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Modified Duhamel's Theorem for Variable Coefficient of Convective Surface Heat Transfer

Kan-ichi Hayakawa & Ernesto B. Giannoni-Succar*

Food Science Department, New Jersey Agricultural Experiment Station, Cook College, Rutgers University, PO Box 231, New Brunswick, NJ 08903, USA

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ABSTRACT

Duhamel's theorem estimates the temperature of a heat conduction body exposed to time variable heat exchange medium temperatures (T_a). Since the theorem is not applicable to variable convective, surface heat conductance (h), new general solutions are obtained for variable T_a and h. These solutions are expressed in terms of normalized temperature response functions of a body exposed to step functional medium temperatures and include continuity constants to ensure temperature continuity at each time for a change in h. Sample applications of the theorem are presented for spherical food utilizing a published analytical temperature response function.

NOTATION

a	Characteristic dimension of a sample body or spherical radius (mm)
bo, b1, b2, b3, b(n−3), b(n−2), b(n−1)	Multiples of Δt which represent times of sudden changes in the coefficient of surface heat transfer, Table 1
Bi	$=ha/k$. Biot number
c1, c2, c3 c(n−3), c(n−2), c(n−1)	Multiple of Δt which represents a time range for a constant h value, see Table 1
D	Domain of a body excluding its surface

*Currently with FMC Corporation, Food Ingredients Division, Philadelphia, PA 19103, USA.

Fo	$=\alpha t/a^2$. Fourier number
$G(t)$	Trial function which is dependent on location in a body
h	When h is used without a subscript, it represents the coefficient of convective surface heat transfer applicable to any time range (w/mm^2 C°)
h_x	xth coefficient of convective surface heat transfer (w/mm^2 C°)
k	Summation index or thermal conductivity (w/mm C°)
$\mathbf{n}$	Outward normal vector on body surface (mm)
p	Number of impulses after the last change in the coefficient of surface heat transfer (−)
r	Radial variable (mm)
s	Summation index
S	Surface of a body
t	Time (s)
t_{bx}	Time of the $(x+1)$st change in the coefficient of surface heat transfer. For example, at t_{b1} the coefficient changes from h_1 to h_2. (The first change at t_{bo} from zero to h_1) (s)
t_x	Time of the $(x+1)$st change in heat exchange medium temperature. For example, T_{a1} changes to T_{a2} at t_1 (the first change at t_o from T_o to T_{a1}) (s)
$t^{y,z}_{bm,x}$	Location dependent continuity constant for thermal influence of $(x+1)$st step-functional surrounding medium temperature, occurred in a time range where the yth coefficient of surface heat transfer, h_y, is applicable. This constant is estimated at the time $t_{b(z-1)}$, when the coefficient of surface heat transfer suddenly changes from $(z-1)$st value, h_{z-1}, to the zth vaue, h_z. When y and z are not adjacent integers, the continuity constant should be estimated successively starting with $t^{y,y+1}_{b,mx}$. Namely, one needs to calculate each of $t_{bm,x}$, where $i=1,2,3\ldots,\ z-y$. For example, $t^{1,3}_{bm,o}$ is the continuity constant for the first step functional surrounding medium temperature present in a time range where the first coefficient of surface heat transfer, h_1, is applicable. This constant is estimated when the second coefficient, h_2, was suddenly changed to the third coefficient, h_3. The value of $t^{1,2}_{bm,o}$ is required before estimating the value of $t^{1,3}_{bm,o}$ (s)
T	Temperature (°C)
T_{ax}	Surrounding medium temperature between $(x-1)$st and xth step changes or of xth impulse (°C)
U	$=T-T_o$ (C°)
v	Function used to solve heat conduction equation
$\mathbf{x}$	Location coordinate vector (mm)
α	Thermal diffusivity (mm^2/s)
β	Characteristic root for spherical heat conduction. β_{xs} signifies the sth root for the xth Biot number, see eqn (70)

γ	Expression defined by eqn (69) Γ_{xs} signifies γ for xth Biot number and sth characteristic room
Γ	Expression defined by eqn (69). Γ_{xs} signifies Γ with xth Biot number and sth characteristic roll
Δt	Uniform time interval (s)
ρ	$=r/a$, dimensionless radial variable
ψ_x	Normalized temperature response function for the xth coefficient of surface heat transfer

Subscripts

a	Surrounding heat exchange medium
bm	Appended to t to represent continuity constant
bo + 1, bo − 2, bo + 3, bo + c1	First, second, third and last impulse in time range of h_1, respectively
b1 + 1, b1 + 2, b1 + c2	First, second and last impulse in time range of h_2, respectively
b3 + 1	First impluse in time range of h_3
b(n − 3) + c(n − 2)	Last impulse in time range of h_{n-2}
b(n − 2) + 1, b(n − 2) + c(n − 1)	First and last impulses in time range of h_{n-1}, respectively
b(n − 1) + 1, b(n − 1) + 2	First and second impulses in time range of h_n, respectively
b(n − 1) + p	Last step change in the time range of h_n (related to last temperature in the time variable T_a)
o	Initial value
x	xth value
1, 2, 3	First, second and third values, respectively

INTRODUCTION

Duhamel's theorem has been used to derive an analytical heat conduction solution for a body exposed to time variable medium temperatures using a normalized analytical solution of the same body (Carslaw & Jaeger, 1972). Several researchers (e.g. Hayakawa, 1971, 1972; de Ruyter & Brunet, 1973; Uno & Hayakawa, 1980; Lekwauwa & Hayakawa, 1986) have applied this theorem to estimate the transient state temperature of food subjected to heating or cooling processes.

Food is frequently exposed to different heat exchange media within one heat transfer process. During this process, an overall coefficient for convective heat transfer (h) changes with time. For example, one heat sterilization process usually consists of heating and cooling phases. Packaged food is heated frequently by steam or a steam–air mixture during the heating phase and cooled by water during the cooling phase. Duhamel's theorem cannot be used to estimate product temperature in this case since the theorem assumes constant h (a heat conduction equation with variable h being mathematically nonlinear while the theorem is applicable only to linear equations). The present paper shows a newly derived theorem applicable to heat transfer processes with variable h and variable medium temperatures.

MODIFIED DUHAMEL'S THEOREM

Derivation of a new theorem is presented below starting with a simple case of two changes in the h value (two h values) and of two step-changes in the heat exchange medium temperature, T_a (two medium temperatures). This is followed by a slightly more complex case of three changes in h and three step changes in T_a. Any smooth change in T_a may be approximated by a series of impulses with time variable h. Therefore, the third case is when one impulse in T_a followed by any number of changes in h is considered. The result obtained is then used for the fourth case, a deviant thermal process with a sudden medium temperature drop in the heating phase. The last case is most general with any number of changes in h and with smooth changes in T_a approximated as a series of gates of uniform width.

1. Two step changes in T_a with two changes in h

Assumed changes in T_a and h are shown in Fig. 1(a). There are sudden changes in the medium temperature at times 0 ($=t_o$) and t_{b1}, increasing to T_{a1} from T_o at 0 and decreasing from T_{a1} to T_{a2} at t_1 (the medium temperature being T_o from $-\infty$ to 0). The coefficients of surface heat conductance, h, changes from zero to h_1 at 0 ($=t_{bo}=t_o$) and from h_1 to h_2 at

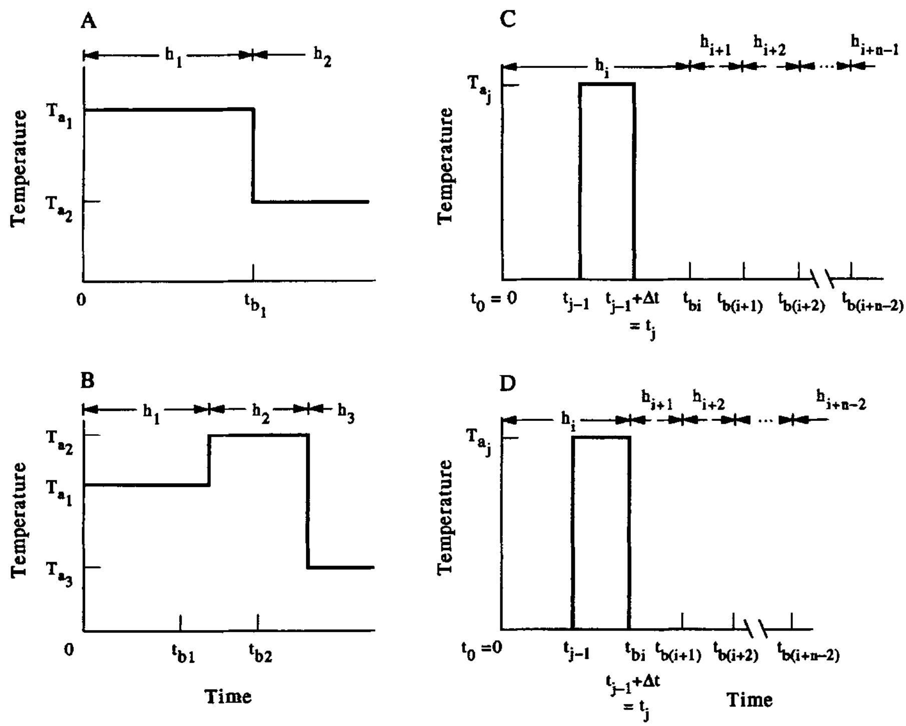

Fig. 1. Assumed changes in heat exchange medium temperature and in coefficient of convective surface heat transfer.

t_{b1} ($t_{b1}=t_1$ and h being zero from $-\infty$ to 0). Any h value may be assumed from $-\infty$ to 0 without loss of the generality, since this does not influence the final results.

Dimensional temperature T is transformed to $U(=T-T_o)$ for convenience. The heat conduction equation and applicable boundary and initial conditions, expressed in U, are:

$$\partial U/\partial t=\alpha\nabla^2 U \quad 0(=t_o=t_{bo})<t \quad \text{and } \mathbf{x}\in D \tag{1}$$

$$k\,U/\partial n=h_1(U-U_{a1}) \quad 0(=t_{bo}) \quad <t<t_{b1}(=t_1) \text{ and } \mathbf{x}\in S \tag{2}$$

$$k\partial U/\partial n=h_2(U-U_{a2}) \quad t_{b1}(=t_1)<t \text{ and } \mathbf{x}\in S \tag{3}$$

$$U=0 \text{ when } t=0 \text{ and } \mathbf{x}\in D\cup S \tag{4}$$

Assume heat conduction function U at any location being a sum of two functions v_1 and v_2.

$$U=v_1+v_2 \tag{5}$$

The functions v_1 and v_2 are the solutions of the following equations:

For v_1:

$$\partial v_1/\partial t=\alpha\nabla^2 v_1 \qquad 0<t \qquad \mathbf{x}\in D \tag{6}$$

$$k\partial v_1/\partial n=h_1(v_1-U_{a1}) \qquad 0<t\leq t_{b1} \qquad \mathbf{x}\in S \tag{7}$$

$$k\partial v_1/\partial n=h_2(v_1-0) \qquad t_{b1}<t \qquad \mathbf{x}\in S \tag{8}$$

$$v_1=0 \qquad t=0 \qquad \mathbf{x}\in D\cup S \tag{9}$$

For v_2:

$$\partial v_2/\partial t=\alpha\nabla^2 v_2 \qquad t_{b1}<t \qquad \mathbf{x}\in D \tag{10}$$

$$k\partial v_2/\partial n=h_2(v_2-U_{a2}) \qquad t_{b1}<t \qquad \mathbf{x}\in S \tag{11}$$

$$v_2=0 \qquad t\leq t_{b1} \qquad \mathbf{x}\in D\cup S \tag{12}$$

It is clear that the sums of corresponding equations [i.e. eqns (6) and (10), (7) and (12), (8) and (11), and (9) and (12)] produce the original eqns (1)–(4) provided that eqn (5) is satisfied. Function v_1 is the temperature response to the thermal environment between 0 and t_{b1}, an impulse represented by a step-up-change from 0 to U_{a1} at the zero time and a step-down-change from U_{a1} to 0 at t_{b1}. Function v_2 is the temperature response to a step functional thermal environment beyond t_{b1}, a step-up-change from 0 to U_{a2}. Note that influence of h_1 exists on v_1 at $t>t_{b1}$ although the thermal environment is removed at t_{b1}, eqn (8).

To solve v_1 and v_2 the following normalized temperature response functions ψ_1 and ψ_2 are used.

For ψ_1:

$$\partial\psi_1/\partial t=\alpha\nabla^2\psi_1 \qquad 0<t \qquad \mathbf{x}\in D \tag{13}$$

$$k\partial\psi_1/\partial n=h_1(\psi_1-1) \qquad 0<t \qquad \mathbf{x}\in S \tag{14}$$

$$\psi_1=0 \qquad t=0 \qquad \mathbf{x}\in D\cup S \tag{15}$$

For ψ_2:

$$\partial\psi_2/\partial t=\alpha\nabla^2\psi_2 \qquad 0<t \qquad \mathbf{x}\in D \tag{16}$$

$$k\partial\psi_2/\partial n=h_2(\psi_2-1) \qquad 0<t \qquad \mathbf{x}\in D \tag{17}$$

$$\psi_2=0 \qquad t=0 \qquad \mathbf{x}\in D\cup S \tag{18}$$

Note that functions ψ_1 and ψ_2 are the normalized functions related to h_1 and h_2, respectively (a body of a zero initial temperature exposed to a heat exchange medium temperature of unity).

When $0<t\leq t_{b1}$, eqns (6), (7) and (9) become identical to those obtained by multiplying both sides of eqns (13), (14) and (15) by U_{a1}. Therefore, one obtains:

$$v_1=U_{a1}\psi_1(t) \quad \text{when } 0<t\leq t_{b1} \tag{19}$$

For brevity, $\psi_1(t)$ is used to imply $\psi_1(t,\mathbf{x})$ and a similar, simplified symbol for ψ_2.

Function v_1 for $t>t_{b1}$ is the temperature response to the surrounding temperature impulse which begins at 0 and ends at t_{b1}. This function may be determined by taking differences of two temperature response functions for these two step functional environmental temperature histories: one beginning at 0 and another beginning at t_{b1}. Therefore, one assumes:

$$v_1=U_{a1}[G(t)-\psi_2(t-t_{b1})] \quad t\geq t_{b1} \tag{20}$$

where $G(t)$ is an unknown, monotonously increasing function. By substituting eqn (20) into eqns (6) and (8), one gets:

$$U_{a1}k\partial[G(t)-\psi_2(t-t_{b1})]/\partial t=U_{a1}\alpha\nabla^2[G(t)-\psi_2(t-t_{b1})] \quad t>t_{b1} \tag{21}$$

$$\begin{aligned} U_{a1}k\partial[G(t)-\psi_2(t-t_{b1})]/\partial n&=U_{a1}h_2[G(t)-\psi_2(t-t_{b1})]\\ &=U_{a1}h_2[\{G(t)-1\}-\{\psi_2(t-t_{b1})-1\}] \quad t>t_{b1} \end{aligned} \tag{22}$$

Since ψ_2 is the solution of eqns (16), (17) and (18), eqns (21) and (22) become as follows:

$$\partial G/\partial t=\alpha\nabla^2 G \qquad t>t_{b1} \qquad \mathbf{x}\in D \tag{23}$$

$$k\partial G/\partial n=h_2(G-1) \qquad t>t_{b1} \qquad \mathbf{x}\in S \tag{24}$$

The values of v_1 estimated by eqns (19) and (20) should be continuous at t_{b1}. Therefore one has:

$$\lim_{t\to t_{b1}}\psi_1(t)=\lim_{t\to t_{b1}}[G(t)-\psi_2(t-t_{b1})]=\lim_{t\to t_{b1}}G(t) \tag{25}$$

Equations (16) and (17) are identical to eqns (23) and (24) except for the applicable time range.

In view of G being a monotonously increasing function which satisfies eqns (23), (24) and (25), G should be nil at a certain value of t. Therefore, t is transformed to t_m which becomes zero at this certain value of t.

$$t_m=t+t_{mx}-t_{b1} \tag{26}$$

Therefore, in terms of the transformed variable t_m, heat conduction around any location is represented by:

$$\partial G/\partial t_m=\alpha\nabla^2 G \qquad t_m>0 \qquad \mathbf{x}\in D$$

$$k\partial H/\partial n=h_2\{G-1\} \qquad t_m>0 \qquad \mathbf{x}\in S$$

$$G=0 \qquad t_m=0 \qquad \mathbf{x}\in D\cup S$$

Since the above equations are identical to eqns (16), (17) and (18), one obtains:

$$G(t_m)=\psi_2(t_m)=\psi_2(t+t_{mx}-t_{b1}) \tag{27}$$

The value of t_{mx} is then estimated by using eqns (25) and (27)

$$\psi_1(t_{b1})=\psi_2(t_{mx})=\psi_2(t^{1,2}_{bm,o}) \tag{28}$$

Note that $t^{1,2}_{bm,o}$ is location dependent since $\psi_1(t_{b1})$ is location dependent.
Therefore

$$G(t_m)=\psi_2(t^{1,2}_{bm,o}+t-t_{b1}) \tag{29}$$

Finally, there is the following solution v_1 using eqns (20) and (29).

$$v_1=U_{a1}[\psi_2(t^{1,2}_{bm,o}+t-t_{b1})-\psi_2(t-t_{b1})] \quad \text{for } t_{b1}<t \tag{30}$$

The solution of eqns (10), (11) and (12) may be easily obtained by comparing them with eqns (16), (17) and (18).

$$V_2=U_{a2}\psi_2(t-t_{b1}) \quad t_{b1}\leq t \tag{31}$$

From eqns (5), (19), (30) and (31), one finally has the following solution for U:

$$U=\begin{cases} U_{a1}\psi_1(t) \quad 0<t\leq t_{b1} & (32a)\\ U_{a1}[\psi_2(t^{1,2}_{bm,o}+t-t_{b1})-\psi_2(t-t_{b1})+U_{a2}\psi_2(t-t_{b1}) \quad t_{b1}\leq t & (32b)\end{cases}$$

where

$$\psi_1(t_{b1})=\psi_2(t^{1,2}_{bm,o}) \tag{32c}$$

The constant $t^{1,2}_{bm,o}$, in eqn (32b) (continuity constant) is required to ensure the temperature continuity at the time of the change in h from h_1 to h_2. This constant may be estimated easily by eqn (32c) because of known temperature response functions ψ_1 and ψ_2 and of the given t_{b1}.

2. Three step changes in T_a with three changes in h

Next is a case for one additional change in h and T_a, Fig. 1(b). Dependent variable T is transformed to U as before.

Food temperatures before the second changes in h, $0<t\leq t_{b1}$, and between the second and third changes in h, $t_{b1}\leq t\leq t_{b2}$, may be estimated by eqns (32a) and (32b), respectively. A solution for estimating food temperature after the third h change, $t\geq t_{b2}$, is derived below.

Heat conduction [eqn (1)] and the initial condition [eqn (4)] are applicable to the present case together with the following boundary conditions:

$$k\partial U/\partial n=h_1(U-U_{a1}) \quad 0(=t_o=t_{bo})\leq t_{b1}(=t_1) \quad \mathbf{x}\in S \tag{33}$$

$$k\partial U/\partial n=h_2(U-U_{a2}) \quad t_{b1}(=t_1)<t\leq t_{b2}(=t_2) \quad \mathbf{x}\in S \tag{34}$$

$$k\partial U/\partial n=h_3(U-U_{a3}) \quad t_{b2}(=t_2)<t \quad \mathbf{x}\in S \tag{35}$$

Normalized temperature response function applicable to h_1, h_2 and h_3 are, respectively, represented by ψ_1, ψ_2 and ψ_3. For example, ψ_3 is the solution of governing equations obtained replacing the subscript '1' of eqn (13)–(15) by '3'.

We assume that the solution U in $t > t_{b2}(=t_2)$ are expressed as the sum of three functions:

$$U = v_1 + v_2 + v_3 \tag{36}$$

These functions are the solutions of the following equations.

For v_1:

$$\begin{array}{lll} kv_1/\partial T = \nabla \alpha \nabla^2 v_1 & t_{bo}(=0) < t & \mathbf{x} \epsilon D \\ kv_1/\partial n = h_1(v_1 - U_{a1}) & t_{bo}(=0) < t \leq t_{b1} & \mathbf{x} \epsilon S \\ kv_1/\partial n = h_2(v_1 - 0) & t_{b1} < t \leq t_{b2} & \mathbf{x} \epsilon S \\ k\partial v_1/\partial n = h_3(v_1 - 0) & t_{b2} < t & \mathbf{x} \epsilon S \\ v_1 = 0 & t_{bo}(=0) & \mathbf{x} \epsilon D \cup S \end{array} \tag{37}$$

For v_2:

$$\begin{array}{lll} \partial v_2/\partial t = \alpha \nabla^2 v_2 & t_{b1} < t & \mathbf{x} \epsilon D \\ k\partial v_2/\partial n = h_2(v_2 - U_{a2}) & t_{b1} < t \leq t_{b2} & \mathbf{x} \epsilon S \\ k\partial v_2/\partial n = h_3(v_2 - 0) & t_{b2} < t & \mathbf{x} \epsilon S \\ V_2 = 0 & t \leq t_{b1} & \mathbf{x} \epsilon D \cup S \end{array} \tag{38}$$

For v_3:

$$\begin{array}{lll} \partial v_3/\partial t = \alpha \nabla^2 v_3 & t_{b2} < t & \mathbf{x} \epsilon D \\ k\partial v_3/\partial m = h_3(v_3 - U_{a3}) & t_{b2} < t \leq t_{b2} & \mathbf{x} \epsilon S \\ v_3 = 0 & t \leq t_{b2} & \mathbf{x} \epsilon D \cup S \end{array} \tag{39}$$

The solution of eqns (37) may be derived through an analysis similar to the one presented previously.

When $t_{b1} < t \leq t_{b2}$, v_1 includes function ψ_2 with one continuity constant applicable at t_{b1}, continuity between ψ_1 and ψ_2. This is identical to eqn (30). When $t_{b2} < t$, v_1 includes ψ_3 with two continuity constants applicable at t_{b2}, continuity between ψ_1 and ψ_3. The solution obtained is:

$$v_1 = U_{a1}[\psi_3(t^{1,3}_{bm,o} + t - t_{b2}) - \psi_3(t^{1,3}_{bm,1} + t - t_{b2})] \quad t > t_{b2} \tag{40}$$

In the above equation, $\psi_3(t)$ is a normalized temperature response function for h_3 [the solution of the equations obtained by replacing h_1 and ψ_1 of eqns (13)–(15) by h_3 and ψ_3, respectively]. Equation (40) represents the thermal response at any time beyond t_{b2} for an impulse that occurred between $0(=t_o)$ and $t_{b1}(=t_1)$.

The value of $t^{1,3}_{bm,o}$ in eqn (40) is a continuity constant related to the thermal influence of the step functional surrounding medium temperature history beginning at the zero time or $t_{bo}(=t_o$, the first step function). This temperature history is indicated by subscript '0' of the continuity constant [zero being equal to 1 (first) -1]. This subscript convention was used since the first step change is usually at a zero time. The step change is within the time range where the first h value, h_1, is applicable. This is signified by superscript '1'. The continuity constant is estimated at the third change in h

(h_2 to h_3) at t_{b2}. This is indicated by superscript '3'. The constant is required to estimate the body temperature in the time range where h_3 is applicable. The constant may be estimated sequentially using the normalized temperature response functions ψ_1, ψ_2 and ψ_3 and using continuity conditions at t_{b1} and t_{b2} as shown below.

$$\psi_1(t_{b1})=\psi_2(t_{bm,o}^{1,2}) \tag{41}$$

$$\psi_2(t_{bm,o}^{1,2}+t_{b2}-t_{b1})=\psi_3(t_{bm,o}^{1,3}) \tag{42}$$

Continuity constant $t_{bm,1}^{1,3}$ is related similarly to the second step-functional medium temperature change at t_1 (subscript $1=2-1$).

$$\psi_1(t_{b1}-t_1)=\psi_2(t_{bm,1}^{1,2})$$

$$\psi_2(t_{bm,1}^{1,2}+t_{b2}-t_{b1})=\psi_3(t_{bm,1}^{1,3})$$

Since the second step change occurs at the end of the h_1 time range, $t_1=t_{b1}$, the first equation becomes $\psi_1(0)=\psi_2(t_{bm,1}^{1,2})$. This equation becomes $1=\psi_2(t_{bm,1}^{1,2})$ because of the initial condition of ψ_1. Furthermore, one obtains that $t_{bm,1}^{1,2}=0$ because of the initial condition of function ψ_2, $\psi_2(0)=1$. The second equation becomes:

$$\psi_2(t_{b2}-t_{b1})=\psi_3(t_{bm,1}^{1,3}) \tag{43}$$

Function v_2 is the thermal response of the impulse, beginning at t_{b1} and ending at t_{b2}, in the time range of h_3. Function v_2 determined through deviations similar to those resulted in eqn (30).

$$v_2=U_{a2}[\psi_3(t_{bm,1}^{2,3}+t-t_{b2})-\psi_3(t-t_{b2})] \quad t>t_{b2} \tag{44}$$

Noting that $t_1=t_{b1}$, one has:

$$\psi_2(t_{b2}-t_{b1})=\psi_3(t_{bm,1}^{2,3}) \tag{44a}$$

Comparing eqns (43) and (44a), one finds:

$$t_{bm,1}^{1,3}=t_{bm,1}^{2,3} \tag{44b}$$

Finally, v_3 is determined through derivations similar to those for eqn (31).

$$v_3=U_{a3}\psi_3(t-t_{b2}) \quad t>t_{b2} \tag{45}$$

Therefore, from eqns (40), (44) and (45), one has:

$$U=U_{a1}[\psi_3(t_{bm,o}^{1,3}+t-t_{b2})-\psi_3(t_{bm,1}^{1,3}+t-t_{b2})]$$
$$+U_{a2}[\psi_3(t_{bm,1}^{2,3}+t-t_{b2})-\psi_3(t-t_{b2})]+U_{a3}\psi_3(t-t_{b2})$$

The above equation is transformed as follows using eqn (44b):

$$U=U_{a1}\psi_3(t_{bm,o}^{1,3}+t-t_{b2})+(U_{a2}-U_{a1})\psi_3(t_{bm,1}^{2,3}+t-t_{b2})$$
$$+(U_{a3}-U_{a2})\psi_3(t-t_{b2}) \tag{46}$$

3. One T_a-impulse and n effective hs

Any curvilinear change in T_a with any number of changes in h may be approximated with a sum of impulses, each of them followed by any number of changes in h. Therefore, one considers an impulse followed by $(n-1)$

changes in h, the n values of h [Fig. 1(c)]. As shown in the figure, the impulse is between t_j and $t_j+\Delta t$ with an applicable h value of h_i and followed by h changes at t_{bi} to h_{i+1}, at $t_{b(i+1)}$ to h_{i+2}, at $t_{b(i+2)}$ to h_{i+3}, ... at $t_{b(i+n-3)}$ to h_{i+n-2}, and at $t_{b(i+n-2)}$ to h_{i+n-1}.

The governing equations are given below.

$$\partial v_m/\partial t=\alpha\nabla^2 v_m \quad x\epsilon D \quad t>t_{j-1} \tag{47}$$

$$\left.\begin{array}{ll} k\partial v_m/\partial n=h_i(v_m-U_{ai}) & t_{j-1}<t\leq t_{j-1}+\Delta t(=t_j) \\ k\partial v_m/\partial n=h_i(v_m-0) & t_j<t\leq t_{bi} \\ k\partial v_m/\partial n=h_{i+1}(v_m-0) & t_{bi}<t\leq t_{b(i+1)} \\ k\partial v_m/\partial n=h_{i+2}(v_m-0) & t_{b(i+1)}<t\leq t_{b(i+2)} \\ \vdots & \\ k\partial v_m/\partial n=h_{i+1}(v_m-0) & t_{b(i+n-2)}<t \end{array}\right\} \tag{48}$$

$$\text{where } \mathbf{x}\epsilon S \quad v_m=0 \quad \mathbf{x}\epsilon S\bigcup S \text{ and } t\leq t_i \tag{49}$$

Without loss of generality, one assumes the impulse is in the time range of the ith value of h. The impulse is defined by the jth step change (step-up) which occurred at t_{j-1} and the $(j+1)$st step change (step-down) which occurred at t_j $(=t_{j-1}+\Delta t)$. This assumption will simplify the application of the solution for v_m to the general case (the fourth case).

The solution of eqn (47) for $t>t_{b(i+n-2)}$ may be determined through an approach similar to the one for deriving eqn (40).

$$v_m=U_{aj}[\psi_{i+n-1}(t_{j-1}^{i,i+n-1}+t-t_{b(i+n-2)})-\psi_{i+n-1}(t_{bm,j}^{i,i+n-1}+t-t_{b(i+n-2)})] \tag{50}$$

The two continuity constants in eqn (50) may be estimated successively using the normalized functions ψ_i, $\psi_{i+1}, \ldots, \psi_{i+n-1}$.

For $t_{bm,j-1}^{i,i+n-1}$:

$$\left.\begin{array}{l} \psi_i(t_{bi}-t_{j-1})=\psi_{i+1}(t_{bm,j-1}^{i,i+1}) \\ \psi_{i+1}(t_{bm,j-1}^{i,j+1}+t_{b(i+1)}-t_{bi})=\psi_{i+2}(t_{bm,j-1}^{i,i+2}) \\ \psi_{i+2}(t_{bm,j-1}^{i,i+2}+t_{b(i+2)}-t_{b(i+1)})=\psi_{i+3}(t_{bm,j-1}^{i,i+3}) \\ \vdots \qquad\qquad\qquad \vdots \\ \psi_{i+n-2}(t_{bm,j-1}^{i,i+n-2}+t_{b(i+n-2)}-t_{b(i+n-3)}=\psi_{i+n-1}(t_{bm,j-1}^{i,i+n-1}) \end{array}\right\} \tag{51}$$

For $t_{bm,j}^{i,i+n-1}$:

$$\psi_i(t_{bi}-t_j)=\psi_{i+1}(t_{bm,j}^{i,i+1}) \tag{52}$$

Other equations for successively estimating $t_{\mathrm{bm},j}^{i,i+n-1}$ are obtained by replacing subscript 'bm,$j-1$' in eqn (51) with 'bm,j'.

When the impulse terminates at t_{bi}, Fig. 1(d), $t_{\mathrm{bm},j}^{i,i+1}$ is equal to zero always as shown below. The left hand side of eqn (52) becomes $\psi_i(0)$ since $t_{\mathrm{b}i}=t_j$. Since $\psi_i(0)=1$ according to the initial condition of ψ_i, one has: $\psi_{i+1}(t_{\mathrm{bm},j}^{i,i+1})=1$. Because of the initial condition of $\psi_{i+1}(\psi_{i+1}(0)=1)$ $t_{\mathrm{bm},j}^{i,i+1}=0$. The remaining continuity constants required to estimate $t_{\mathrm{bm},j}^{i,i+n-1}$ are nonzero.

4. Divided or deviant thermal process

Equations (50)–(52) are applied to derive formulae for estimating temperature responses of conduction heating of food undergoing thermal processes. The first sample application is for a process with one sudden medium temperature change in the heating phase without a change in h (a divided process) and the second for a process with a sudden drop from the holding level of a heating medium temperature followed by a temperature increase to the holding level, the drop due to a malfunctioned temperature control system (a deviant process).

The equations derived previously for the three step T_{a} changes are applicable to the divided process. However, h does not change with the T_{a} change in the heating phase for some processes as assumed previously. Therefore, one assumes as follows for the present sample application. The medium temperature changes from T_{o} to T_{a1} at $t_{\mathrm{o}}(=t_{\mathrm{bo}})$, T_{a1} to T_{a2} at t_1 and T_{a2} to T_{a3} at $t_2(=t_{\mathrm{b1}})$. Additionally, the h value changes from zero to h_1 to $t_{\mathrm{bo}}(=t_{\mathrm{o}})$ and h_1 to h_2 at $t_{\mathrm{b1}}(=t_2)$.

Equations for estimating the temperature response when $t\leq t_{\mathrm{b1}}$ may be derived applying the standard superposition theorem.

$$U=\left\{\begin{array}{ll} U_{\mathrm{a1}}\psi_1(t) & 0\leq t\leq t_1 \\ U_{\mathrm{a1}}\psi_1(t)+(U_{\mathrm{a2}}-U_{\mathrm{a1}})\psi_1(t-t_1) & t_1\leq t\leq t_2 \end{array}\right\} \tag{53}$$

When $t\geq t_2$, the assumed heat exchange medium temperature is divided into two gates located between 0 and t_1 and t_1 and t_2 $(=t_{\mathrm{b1}})$ before the change in h and one step after the change. The following equation is obtained by applying eqn (50) to each gate, noting $i=1$ and $n=2$ for each, and applying an equation similar to eqn (19) to the step.

$$U=U_{\mathrm{a1}}\{\psi_2(t_{\mathrm{bm,o}}^{1,2}+t-t_{\mathrm{b1}})-\psi_2(t_{\mathrm{bm,1}}^{1,2}+t-t_{\mathrm{b1}})\}+U_{\mathrm{a2}}\{\psi_2(t_{\mathrm{bm,1}}^{1,2}+t-t_{\mathrm{b1}}) \\ -\psi_2(t_{\mathrm{bm,2}}^{1,2}+t-t_{\mathrm{b1}})\}+U_{\mathrm{a3}}\psi_2(t-t_{\mathrm{b}}) \tag{54}$$

The continuity constants may be estimated as follows, applying eqns (51) and (52):

$$\left.\begin{array}{ll} t_{\mathrm{bm,o}}^{1,2}: & \psi_1(t_{\mathrm{b1}}-t_{\mathrm{o}})=\psi_1(t_{\mathrm{b1}})=\psi_2(t_{\mathrm{bm,o}}^{1,2}) \\ t_{\mathrm{bm,1}}^{1,2}: & \psi_1(t_{\mathrm{b1}}-t_1)=\psi_2(t_{\mathrm{bm,1}}^{1,2}) \\ t_{\mathrm{bm,2}}^{1,2}: & \psi_1(t_{\mathrm{b1}}-t_2)=\psi_2(t_{\mathrm{bm,2}}^{1,2}) \end{array}\right\} \tag{55}$$

Since $t_{\mathrm{b1}}=t_2$, one finds that $t_{\mathrm{bm,2}}^{1,2}=0$. Therefore, eqn (54) becomes as follows.

$$U=U_{a1}\psi_2(t_{bm,o}^{1,2}+t-t_{b1})+(U_{a2}-U_{a1})\psi_2(t_{bm,1}^{1,2}+t-t_{b1}) \\ +(U_{a3}-U_{a2})\psi_2(t-t_{b1}) \quad (56)$$

Deviant process

An assumed medium temperature history is a step-functional medium temperature increase to T_{a1} at a zero time ($0=t_o=t_{bo}$), a sudden drop to T_{a2} at t_1, an increase to T_{a1} ($=T_{a3}$) at t_2, and a sudden drop to T_{a4} at t_3 ($=t_{b1}$) for ending the heating cycle and starting the cooling cycle. The convective surface heat transfer coefficient up to t_3 is h_1 and beyond t_3 is h_2.

The food temperature (U) up to t_3 may be estimated by applying the standard superposition theorem.

$$U=\begin{cases} U_{a1}\psi_1(t) & 0\leq t\leq t_1 \\ U_{a1}\psi_1(t)+(U_{a2}-U_{a1})\psi_1(t-t_1) & t_1\leq t\leq t_2 \\ U_{a1}\psi_1(t)+(U_{a2}-U_{a1})\psi_1(t-t_1)+(U_{a1}-U_{a2})\psi_1(t-t_2) & \\ & t_2\leq t\leq t_3(=t_{b1}) \end{cases} \quad (57)$$

The temperature response formula, when $t_3\leq t$, is derived through an approach similar to the last example, dividing the assumed medium temperature history into three gates and one step. One obtains the following equation by applying eqn (50) to each gate ($i=1$ and $n=2$ for all gates) and applying an equation similar to eqn (19) to the step:

$$U=U_{a1}\psi_2(t_{bm,o}^{1,2}+t-t_{b1})+(U_{a2}-U_{a1})\psi_2(t_{bm,1}^{1,2}+t-t_{b1}) \\ +(U_{a1}-U_{a2})\psi_2(t_{bm,1}^{1,2}+t-t_{b1})+(U_{a4}-U_{a1})\psi_2(t-t_{b1}) \quad (58)$$

where

$$\left.\begin{aligned} \psi_1(t_{b1}-t_o)&=\psi_2(t_{bm,o}^{1,2}) \\ \psi_1(t_{b1}-t_1)&=\psi_2(t_{bm,1}^{1,2}) \\ \psi_1(t_{b1}-t_2)&=\psi_2(t_{bm,2}^{1,2}) \end{aligned}\right\} \quad (59)$$

The next example assumes a change in h at each heat exchange medium temperature change considered in the above. Symbols representing the times of h changes are different from the above to reflect the assumed h changes. The medium temperature and h change from T_o to T_{a1} and from zero to h_1 at a zero time ($=t_{bo}=t_o$), T_{a1} to T_{a2} and h_1 to h_2 at t_{b1} ($=t_1$), T_{a2} to T_{a1} ($=T_{a3}$) and h_2 to h_1 ($=h_3$) at t_{b2} ($=t_2$), T_{a1} to T_{a4} and h_1 to h_4 at t_{b3} ($=t_3$).

The food temperature between 0 and t_{b1}, t_{b1} and t_{b2}, and t_{b3} and t_{b3} are estimated applying eqns (32a), (32b) and (46), respectively.

$$U=\begin{cases} U_{a1}\psi_1(t) \quad 0\leq t\leq t_{b1}(=t_1) & (60a) \\ U_{a1}\psi_2(t_{bm,o}^{1,2}+t-t_{b1})+(U_{a2}-U_{a1})\psi_2(t-t_{b1}) \quad t_{b1}\leq t\leq t_{b2}(=t_2) & (60b) \\ U_{a1}\{\psi_1(t_{bm,o}^{1,3}+t-t_{b2})-\psi_1(t_{bm,1}^{1,3}+t-t_{b2})\}+U_{a2}\psi_1(t_{bm,1}^{2,3}+t-t_{b2}) & \\ \quad +(U_{a1}-U_{a2})\psi_1(t-t_b) \quad t_{b2}\leq t\leq t_{b3}(=t_3) & (60c) \end{cases}$$

Note that function ψ_1 was used in the third time range [eqn (60c)] since the applicable h is h_1. The continuity constants in eqns (60b) and (60c) may be estimated as follows.

$$t_{\mathrm{bm,o}}^{1,2}: \quad \psi_1(t_{\mathrm{b1}}-t_{\mathrm{o}})=\psi_1(t_{\mathrm{b1}})=\psi_2(t_{\mathrm{bm,o}}^{1,2}) \tag{61a}$$

$$t_{\mathrm{bm,o}}^{1,3}: \quad \psi_2(t_{\mathrm{bm,o}}^{1,2}+t_{\mathrm{b2}}-t_1)=\psi_2(t_{\mathrm{bm,o}}^{1,2}+t_{\mathrm{b2}}-t_{\mathrm{b1}})=\psi_1(t_{\mathrm{bm,o}}^{1,3}) \tag{61b}$$

$$t_{\mathrm{bm,1}}^{1,3}: \quad \psi_1(t_{\mathrm{b1}}-t_1)=\psi_1(t_{\mathrm{b1}}-t_{\mathrm{b1}})=\psi_1(0)=\psi_2(t_{\mathrm{bm,1}}^{1,2})$$

Therefore

$$t_{\mathrm{bm,1}}^{1,2}=0$$

$$\psi_2(t_{\mathrm{bm,1}}^{1,2}+t_{\mathrm{b2}}-t_1)=\psi_2(t_{\mathrm{b2}}-t_{\mathrm{b1}})=\psi_1(t_{\mathrm{bm,1}}^{1,3}) \tag{61c}$$

$$t_{\mathrm{bm,1}}^{2,3}: \quad \psi_2(t_{\mathrm{b2}}-t_1)=\psi_2(t_{\mathrm{b2}}-t_{\mathrm{b1}})=\psi_1(t_{\mathrm{bm,1}}^{2,3}) \tag{61d}$$

In the above equation, t_i and $t_{\mathrm{b}i}$ were used although both are the same in this case.

From eqns (61c) and (61d), one has:

$$t_{\mathrm{bm,1}}^{1,3}=t_{\mathrm{bm,1}}^{2,3} \tag{61e}$$

Therefore, eqn (60c) becomes

$$U=U_{\mathrm{a1}}\psi_1(t_{\mathrm{bm,o}}^{1,3}+t-t_{\mathrm{b2}})+(U_{\mathrm{a2}}-U_{\mathrm{a1}})\psi_1(t_{\mathrm{bm,1}}^{2,3}+t-t_{\mathrm{b2}})+(U_{\mathrm{a1}}-U_{\mathrm{a2}})\psi_1(t-t_{\mathrm{b}})$$

$$t_{\mathrm{b2}}\le t\le t_{\mathrm{b3}}(=t_3) \tag{62}$$

When $t>t_{\mathrm{b3}}$, the food temperature may be estimated through an approach similar to the last example [$i=1$, 2 or 3 and $n=4$ in the general gate equation, eqn (50)].

$$\begin{aligned} U=&U_{\mathrm{a1}}[\psi_4(t_{\mathrm{bm,o}}^{1,4}+t-t_{\mathrm{b3}})-\psi_4(t_{\mathrm{bm,o}}^{1,4}+t-t_{\mathrm{b3}})]+U_{\mathrm{a2}}[\psi_4(t_{\mathrm{bm,1}}^{2,4}+t-t_{\mathrm{b3}}) \\ &-\psi_4(t_{\mathrm{bm,2}}^{2,4}+t-t_{\mathrm{b3}})]+U_{\mathrm{a1}}[\psi_4(t_{\mathrm{bm,2}}^{3,4}+t-t_{\mathrm{b3}})-\psi_4(t_{\mathrm{bm,3}}^{3,4}+t-t_{\mathrm{b3}})] \\ &+U_{\mathrm{a4}}\psi_4(t-t_{\mathrm{b3}}) \end{aligned} \tag{63}$$

The continuity constants in the above equations may be estimated applying eqns (51) and (52).

$$\begin{aligned} t_{\mathrm{bm,o}}^{1,4}: \quad &\psi_1(t_{\mathrm{b1}}-t_{\mathrm{o}})=\psi_1(t_{\mathrm{b1}})=\psi_2(t_{\mathrm{bm,o}}^{1,2}) \\ &\psi_2(t_{\mathrm{bm,o}}^{1,2}+t_{\mathrm{b2}}-t_{\mathrm{b1}})=\psi_1(t_{\mathrm{bm,o}}^{1,3}) \\ &\psi_1(t_{\mathrm{bm,o}}^{1,3}+t_{\mathrm{b3}}-t_{\mathrm{b2}})=\psi_3(t_{\mathrm{bm,o}}^{1,4}) \end{aligned} \tag{64a}$$

$$\begin{aligned} t_{\mathrm{bm,1}}^{1,4}: \quad &\psi_1(t_{\mathrm{b1}}-t_1)=\psi_1(0)=\psi_2(t_{\mathrm{bm,1}}^{1,2}), \text{ thus } t_{\mathrm{bm,1}}^{1,2}=0 \\ &\psi_2(t_{\mathrm{bm,1}}^{1,2}+t_{\mathrm{b2}}-t_{\mathrm{b1}})=\psi_2(t_{\mathrm{b2}}-t_{\mathrm{b1}})=\psi_1(t_{\mathrm{bm,1}}^{1,3}) \\ &\psi_1(t_{\mathrm{bm,1}}^{1,3}+t_{\mathrm{b3}}-t_{\mathrm{b2}})=\psi_4(t_{\mathrm{bm,1}}^{1,3}) \end{aligned} \tag{64b}$$

$$\begin{aligned} t_{\mathrm{bm,1}}^{2,4}: \quad &\psi_2(t_{\mathrm{b2}}+t_1)=\psi_1(t_{\mathrm{bm,1}}^{2,3}) \\ &\psi_1(t_{\mathrm{bm,1}}^{2,3}+t_{\mathrm{b3}}-t_{\mathrm{b2}})=\psi_4(t_{\mathrm{bm,1}}^{2,3}) \end{aligned} \tag{64c}$$

Comparing the second equation for $t_{\mathrm{bm,1}}^{1,4}$ and the first equation for $t_{\mathrm{bm,1}}^{2,4}$ and the first equation for $t_{\mathrm{bm,1}}^{2,4}$, one finds $t_{\mathrm{bm,1}}^{1,3}=t_{\mathrm{bm,1}}^{2,3}$. Therefore, comparing eqns (64b) and (64c), one finds $t^{1,4}_{\mathrm{bm},1}=t^{2,4}_{\mathrm{bm},1}$.

$$t^{2,4}_{\mathrm{bm},2}\text{:}\quad \psi_2(t_{b2}-t_2)=\psi_2(0)=\psi_1(t^{2,3}_{\mathrm{bm},2}),\ \text{thus,}\ t^{2,3}_{\mathrm{bm},2}=0$$

$$\psi_1(t^{2,3}_{\mathrm{bm},2}+t_{b3}-t_{b2})=\psi_1(t_{b3}-t_{b2})=\psi_4(t^{2,4}_{\mathrm{bm},2}) \tag{65a}$$

$$t^{3,4}_{\mathrm{bm},2}\text{:}\quad \psi_1(t_{b3}-t_2)=\psi_4(t^{3,4}_{\mathrm{bm},2}) \tag{65b}$$

Comparing eqns (65a) and (65b), one finds $t^{2,4}_{\mathrm{bm},2}=t^{3,4}_{\mathrm{bm},2}$.

$$t^{3,4}_{\mathrm{bm},3}\text{:}\quad \psi_1(t_{b3}-t_3)=\psi_1(0)=\psi_4(t^{3,4}_{\mathrm{bm},3}) \tag{65c}$$

Thus, $t^{3,4}_{\mathrm{bm},3}=0$

Equation (63) becomes as follows applying the continuity constant equality found above ($t^{1,4}_{\mathrm{bm},1}=t^{2,4}_{\mathrm{bm},1}$ and $t^{2,4}_{\mathrm{bm},2}=t^{3,4}_{\mathrm{bm},2}$).

TABLE 1
Assumed Time Variable Surrounding Medium Temperature and Surface Heat Conductance

Time range[a]	*Medium temp.*	*Applicable h*	*Normalized function*
bo ~ bo + 1	$U_{a(bo+1)}$	h_1	ψ_1
bo + 1 ~ bo + 2	$U_{a(bo+2)}$		
bo + 2 ~ bo + 3	$U_{a(bo+3)}$		
⋮	⋮		
bo + c1 ~ b1 = bo + c1	$U_{a(bo+c1)}$		
b1 ~ b1 + 1	$U_{a(b1+1)}$	h_2	ψ_2
b1 + 1 ~ b1 + 2	$U_{a(b1+2)}$		
⋮	⋮		
b − 1 + c2 − 1 ~ b2 = b1 + c2	$U_{a(b1+c2)}$		
b2 ~ b2 + 1	$U_{a(b2+1)}$	h_3	ψ_3
b2 + 1 ~ b2 + 2	$U_{a(b2+2)}$		
⋮	⋮		
b2 + c3 − 1 ~ b3 = b2 + c3	$U_{a(b2+c3)}$		
b3 ~ b3 + 1	$U_{a(b3+1)}$	h_4	ψ_4
⋮	⋮	⋮	⋮
b(n − 3) + c(n − 2) − 1 ~ b(n − 2)[b]	$U_{a(b(n-3)+c(n-2))}$	h_{n-2}	ψ_{n-2}
b(n − 2) ~ b(n − 2) + 1	$U_{a(b(n-2)+1)}$	h_{n-1}	ψ_{n-1}
⋮	⋮		
b(n − 2) + c(n − 1) − 1 ~ b(n − 1)[c]	$U_{a(b(n-2)+c(n-1))}$		
b(n − 1) ~ b(n − 1) + 1	$U_{a(b(n-1)+1)}$	h_n	ψ_n
	$U_{a(b(n-1)+2)}$		
⋮	⋮		
b(n − 1) + p − 1 ~ b(n − 1) + p	$U_{a(b(n-1)+p)}$		

[a] *Multiples of* Δt. Generally, bo = 0.
[b] b(n − 2) = b(n − 3) + c(n − 2).
[c] b(n − 1) = b(n − 2) + c(n − 1).

$$U = U_{a1}\psi_4(t^{1,4}_{bm,o} + t - t_{b3}) + (U_{a2} - U_{a1})\psi_4(t^{2,4}_{bm,1} + t - t_{b3})$$
$$+ (U_{a1} - U_{a2})\psi_4(t^{3,4}_{bm,2} + t - t_{b3}) + (U_{a4} - U_{a1})\psi_4(t - t_{b3}) \qquad (66)$$

5. Time variable T_a and n effective hs

The last case is for a body exposed to time variable surrounding medium temperature with any number of changes in the surface heat conductance as summarized in Table 1. The medium temperature history between (bo)Δt and (bo+c1)Δt is approximated by (c1+1) step changes at uniform time intervals, the first one being (bo)Δt (normally 'bo' being zero). This history is mathematically equal to the sum of c1 impulses of the same widths and of different heights. The h value and normalized response function in this time range are h_1 and ψ_1, respectively. The medium temperature histories in other time ranges are approximated in a similar way and n different surface coefficients and response functions are assigned to these ranges as shown in the table (e.g. h changed at b1$\cdot\Delta t$, b2Δt,... b$(n-2)\cdot\Delta t$, and b$(n-1)\cdot\Delta t$).

The temperature response of the body at (b$(n-1)+p)\Delta t$ may be estimated by the repeated use of eqn (50) and a published equation (Hayakawa, 1971) as shown below:

$$U = \sum_{s=0}^{n-2} \sum_{k=1}^{c(s+1)} U_{a(bs+k)}[\psi_n(t^{s+1,n}_{bm,b(k-1)} + p\Delta t) - \psi_n(t^{s+1,n}_{bm,b(s+k)} + p\Delta t)]$$

$$+ \sum_{k=1}^{p-1} U_{a(b(n-1)+k)}[\psi_n((p-k+1)\Delta t) - \psi_n((p-k)\Delta t)]$$

$$+ U_{a(b(n-1)+p)} \cdot \psi_n(\Delta t) \qquad (67)$$

The upper limit of the inner summation of the first term, c$(s+1)$, is related to the number of impulses within the time range for each h. It should be noted that the parenthesis in the expression c$(s+1)$ should be removed when $s+1$ is a definite integer. For example, when $s=1$, it is c2. The same should be practiced with the subscripts of U_a, i.e. (b$s+k$), and of continuity constants, i.e. b$(k-1)$ and b$(s+k)$. The upper limit of the outer summation of the same group is related to the number of changes in h. The continuity constants may be determined by applying eqns (51) and (52). For example, $t^{1,n}_{bm,bo}$ (the first constant when $s=0$ and $k=1$) may be estimated as follows.

$$\psi_1((c1)\Delta t) = \psi_2(t^{1,2}_{bm,bo})$$
$$\psi_2(t^{1,2}_{bm,bo} + (c2)\Delta t) = \psi_3(t^{1,3}_{bm,bo})$$
$$\vdots$$
$$\psi_{n-1}(t^{1,n-1}_{bm,bo} + (c(n-1))\Delta t) = \psi_n(t^{1,n}_{bm,bo})$$

The second summation series and the last term in eqn (61) are related to responses from the surrounding medium temperature history in $t > (b(n-1))\Delta t$ and were obtained by applying a method developed previously (Hayakawa, 1971). Because of no change in the heat conductance beyond $(b(n-1))\Delta t$, no continuity constant is included in these expressions. They become identical to Duhamel's theorem integration when Δt approaches zero (Carslaw & Jaeger, 1972). However, the first term cannot be reduced to a simple integration because of the continuity constants.

DISCUSSION

Integral transforms (e.g. Laplace transform) have been used to derive temperature response solutions for the time variable heat exchange medium temperature (Hayakawa & Ball, 1971). However, this is not applicable to a problem of time variable h since the problem becomes mathematically nonlinear (all integral transformation methods are applicable only to linear equations). Therefore, the equations given above will provide an invaluable means for deriving analytical solutions for heat conduction with time variable h.

Two sample applications of the above derived equations are given below. One example is for the thermal processing of a spherical food approximated by two changes in T_a and h (two values each of T_a and h). The first is for a heating medium and the second for a cooling medium. Another example is for the thermal processing of the same food with three changes in T_a and h (one additional change in the heating medium T_a and h before cooling).

The normalized temperature response function of a sphere, which is expressed in terms of Biot number, Bi instead of h ($Bi=ha/k$) is used for both sample applications.

$$\psi_x=1-(2Bi_x/\rho)\sum_{s=1}^{\infty}\Gamma_{xs}\exp(-\gamma_{xs}t) \tag{68}$$

where $Bi=h_xa/l$ $\gamma_{xs}=\alpha\beta_{xs}^2/a^2$

$$\Gamma_{xS}=[\beta_{xs}^2+(Bi_x^2-1)^2]\sin\beta_{xs}\,sin(\rho\beta_{xs})/\{\beta_{xs}^2[\beta_{xs}^2+Bi_x(Bi_x-1)]\} \tag{69}$$

$$\beta_{xs}\cot\beta_{xs}+Bi_x-1=0 \tag{70}$$

Using eqn (32a) and noting $U=T-T_o$, one obtains eqn (71) for estimating the temperature response of a spherical food during heating.

$$T=T_{a1}-(T_{a1}-T_o)(2Bi_1/\rho)\ \Gamma_{1s}\exp(-\gamma_{1s}t)\ \text{ for }\ 0<t\leq t_{b1} \tag{71}$$

Equation (72) is used to estimate the temperature response during the cooling obtained using eqn (32b).

$$T=T_{a2}+(2Bi_2/\rho)(T_{a1}-T_{a2})\sum_{s=1}^{\infty}\Gamma_{2s}\exp[-\gamma_{2s}(t-t_{b1})]$$

$$-(2Bi_2/\rho)(T_{a1}-T_o)\sum_{s=1}^{\infty}\Gamma_{2s}\exp[-\gamma_{2s}(t_{bm,o}^{1,2}+t-t_{b1})]$$

$$\text{for }\ t_{b1}\leq t \tag{72}$$

The continuity constant in eqn (72) is determined using eqns (32c) and (68).

$$Bi_1 \sum_{s=1}^{\infty} \Gamma_{1s} \exp(-\gamma_{1s} t_{b1}) = Bi_2 \sum_{s=1}^{\infty} \Gamma_{2s} \exp(-\gamma_{2s} t_{bm,o}^{1,2}) \tag{73}$$

Since Γ_{1s} and Γ_{2s} are dependent on ρ, $t_{bm,o}^{1,2}$ is location dependent as stated previously.

For a moderately large t_{b1} values, the left side of eqn (73) may be approximated by the first term of the summation series. Additionally assuming the first term approximation of the right side, one obtains:

$$t_{bm,o}^{1,2} = (1/\beta_{21}^2)\{\beta_{11} t_{b1}^2 + (a^2/\alpha)\ln[h_2\Gamma_{21}/(h_1\Gamma_{11})]\} \tag{74}$$

The assumption of the first term approximation should be validated estimating values of the first and second terms of the series using the estimated $t_{bm,o}^{1,2}$. If the second terms are not negligible, compared to the respective first terms, the constant should be recalculated using eqn (73).

A temperature response chart of a spherical body (Schneider, 1963) provides values of the normalized temperature response function at the spherical center. Therefore, this chart simplifies the application of eqns (32a), (32b) and (32c). As an example, one estimates $t_{bm,o}^{1,2}$, using the response chart, for water cooking of a 10 mm radius spherical food followed by air cooling. The assumed k and α of the food are $5{\cdot}00 \times 10^{-4}$ W/(mm C) and $0{\cdot}128$ mm^2/s, respectively. The h values during water cooking (h_1) and air cooling (h_2) are $1{\cdot}42 \times 10^{-3}$ and $7{\cdot}38 \times 10^{-6}$ W/(mm^{2}C), respectively. The values of Bi for the cooking and cooling processes are then 28·2 (Bi_1) and 0·15 (Bi_2), respectively ($Bi = ha/k$).

For an assumed water cooking time (t_{b1}) of 100 s, the value of the Fourier number, Fo, is 0·128 ($Fo = \alpha t/a^2$). The value of the temperature response function ψ_1 obtained from the chart for the Bi_1 and Fo values is 0·43. The Fo value corresponding to this response value, 0·43, is 1·65 for a Bi_2 value of 0·150. Therefore, one obtains $t_{bm,o}^{1,2}$ as follows using the definition of Fo.

$$t_{bm,o}^{1,2} = Fo\,a^2/\alpha = 1{\cdot}65 \times 10^2/0{\cdot}128 = 1290 \text{ s}$$

Because h_2 is much smaller than h_1, the continuity value is much larger than t_{b1}. Similarly, $t_{bm,o}^{1,2}$ values for assumed t_{b1} values of 200 and 300 s are 3710 and 6250 s, respectively.

One obtains eqn (75) when the published superposition theorem is applied incorrectly to the present problem (removing T_{a1} environment for $t \geq t_{b1}$, in the h_2 time range, and adding T_{a2} environment in the same time range).

$$= T_{a2} + \frac{2Bi_2}{\rho}(T_{a1} - T_{b2}) \sum_{s=1}^{\infty} \Gamma_{2s} \exp[-\gamma_{2c}(t - t_{b1})]$$

$$- \frac{2Bi_2}{\rho}(T_{a1} - T_o) \sum_{s=1}^{\infty} \Gamma_{2s} \exp(-\gamma_{2s} t) \tag{75}$$

Temperature discontinuities at t_{b1} are estimated, using eqn (75), for the processing of spherical food considered above using the same temperature response chart. The central temperature at the end of 100 s water cooking (t_{b1}) is estimated, using eqn (71), as 44·4°C. The central temperature at the beginning of the air cooling (t_{b1}) is 11·2°C when eqn (75) is used. The discontinuity is −33·2°C. No discontinuity is obtained when eqn (72) is applied. Similarly, the temperatures at the end of cooking and beginning of cooling, when $t_{b1}=200$ s, are 76·0 and 14·0°C, respectively (−62·0°C discontinuity). The discontinuity is −68·8°C when $t_{b1}=300$ s. This clearly shows considerable errors when the existing superposition theorem is applied to a thermal process of variable h.

For a problem of three changes in each of T_a and h, eqn (64) estimates the food temperature when $0 \leq t \leq t_{b1}$ and eqn (72) when $t_{b1} \leq t \leq t_{b2}$. Equation (46) is applied when $t_{b2} \leq t$. From this equation, one has:

$$U = U_{a1}\psi_3(t_{bm,o}^{1,3}+t-t_{b2}) + (U_{a2}-U_{a1})\psi_3(t_{bm,1}^{2,3}+t-t_{b2}) + (U_{a3}-U_{a2})\psi_3(t-t_{b2}) \quad t \geq t_{b2} \tag{76}$$

Substituting eqn (68) into eqn (76), one obtains:

$$\begin{aligned} T = T_{a3} &+ (T_{a1}-T_{a2})(2Bi_3/\rho) \sum_{s=1}^{\infty} \Gamma_{3s} \exp[-\gamma_{3s}(t_{bm,1}^{2,3}+t-t_{b2})] \\ &+ (T_a - T_{a3})(2Bi_3)/\rho \sum_{s=1}^{\infty} \Gamma_{3s} \exp[-\gamma_{3s}(t-t_{b2})] \\ &- (T_{a1}-T_o)(2Bi_3/\rho) \sum_{s=1}^{\infty} \Gamma_{3s} \exp[-\gamma_{3s}(t_{bm,o}^{1,3}+t-t_{b2})] \\ & \qquad t \geq t_{b2} \end{aligned} \tag{77}$$

The continuity constants in eqn (77) are determined using eqns (41)–(43).

$$Bi_2 \sum_{s=1}^{\infty} \Gamma_{2s} \exp[-\gamma_{2s}(t_{bm,o}^{1,2}+t_{b2}-t_{b1})] = Bi_3 \sum_{s=1}^{\infty} \Gamma_{3s} \exp[-\gamma_{3s}(t_{bm,o}^{1,3})] \tag{78}$$

$$Bi_2 \sum_{n=1}^{\infty} \Gamma_{2s} \exp[-\gamma_{2s}(t_{b2}-t_{b1})] = Bi_3 \sum_{n=1}^{\infty} \Gamma_{3s} \exp[-\gamma_{3s} t_{bm,1}^{2,3}] \tag{79}$$

Constant $t_{bm,o}^{1,2}$ in eqn (78) is determined using eqn (73). The published spherical temperature response chart simplifies applications of eqns (77), (78) and (79) as shown above.

Analytical equations for the bodies of other, simple shapes may be obtained through similar derivations using published analytical solutions. The body shapes of available analytical solutions include an infinite plate, infinite cylinder, infinite rectangular column, finite cylinder, circular cone, and rectangular parallelepiped (Carslaw & Jaeger, 1972).

The shapes of many foods are irregular. The theorems derived above are not applicable to these foods because there are no analytical solutions available. In this case, empirical temperature response functions may be used to apply the theorems. This will be presented in a future paper.

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