991. Back Action Evasion as an Alternative to Impedance Matching, *Science*, Vol. 252, pp. 528–532.
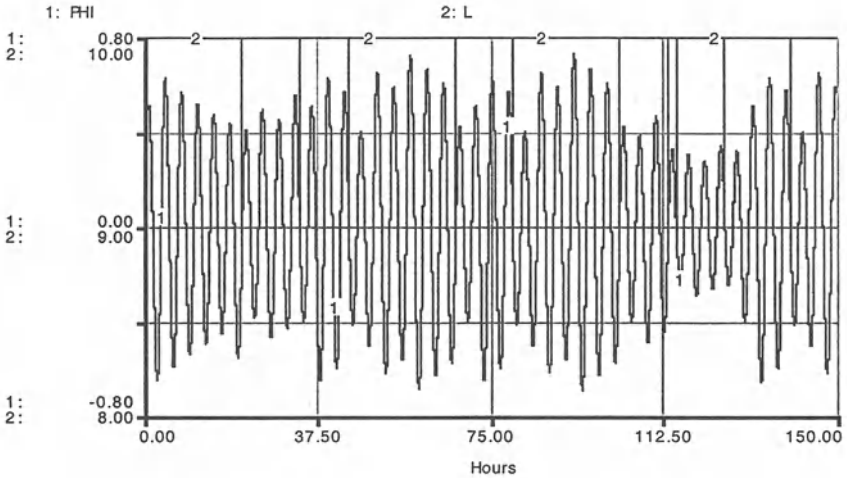
FIGURE 36.14

```
                    MECHANICAL AMPLIFIER

PHI(t) = PHI(t - dt) + (d_PHI\dt) * dt
INIT PHI = .5
INFLOWS:
d_PHI\dt = Y


PHI_MAX(t) = PHI_MAX(t - dt) + (D_PHI_MAX) * dt
INIT PHI_MAX = .01
INFLOWS:
D_PHI_MAX = IF (d_PHI\dt < .03) AND (d_PHI\dt > 0)
THEN (ABS(PHI) - PHI_MAX)/dt ELSE 0
Y(t) = Y(t - dt) + (dY\dt) * dt
INIT Y = .2
INFLOWS:
dY\dt = - 32.2/L*SIN(PHI) - Y*(Q/M+2*DERIVN(L,1)/L)


L = if (PHI_MAX < .6 ) AND ((PHI>-.001) AND (PHI
<.001)) THEN L_MEAN - 1 ELSE L_MEAN
L_MEAN = 10
M = 10
Q = .5
```

Part 8

Conclusion

37
Chaos

37.1 A New Paradigm

Before the advent of modern computer technology and software, many modeling efforts and scientific experiments were designed for linear, often static systems, that had the advantage to be analytically solvable. The ways of thinking about system behavior and the tools applied to describe that behavior were rooted deeply in classical mechanics. This science was used to describe the behavior of whole classes of moving objects, such as pendulums, falling rocks, or projectiles. We modeled those systems in the previous chapter. The scientific paradigms associated with classical mechanics were not only applied in the realm of the natural sciences but increasingly influenced models of economic and ecological systems as well.

The strength of these paradigms lies in their view of systems as predictable, well-described entities that can be analyzed with available mathematical tools. Students were told that nonlinear systems are generally unsolvable and that such systems are exceptions. The first of these statements is true; nonlinear systems, some of which we modeled in the previous chapters, generally do not have an explicit mathematical solution. However, the second statement, that nonlinear systems are exceptions, is false. Rather, many real systems are governed by nonlinearities. These systems frequently exhibit characteristics that were previously not anticipated or have been identified incorrectly.

The emergence of chaos theory made us aware of the importance of nonlinearities, a lack of predictability that is inherent in many of these nonlinear systems, the sensitivity of model results to small changes in initial conditions, and therefore, the need for increased computer modeling efforts. Today, chaos theory begins to protrude thinking in modern natural sciences as well as in the social sciences.

In the following sections we develop models with potentially chaotic behavior. Before we do so, it is worth noting that among the completely deterministic models, such as those in Chapter 36, lie models whose steady state condition is dependent on the initial conditions of the state variable. The behavior of the key state variables is interesting, involving cusps and bifurcations. For example, the spruce budworm can switch dramatically between two very different population equilibrium levels as the spruce are defoliated by the budworm's action. The tree foliage grows and the insect population grows with it, but at a critical level the insect population jumps

to a much higher level, which causes slow decline in the foliage. Again, at a critical foliage level, the insect population drops suddenly down to its former, lower equilibrium level, allowing the foliage to begin growing again. This is a recurring cycle, and it is the long-term equilibrium condition. But it is deterministic, unlike the chaotic results that we will turn to now.[1]

There are two different styles of modeling chaos in STELLA. The first are discrete models that are sensitive to the time step DT, the second are continuous function models. We discuss these in the following sections.

37.2 Jenson Chaos

In order to show that chaotic system behavior can occur even in some of the simplest models, we return to a population model similar to the one discussed at the beginning of this book. Let us be more general and refer to the state variable as X and the net flow as DELTA X. The exogenous parameter influencing the net flow is A. The net flow DELTA X updates the stock X:

$$\text{DELTA X} = X(t + DT) - X(t) \tag{1}$$

The "reproductive rule" in this model is

$$X(t + DT) = A*X(t)*(1 - X(t)) \tag{2}$$

and, consequently,

$$\text{DELTA X} = A*X(t)*(1 - X(t)) - X(t) \tag{3}$$

The STELLA model of this problem has as its main component the DELTA X equation updating the stock X at each period of time. We also calculate the stock X delayed by DT as LAG X. To compare the dynamics of this model with a totally random process we further calculate a random number RAND and its delayed value (Fig. 37.1).

Run the model with an initial X of 0.1, DT = 1 and A = 1. Then, raise for subsequent runs A to 1, 2 , 3, and then 4, and watch X (Fig. 37.2). Now lower the time step to DT = 0.5 and find the value for A at which chaos begins again. Keep shortening DT until you note that the critical A gets extremely large. This model shows you that the A for DT = 0 is infinity. This result is correct since chaos is typically not noticed on the continuous level. Chaos occurs on the continuous level only if you are stuck with a specific DT in your particular problem and the parameters lie within the critical range.

Use the model with DT = 0.5 to identify the critical A values. You will find that for A < 3.0 there is one solution, namely the logistic; for 3.0 < A < 3.4 there are two solutions; for 3.4 < A < 3.5 there are four solutions; for A

---

[1]For a model of spruce budworm dynamics and other examples of catastrophe see Hannon, B. and M. Ruth (1997) *Modeling Dynamic Biological Systems*, Springer-Verlag, New York.
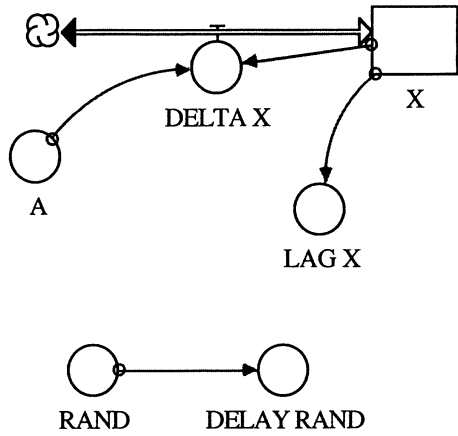
FIGURE 37.1

> 3.57 there is essentially an infinite number of solutions (chaos) with nota-
ble exceptions (try A = 3.83). The upshot of this model is that as A rises to
3.57 (and DT = 1), increasing bifurcation is seen but the number of solu-
tions for X is finite. For A > 3.57, one cannot predict what the final equilib-
rium value will be and even whether there will be a finite or infinte number
of solutions: there is no definite set of equilibrium values. However, there
seem to be values that are "visited" more frequently than others.[2]
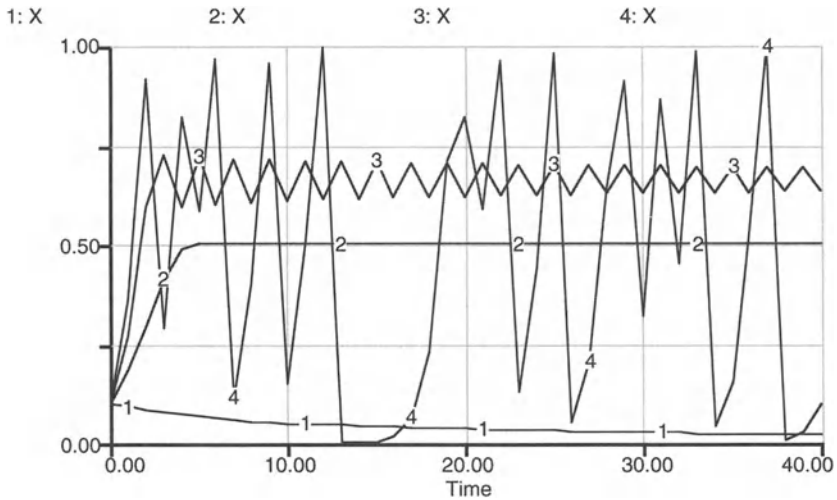
FIGURE 37.2

[2]For a full discussion of this and other versions of chaos, see Jenson, R.V. 1987.
Classical Chaos, *American Scientist*, Vol. 75, pp. 168–181.

1: X v. LAG X
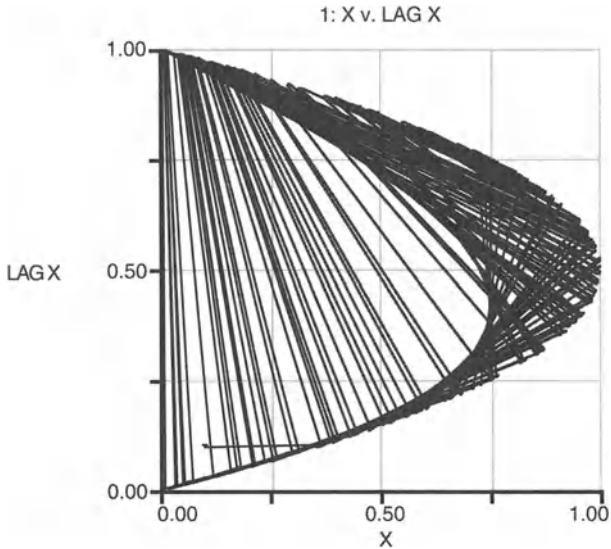
FIGURE 37.3

So, how do you tell if the phenomena you note from a model is chaotic or random? After all they both seem equally unpredictable. Try this: Plot X against X(t-DT) for the above problem and for the function X = RAN-DOM(0,1). The first plot (Fig. 37.3) reveals the parabolic nature of the population function, and the second plot (Fig. 37.4) is truly random. So chaos is unpredictable but not so unpredictable as a truly random process.

1: RAND v. DELAY RAND
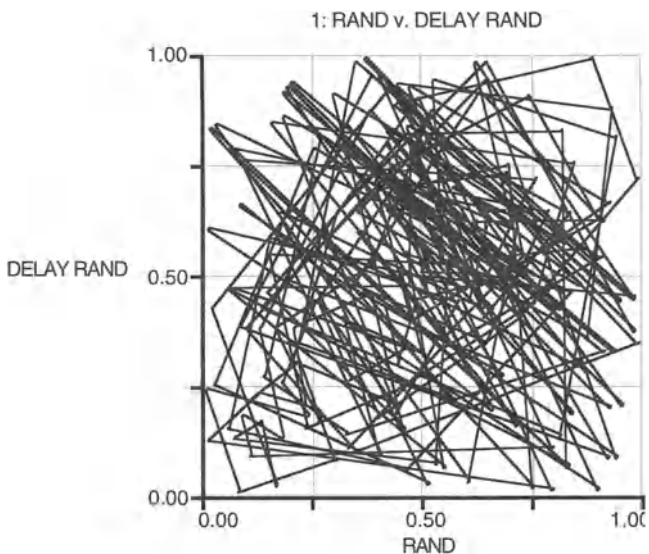
FIGURE 37.4

It is rather like saying that Anne has a travel schedule that is so unpredictable that I don't know whether she will be in California, South Dakota, or West Virginia next week, versus, I don't know which of the 50 states she will be in next week.

JENSON CHAOS

```
X(t) = X(t - dt) + (DELTA_X) * dt
INIT X = .1
INFLOWS:
DELTA_X = A*X*(1 - X) - X   {X_t plus one - X = delta
X}


A = 4
DELAY_RAND = DELAY(RAND,DT)
LAG_X = DELAY(X,DT)
RAND = RANDOM(0,1)
```

37.3 Lorenz Chaos

Much of the effort of dealing with nonlinearities goes back to Edward Lorenz, who simulated weather patterns on a computer. In 1961, Lorenz found inconsistencies in his weather models that are now known as the first recognized instances of chaos. He simplified his equations into the three shown in the model of this section.[3] They are realized now as chaos in continuous functions as opposed to the previous example, which is tied to discrete phenomena.

For the model, the parameters P, R, and B are given exogenously together with the initial conditions of the stocks X, Y, and Z (Fig. 37.5). The flows are specified by the following differential equations:

$$dX/dt = P*(Y - X) \tag{4}$$

$$dY/dt = (R-Z)*X - Y \tag{5}$$

$$dZ/dt = X*Z - B*Z \tag{6}$$

---

[3]For a good description of Lorenz's encounter of chaos see Gleick, J. 1988. *Chaos: Making a New Science*, Penguin Books, New York. For more detail see: Grebogi, C., E. Ott, and J. Yorke. 1987. Chaos, Strange Attractors and Fractal Basin Boundaries in Non Linear Dynamics, *Science*, Vol. 238, pp. 632–638.
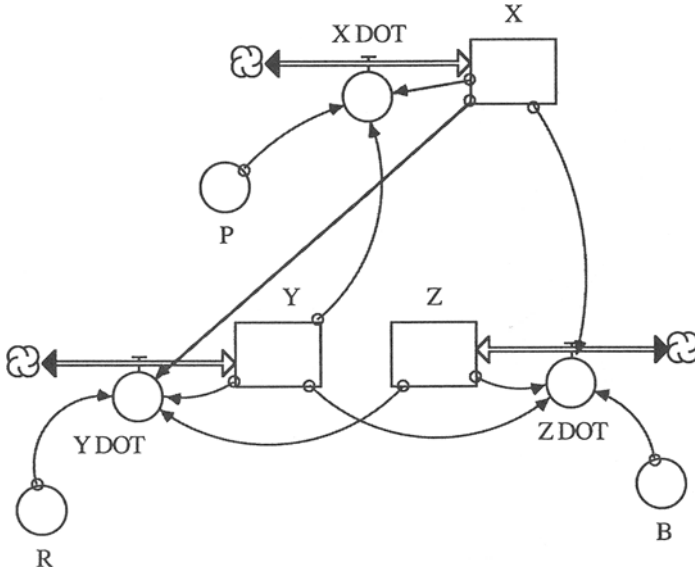
FIGURE 37.5

Run the model with a DT of .0005 and set for consecutive runs R = 16, 22, 28, 34, 40. The results are shown in Figure 37.6. Then choose a DT = .005, run the model again with the previous series of values for R and compare the results. Can you make a qualitative statement about the system behavior as R increases and/or DT decreases? Assess the validity of your statement with further model runs.
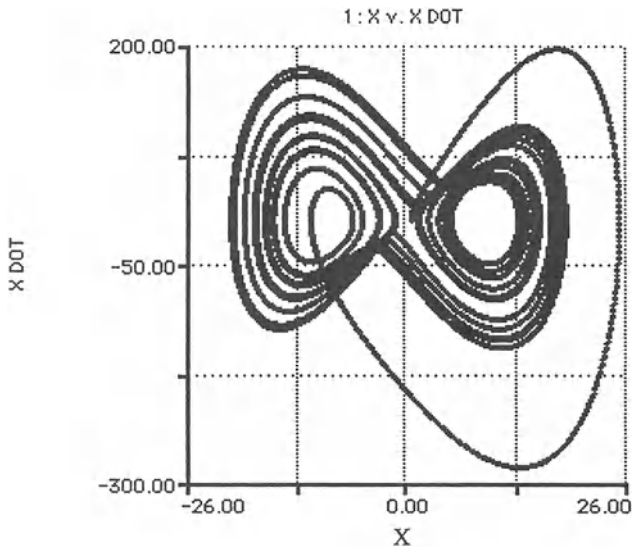
FIGURE 37.6

LORENZ CHAOS

```
X(t)  = X(t - dt) + (X_DOT) * dt
        INIT X = 1
        INFLOWS:
        X_DOT = P*(Y-X)
Y(t)  = Y(t - dt) + (Y_DOT) * dt
        INIT Y = 1
        INFLOWS:
        Y_DOT = (R-Z)*X-Y
Z(t)  = Z(t - dt) + (Z_DOT) * dt
        INIT Z = 0
        INFLOWS:
        Z_DOT = X*Y-B*Z
B = 8/3
P = 10
R = 40
```

37.4 Two-Well Chaos

If we take the mass-spring-damper problem of Chapter 36.3 and make the spring nonlinear, we see again what appears to be the end of Newton's hold on the world of science. It is Newton-with-history. This model, like the two simple ones above, enables us to produce a reproducible but unpredictable result in what was formerly a well-behaved variable. We encounter chaos.

To get a hands-on experience with chaos, take a very thin sheet of steel about one inch wide and 10 to 20 inches long. Hold it at the bottom end and shake your hand rapidly in a horizontal direction. You can feel the sheet flexing evenly at lower frequencies but it becomes erratic at higher ones. This is chaotic behavior. The sheet begins to jump between lateral and torsional vibration. These jumps are seemingly random.

We can show chaos in the mass spring problem by replacing the linear with a *nonlinear* spring. All other features of the problem are of the same style as the earlier problem. The model developed here thus builds directly on the deterministic, linear-spring model of Chapter 36.3. The model is shown in Figure 37.7 and is sometimes referred to as the Duffing equation.

The model can also be invisioned as a little cart in which there are two semicircular depressions, separated by a hill. The cart is shaken in line with the two depressions by an oscillating external force. A marble is placed in one of the wells. With the right frequency and size of the shaking force, the marble will oscillate gently in one of the wells but suddenly without expectation, shift over the interlying hill into the other well, where it will os-
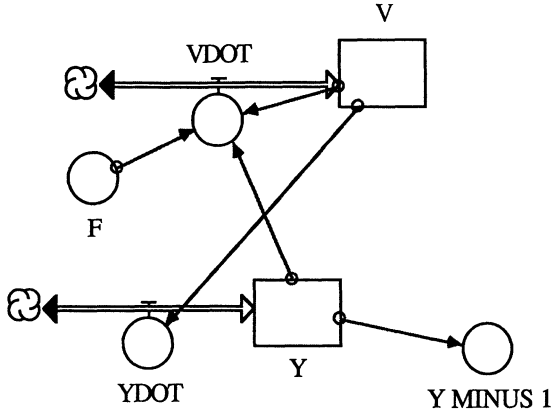
FIGURE 37.7

cillate gently again, for awhile, then jump back. The shifting between wells is unpredictable so we say that the position of the marble is chaotic. In the original interpretation of the model, we say that the weight oscillates about one point and then suddenly shifts to oscillate about another point. Run the model to see this phenomenon (Fig. 37.8). Change the parameters to see how narrow the range is for such behavior.

Figure 37.9 shows the position of the ball in the cart. Every time there is a change in the sign of Y, the ball jumped to the other depression in the cart. If there is no change in sign, the ball merely moved within a depression.

FIGURE 37.8

1: Y v. Y MINUS 1
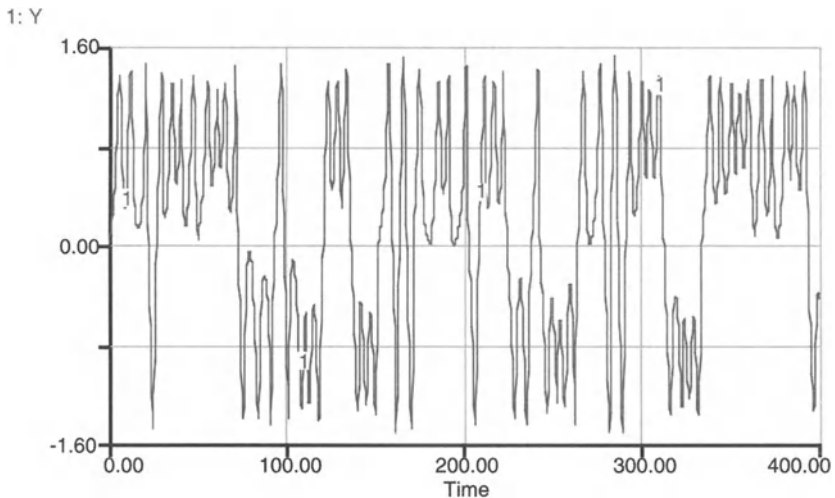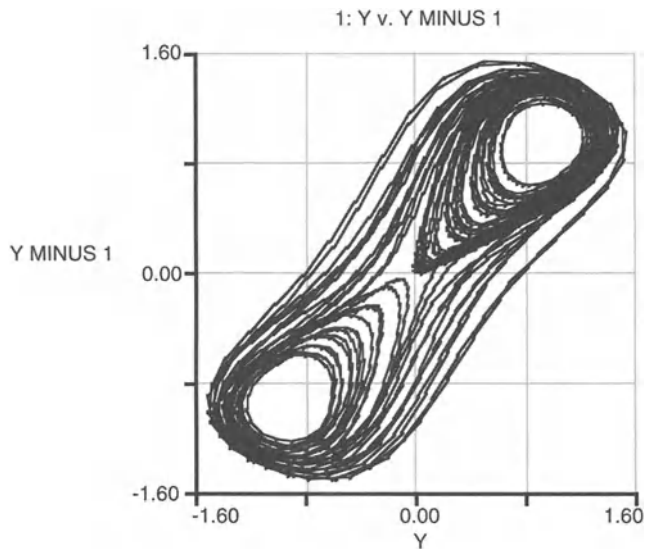
FIGURE 37.9

TWO WELL CHAOS

```
V(t) = V(t - dt) + (VDOT) * dt
INIT V = 0
INFLOWS:
VDOT = -V/3 +Y-Y^3 + F*COS(TIME)

Y(t) = Y(t - dt) + (YDOT) * dt
INIT Y = 0
INFLOWS:
YDOT = V

F = .1
Y_MINUS_1 = DELAY(Y,1)
```

38
Beginning a Dialog

I know it well sir, you always end ere you begin.
—William Shakespeare, *Two Gentlemen of Verona*

We all have different ways of looking at the world that surrounds us, of interpreting our observations and communicating them to others. Models facilitate the process of interpretation and communication. At the same time, models open up new perspectives on the world and stimulate new questions about its workings.

In this book we provided one particular, and very powerful, way of looking at the world. We concentrated on the forces that underlie different dynamic systems. Others, with different educational or cultural backgrounds, may choose a different approach and may develop different models. The potential diversity of perspectives and approaches in modeling is a challenge for all of us and should be perceived as an opportunity to engage in cross-disciplinary and cross-cultural dialogue about the world in which we live.

STELLA, through its use of graphics, is an excellent tool to organize and communicate model assumptions, structure, and results among individuals with different backgrounds. You will soon find that your models become increasingly involved. Frequently, model efforts become large-scale multi-disciplinary endeavors. STELLA is sufficiently versatile to enable development of complex, large-scale dynamic models. Such models can include a variety of features that are typically not dealt with by an individual modeler. Through easy incorporation of new modules into existing dynamic models and flexibility in adjusting models to specific real-world problems, STELLA fosters dialog and collaboration among modelers. It is a superb organizing and knowledge-capturing device for model building in an inter-disciplinary arena. Individuals can easily integrate their knowledge into a STELLA model without "losing sight" of, or influence on, their particular part of the model.

Even though the models developed in this book were guided toward an explanation of real-world phenomena, empirical applications are not the focus of this book. Nevertheless, once we developed sufficiently elaborate models, we made intensive use of real-world data, such as in the models of population dynamics in the United States (Chapter 14), the reestablishment of wolves (Chapter 20), Lyme disease (Chapter 21), optimal tree cutting (Chapter 30), a fishery reserve (Chapter 31), or the pig cycle (Chapter 34). These and other models illustrate the applicability to real-world data and, in general, the power of the dynamic modeling approach chosen in this book.

We highly encourage you to take up and refine some of the models presented in this book to further accommodate data from real systems.

We selected a variety of different systems, spanning the disciplines of physics, genetics, biology, ecology, economics, and engineering, to illustrate the power of dynamic modeling and the multitude of possible applications. Other dynamic modeling books geared specifically toward some of these and other disciplines are published in this book series.

With this book, and the series as a whole, we wish to initiate a dialogue with (and among) you and other modelers. We invite you to share with us your ideas, suggestions and criticisms of the book, its models, and its presentation format. We also encourage you to send us your best STELLA models. We intend to make the best models available to a larger audience, possibly in the form of books, acknowledging you as one of the selected contributors. The models will be chosen based on their simplicity and their application to an interesting phenomenon or real-world problem. Keep in mind that these models are mainly used for educational purposes.

So register now by (e-) mailing or faxing us your name and address and possibly something about your modeling concerns. Invite your interested colleagues and students to also register with us now. We can build a modeling community only if we know how to make, and maintain, contact with you. We believe that the dynamic modeling enthusiasm, the ecolate skill, spreads by word of mouth, by people in groups of two or three sitting around a computer doing this modeling together, building a new model or reviewing one by another such group. Share your thoughts and insights with us, and through us, with other modelers. Here is how you can reach us:

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Appendixes

A1
System Requirements

A1.1. Macintosh

Macintosh Minimum Requirements

Mac OS 7.1
68040 Processor
16MB RAM
35 MB Hard Disk Space
Color Monitor running at least 256 colors

Macintosh Recommended Requirements

120 MHz Power PC or better
32MB RAM
Color Monitor running thousands of colors
QuickTime™ 3.0 or higher

A1.2. Windows

Windows Minimum Requirements

Windows 95, Windows 98, Windows NT 4.0, or Windows 2000
Pentium-Class Processor
16MB RAM
30MB Hard Disk Space
VGA Display with at least 256 colors

Windows Recommended Requirements

233 MHz Pentium Processor or better
64MB RAM
SVGA Display with 16-bit/high color
SoundBlaster or compatible sound card
QuickTime™ 3.0 or higher

A2
Quick Help Guide[1]

A2.1. Overview of STELLAR Operating Environment
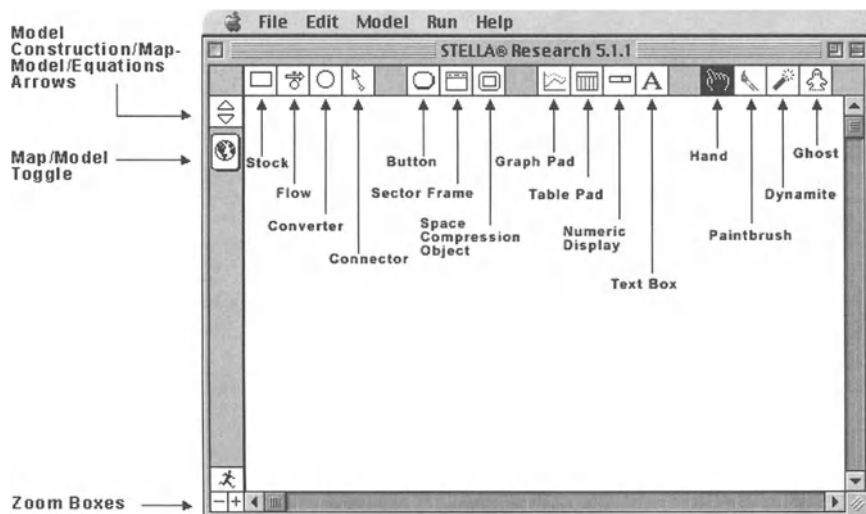
A2.2. Drawing an Inflow to a Stock
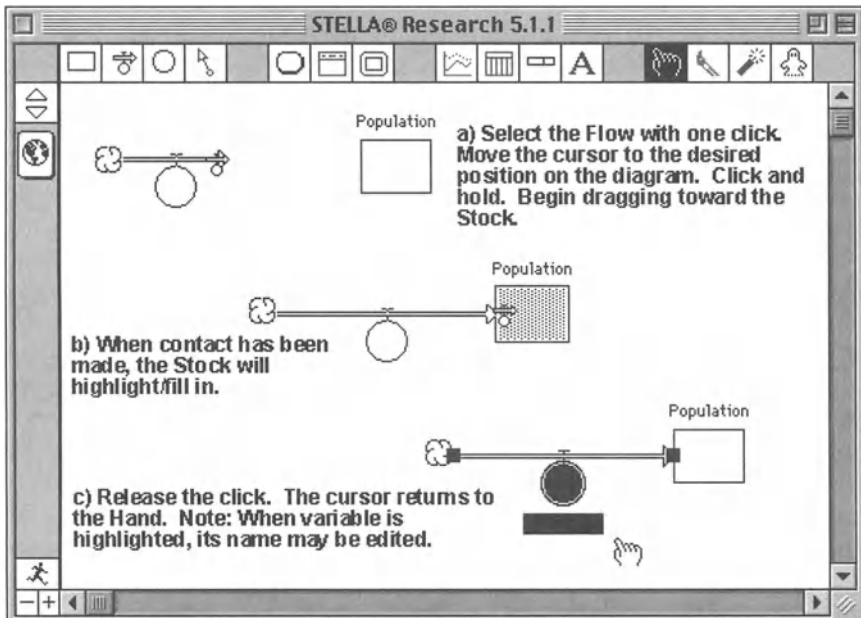
A2.3. Drawing an Outflow from a Stock
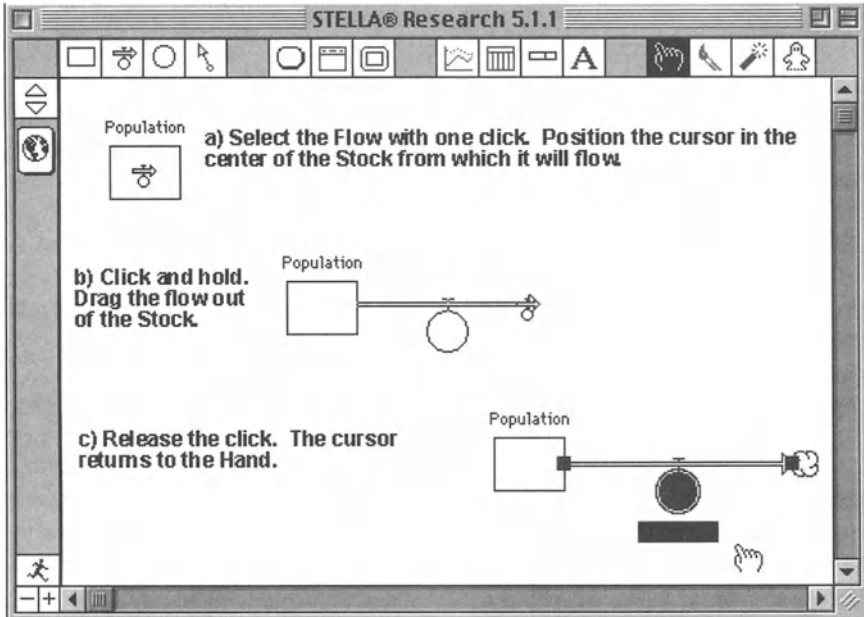
A2.4. Replacing a Cloud with a Stock

A2.5. Bending Flow Pipes

While drawing a Flow, depress the shift key to create a bend in the flow pipe. (You can create as many bends as you like)
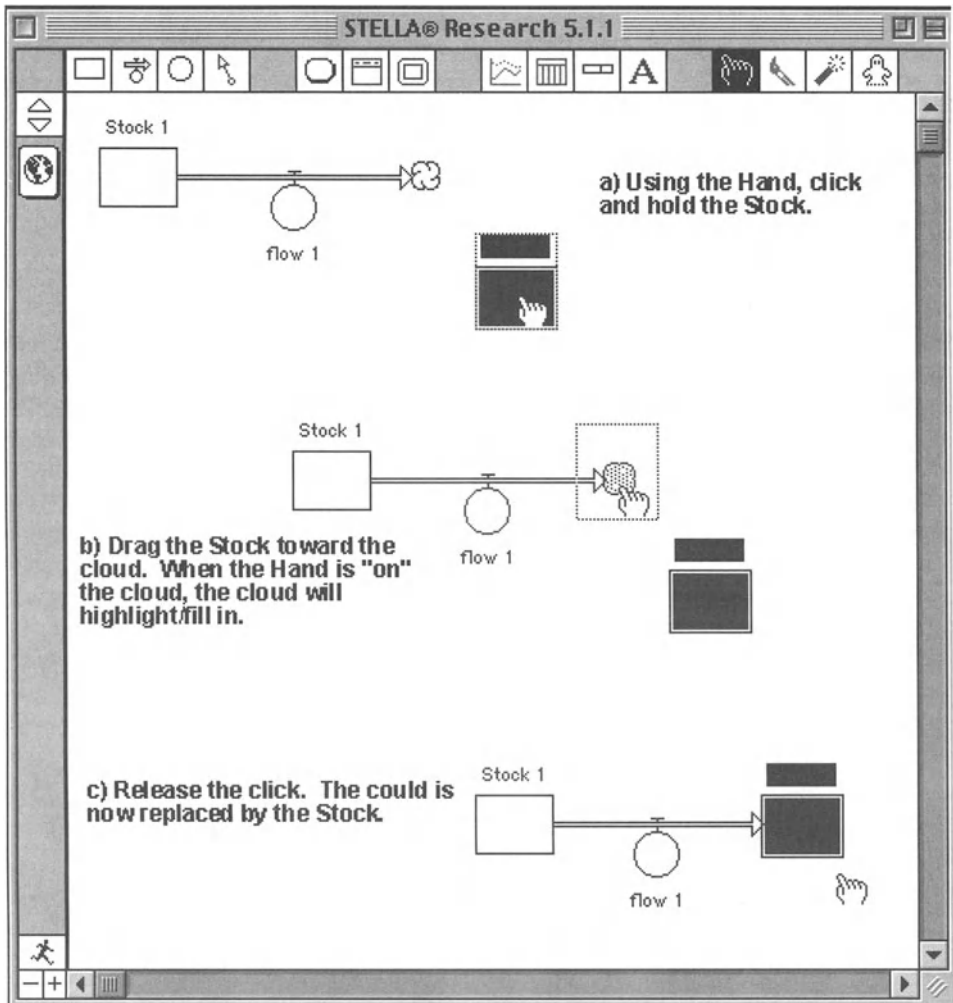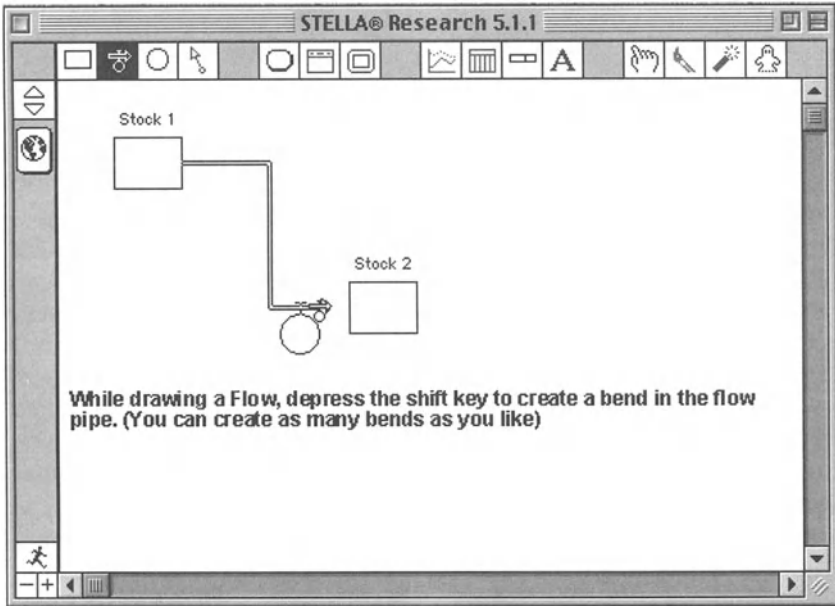
A2.6. Repositioning Flow Pipes

Select the Flow by clicking on it's circle using the Hand. Click and drag the "handles" to reposition the flow pipe.
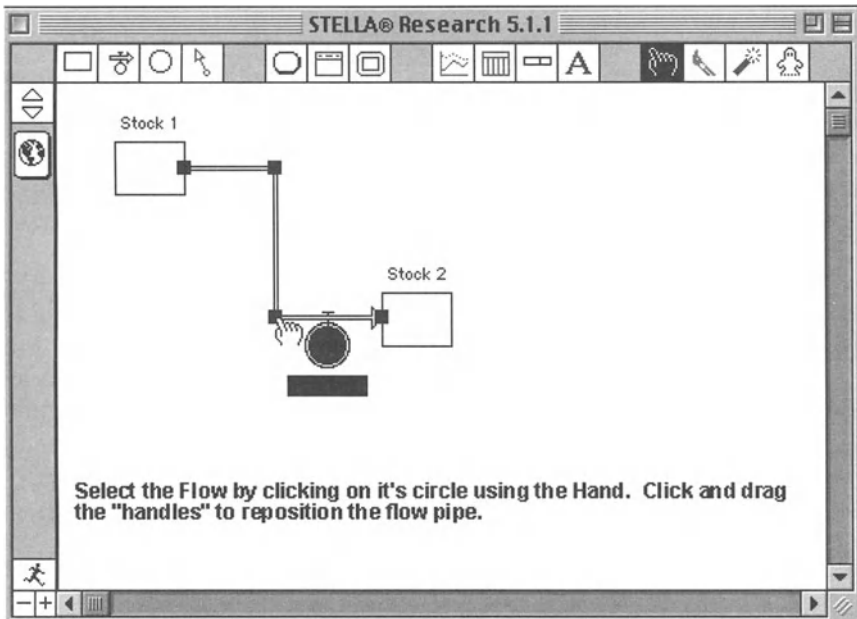
A2.7. Reversing Direction of a Flow
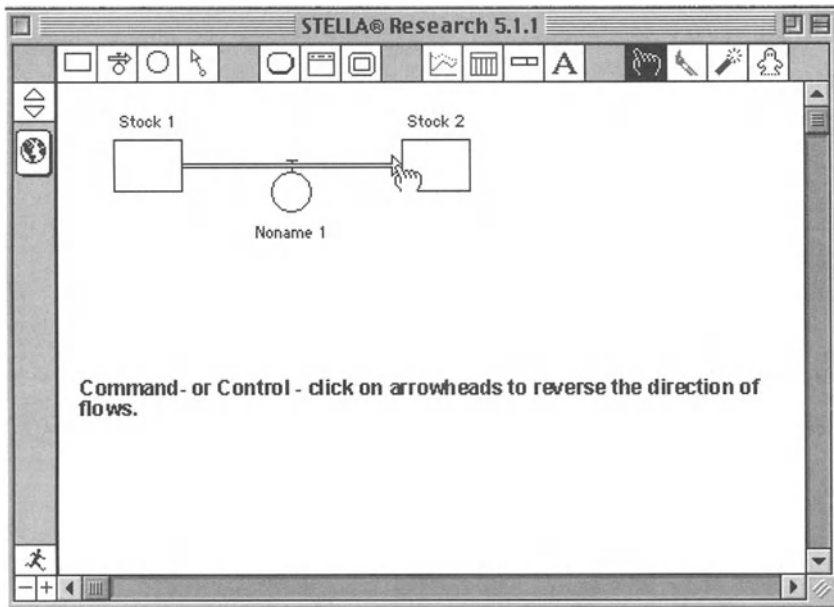
A2.8. Flow Define Dialog—Builtins
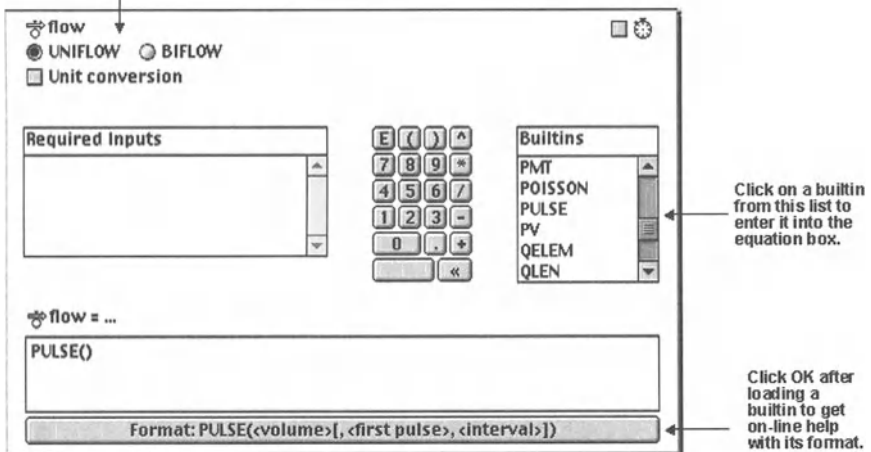
A2.9. Moving Variable Names
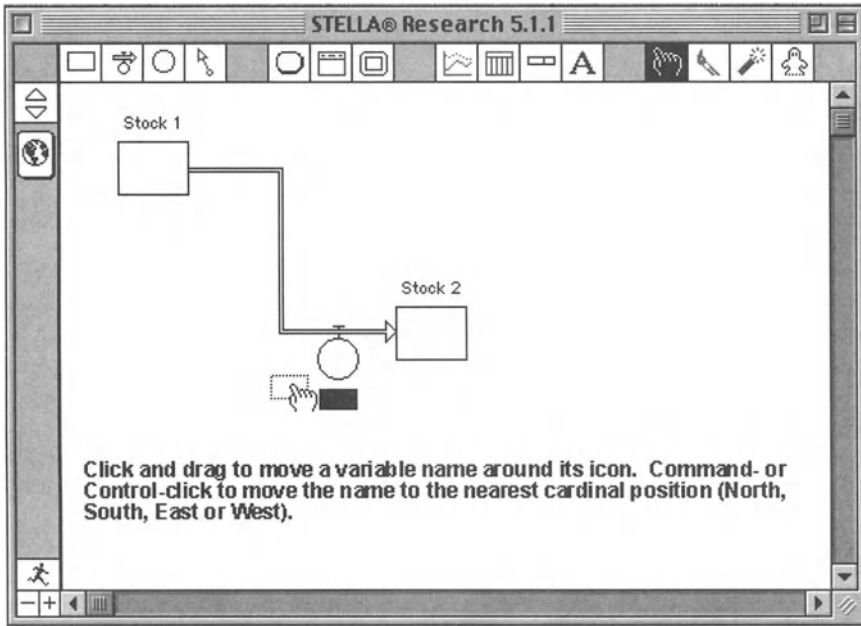
A2.10. Drawing Connectors
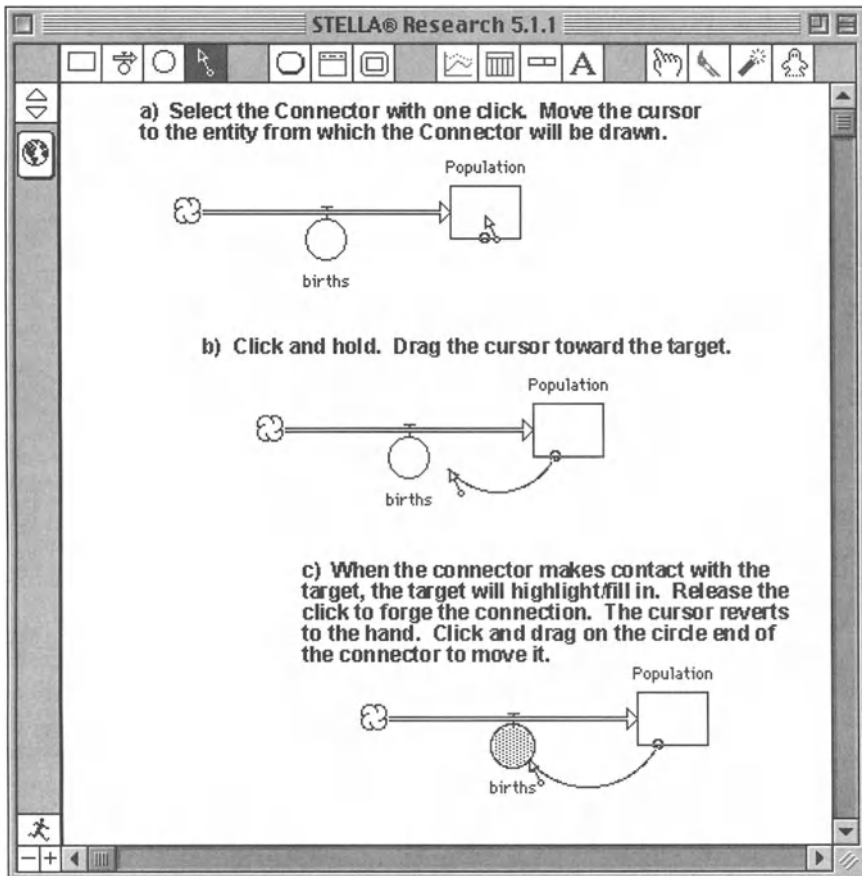
A2.11. Defining Graphs and Tables

Select the graph pad or table pad icon, then deposit it on
the diagram. An empty graph or table page will appear.

Double-click the graph or table page to enter the
Define dialog.

Untitled

Graph Type: ● Time Series   ○ Scatter   ○ Bar   ○ Sketchable
☐ Comparative   ☐ Connect Dots   ☐ Benchmark

Allowable   **Selected**

☐ Population   1.   Select the desired variable(s)
births   2.   from the Allowable list. Hit
                                       the >> button to enter them
                                3.   into the Selected list.
                                4.   Reverse the procedure to
                                       remove items from the
                                5.   Selected list.

☑ Show Numbers On Plots   **Title:** Untitled
☑ Show Grid ☐ Thick Lines
☐ Make 5 Grid Segments

                Min   Max   △ New
Scale: Set   **Page:** 1
                                                            ▽
            From   To
Display: 0   12   Cancel   OK

Note: Analogous operations for Table Pads.   Create as many new graph
                                                      pad pages as you'd like.

A2.12. Dynamite Operations on Graphs and Tables

Dynamite variable name to
clear variable from the graph.

Dynamite here to clear data
from the graph pad page.

Dynamite max or min value to revert to default scale for variable.

Dynamite here to delete page from graph pad.
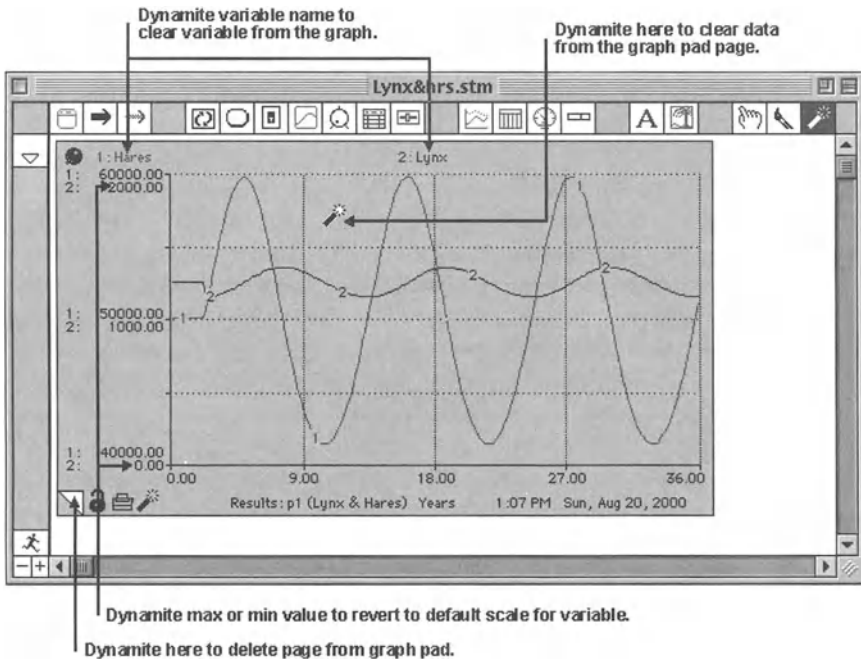
Note: Analogous operations for Table Pads.

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