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Measuring thermal properties with the parallel wire method: a comparison of mathematical models

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Abstract—The accuracy of thermal conductivity and diffusivity measurements using the parallel wire method depends on the mathematical model used. This work presents an analytical solution of a model that takes into account the probe radius and the thermal contact resistance between the material and the probe. The model is compared to the classical hot wire model on the basis of the acquired experimental data. Copyright © 1996 Elsevier Science Ltd.

INTRODUCTION

Among the transient techniques commonly used to measure the thermal properties of a wide range of materials, the hot wire method offers the dual advantage of speed (on the order of a few minutes) and not mandating large-size specimens. It requires recording of the punctual time evolution of the temperature and the spatial position of the measuring point and, in addition, the thermal power assigned to the wire must be known.

The acquired data can be analyzed using various mathematical models that are capable of providing the thermal conductivity and diffusivity values. Owing to its notable effects on data precision, computational effort, and time requirements, the mathematical model must be carefully chosen. In this work, a new mathematical model is presented and compared with the conventional hot wire model. The reliability and usefulness of the two models in calculating thermal properties using the same experimental data are evaluated and compared.

THE MATHEMATICAL MODELS

In all the models, the material was assumed isotropic with two-dimensional conduction in the cylindrical coordinates and the z axis coincident with the heating wire. The temperature was assumed constant and uniform initially, when constant for radius approaching infinite. With these hypotheses, a geometrical straight line heated with power q per unit of length causes the following rise in temperature at radius r and time t [1]:

$$\Theta(r,t) = \frac{q}{4\pi\lambda}\int_{r^2/4at}^{\infty}\frac{\mathrm{e}^{-u}}{u}\,\mathrm{d}u = -\frac{q}{4\pi\lambda}Ei\left(-\frac{r^2}{4at}\right). \quad (1)$$

Defining the Fourier number as $Fo = at/r^2$ for high values (i.e. $Fo > 10$ [1, 2]), equation (1) is usually written as

$$\Theta(r,t) = (q/4\pi\lambda)[\ln(4Fo/C) + O(1/Fo)] \quad (2)$$

with an approximation of $O(1/Fo)$, where C is a constant whose logarithm gives Euler constant γ. It should be stressed that equations (1) and (2) work only with $r > 0$. Since high Fo values can easily be attained with low values of r, equation (2) is useful only for real wire temperatures. With logarithm properties

$$\Theta(r_b,t) - \Theta(r_b,t_0) = (q/4\pi\lambda)\ln(t/t_0), \quad (3)$$

where r_b is the actual wire radius. Obviously, only thermal conductivity can be measured in this way.

Blackwell [3] provides a more complex model which accounts for the wire's mass and thermal contact resistance to the sample, assuming infinite thermal conductivity for the wire. His solution, which is valid only for high values of Fo with wire radius r_b (Fo_b), is

$$\Theta(r,t) = \frac{q}{4\pi\lambda}\left[\ln\left(\frac{4Fo_b}{C}\right) + \frac{2}{Bi} + \frac{1}{2Fo_b}\right]\left\{\ln\left(\frac{4Fo_b}{C}\right) + 1 - \frac{\alpha k_b}{\alpha_b\lambda}\left[\ln\left(\frac{4Fo_b}{C}\right) + \frac{2}{Bi}\right] + O\left(\frac{1}{Fo_b}\right)^2\right\}. \quad (4)$$

Using Carslaw and Jaeger's analytical solution for a physical model that takes into account the wire mass [1], Håkansson *et al.* [4] and Pettersson [5] show how diffusivity can be calculated along with thermal conductivity. However, their method requires lengthy calculation times and does not include the thermal resistance between the wire and the sample. Davis *et al.* [6] recommend the 'parallel wire' configuration, in which the thermocouple is located parallel to the heating wire to avoid perturbation in the cylindrical field.

NOMENCLATURE

a	thermal diffusivity [$m^2\ s^{-1}$]
c	specific heat [$J\ kg\ K^{-1}$]
C	constant
Ei	integral-exponential function ($Ei(x) = \int_x^{\infty} (e^{-u}/u)\,du$)
Fo	Fourier number $= at/r^2$
g^2	p/a
I_0	first-type modified Bessel function, zero order
K_0	second-type modified Bessel function, zero order
K_1	second-type modified Bessel function, first order
m	mass [kg]
p	Laplace parameter
q	power per unit of length [$W\ m^{-1}$]
q'	$= q/2\pi r_b$ [$W\ m^{-2}$]
r	radius [m]
R	thermal resistance [$m^2\ K\ W^{-1}$]
t	time [s]
z	$= r_b^2 c^2/4a$.

Subscripts

b	probe.

Greek symbols

α	$= m_b c_b/\pi r_b$ [$J\ K\ m^{-1}$]
γ	Euler number
Θ	temperature [K]
λ	thermal conductivity [$W\ K\ m^{-1}$]
ρ	density [$kg\ m^{-3}$]
σ	standard deviation.

Despite requiring a significant increase in calculation time, Laurent's method [7] allows the taking into account of the mass of the wire.

Very thin wire is useful in reducing mass, with the wire and thermocouple positions maintained by the specimen itself. In addition, it is useful in easily obtaining high Fourier numbers and avoiding end effects which become negligible when the ratio of the heating wire's length to diameter exceeds 100 [9].

In Fig. 1, illustrating the thermal resistance effects, the results of equation (1) are compared with those of a numerical program in which finite elements are used to solve the energy balance according to Blackwell's physical model. The exponential integral Ei in equation (1) is evaluated using the polynomials provided by Abramowitz [8]. The wire temperature is assumed as the temperature calculated at a radius equal to that of the heating wire. It can be readily seen that the main influence at work is the curve translation.

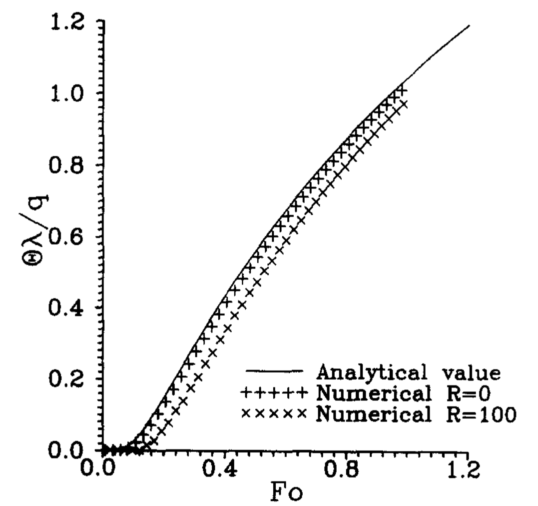

Fig. 1. Nondimensional temperature variation at distance r from the heating wire for two models and different values of thermal resistance R [$m^2\ K\ W^{-1}$].

Let us now consider the solution at $r > r_b$. The Blackwell model describes a probe with radius b, thermal conductivity λ_b, mass m_b per unit of length, specific heat c_b, and boundary surface temperature Θ_b. The material is assumed infinite, with thermal conductivity λ and thermal diffusivity a. In evaluating the temperature variation at radius r from the probe axis, it can be assumed that the probe thermal conductivity λ_b is infinite owing to the smallness of the probe radius r_b in relation to that of the thermocouple location. Hence, the following differential equation expresses the heat transfer in the system:

$$\frac{\partial^2\Theta}{\partial r^2} + \frac{1}{r}\frac{\partial\Theta}{\partial r} = \frac{1}{a}\frac{\partial\Theta}{\partial t} \quad \text{with} \quad \begin{cases} b < r < \infty \\ t > 0 \end{cases} \tag{5}$$

$$-\lambda\frac{\partial\Theta}{\partial r} = \frac{(\Theta_b - \Theta)}{R} \quad \text{with} \quad \begin{cases} r = r_b \\ t > 0 \end{cases} \tag{6}$$

$$-\lambda\frac{\partial\Theta}{\partial r}(2\pi r_b) = q - m_b c_b \frac{\partial\Theta_b}{\partial t} \quad \text{with} \quad \begin{cases} r = r_b \\ t > 0 \end{cases}, \tag{7}$$

where R is the thermal resistance calculated in $r = r_b$. The boundary conditions are

$$\Theta_b(t = 0) = \Theta(t = 0) = 0$$

$$\lim_{r\to\infty} \Theta(r) = 0.$$

If the product $m_b c_b$ in equation (7) is assumed constant so that $\alpha = m_b c_b/\pi r_b$, the result is

$$-\lambda\frac{\partial\Theta}{\partial r} = q' - \frac{\alpha}{2}\frac{\partial\Theta_b}{\partial t} \quad \text{with} \quad \begin{cases} r = r_b \\ t > 0 \end{cases}, \tag{8}$$

where $q' = q/2\pi r_b$. Using the Laplace transformer solving method with parameter p, equations (5), (6) and (8) give

$$\frac{d^2\Theta}{dr^2}+\frac{1}{r}\frac{d\Theta}{dr}=g^2\Theta \quad (9)$$

$$-\lambda\frac{d\Theta}{dr}=\frac{(\Theta_b-\Theta)}{R} \quad (10)$$

$$-\lambda\frac{d\Theta}{dr}=\frac{q'}{p}-\frac{\alpha p}{2}\Theta_b, \quad (11)$$

where $g^2 = p/a$.

Equation (12), the properties of the modified Bessel function, and the boundary conditions can be used to find solutions Θ having the analytical form

$$\Theta = C_1 I_0(rg)+C_2K_0(rg), \quad (12)$$

where I_0 is the zero order of the first-type Bessel function, K_0 is the zero order of the second-type modified Bessel function, and C_1 and C_2 are constants.

Solving the three-equation system [(9)–(11)] gives the following solutions [3]:

$$\Theta(r_b,p)=\frac{2r_bq'\Delta}{p(r_b\alpha p\Delta+2\lambda K_1(gr_b))} \quad (13)$$

$$\Theta(r,p)=\frac{2r_bq'K_0(gr)}{pgr_b(r_b\alpha p\Delta+2\lambda K_1(gr_b))}, \quad (14)$$

where

$$\Delta = K_0(gr_b)/gr_b+(\lambda R/r_b)K_1(gr_b).$$

Blackwell gave equation (4) as the approximate solution to equation (13). The method, as illustrated in Appendix A, was applied to equation (14) to evaluate temperature variations at distance r from the hot wire with the following results:

$$\Theta(r,t)=-\frac{q}{4\pi\lambda}Ei\left(-\frac{1}{4Fo}\right)+\frac{q}{4\pi\lambda}\frac{1}{4Fo}\exp\left(-\frac{1}{4Fo}\right)\times\left[1-\ln(4Fo)-4\frac{\rho_bc_b}{\rho c}\left(\frac{2\lambda R}{b}-\ln(4Fo)\right)\right]. \quad (15)$$

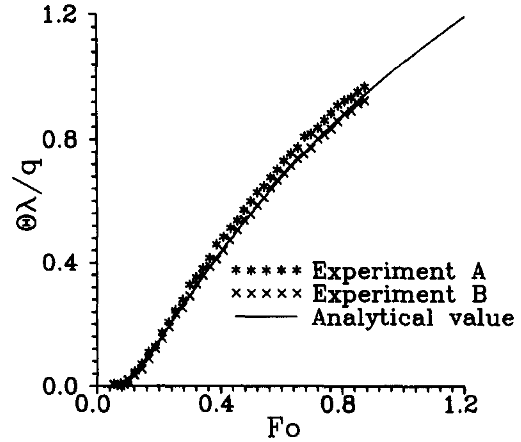

Fig. 3. Nondimensional temperature variation vs Fo at the distance $r = 20.1$ mm for polystyrene; analytical [equation (1)] and two sets of data (experiment A and experiment B).

THE TEST RIG

The test rig is schematically illustrated in Fig. 2. A Teflon-insulated constantan wire with a diameter of 0.075 mm was used for the line source. The wire was combined with an enamelled copper wire of the same diameter to produce the thermocouple for measuring temperature variations. The initial sampling condition was obtained at the reference junction and room temperature. The thermocouple voltage, measured by an HP-3478A multimeter with a resolution of 100 nV, was 0.036 mV K^{-1} in the measuring range. d.c. heating current was provided by a Philips PE1537 stabilized power supply.

An ammeter was used to control the current intensity during measurement of the wire's electrical resistance. Time and voltage were measured by a personal computer, which also provided the initial measurement point.

Two polystyrene parallelepipeds measuring $50\times140\times300$ mm and two rubber parallelepipeds measuring $30\times140\times310$ mm were used as specimens. The density of the materials was measured in relation to water using a scale with an accuracy of 0.0001 g. The wire and thermocouple were sandwiched in between the elements, with contact maximized by loading the specimens with pieces of iron of known weight. The distance between the heating wire and the thermocouple varied according to the specimen material. Figures 3 and 4 show the analytical evaluation of nondimensional temperatures vs Fo in comparison with the experimental data as provided by equation (1). According to ref. [10], the pressure has no effect in the range considered.

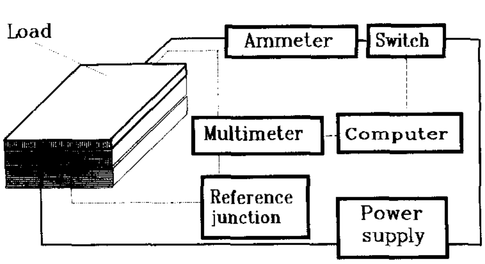

Fig. 2. Schematic of the test rig.

THE EVALUATION OF THE PARAMETERS

The solutions provided by equations (1) and (15) were compared. The parallel wire enabled evaluation of the thermal diffusivity with the ratio between temperatures evaluated at different times [6]

$$\frac{\Theta(r,t)}{\Theta(r,t_0)}=\frac{Ei(-r^2/4at)}{Ei(-r^2/4at_0)} \quad (16)$$

and fitting the experimental data by the least-squares

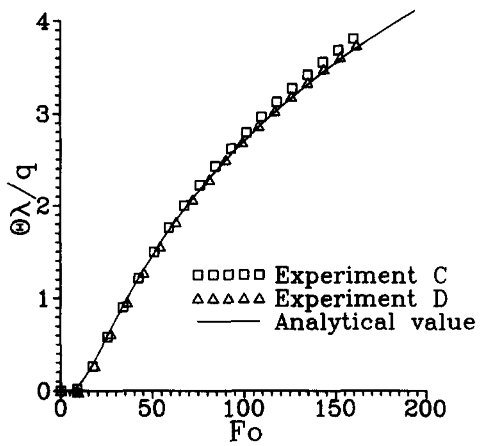

Fig. 4. Nondimensional temperature variation vs *Fo* at distance $r = 4.5$ mm for rubber; analytical [equation (1)] and two sets of data (experiment C and experiment D).

Table 1. Values of the test materials' thermal properties according to ref. [11]

	Material	
	Rubber	Polystyrene
a 10^7 [m^2 s^{-1}]	0.9954	6.424
λ [W m^{-1} K^{-1}]	0.163	0.029
ρ [kg m^{-3}]	1150	29–56
c [J kg^{-1} K^{-1}]	1424	1220

Table 2. Mean values of the thermal properties obtained for the rubber specimen in five trials

	Model	
	Hot wire equations (1) and (16)	Proposed equation (15)
$a \pm \sigma$ 10^7 [m^2 s^{-1}]	2.1052 ± 0.0005	2.0092 ± 0.0080
$\lambda \pm \sigma$ [W m^{-1} K^{-1}]	0.2838 ± 0.0005	0.2999 ± 0.0148
$\rho \pm \sigma$ [kg m^{-3}]	1170 ± 13	1170 ± 13
$c \pm \sigma$ [J kg^{-1} K^{-1}]	1152 ± 15	1276 ± 82

method. Once a was determined, the thermal conductivity could be evaluated by the least-squares method with equation (1). In the proposed model, parameters a, λ and R were obtained in equation (15) by fitting the data by a nonlinear approximated method. The variables producing the χ^2 minimum starting from a random value in the trial range were determined with the algorithm illustrated in Appendix B. The literature values for the test materials are shown in Table 1.

Tables 2 and 3 show the results obtained at room temperature ($\approx 21°$C).

RESULTS

When applied to the same set of data, the analytical solutions of the classical hot wire and Blackwell physical models give very close results. Tables 2 and 3 show that the standard deviation due to trial repetition is very similar. While greater σ for a was attained with the hot wire method for the polystyrene specimen, it is of the same order as that obtained by the proposed method. At the same time, Figs 3 and 4 show a good fitting of equation (1) with the experimental data. In other words, the precision attainable with the Blackwell model is comparable or less than that provided by the simpler classical hot wire model, probably because the analytical solution requires numerous approximations to use the Laplace transform. Also, the nonlinear method, which must be used to fit data, requires a much greater computation effort than that required by linear methods and, in addition, does not always produce convergence.

Table 3. Mean values of the thermal properties obtained for the polystyrene specimen in five trials

	Model	
	Hot wire equations (1) and (16)	Proposed equation (15)
$a \pm \sigma$ 10^7 [m^2 s^{-1}]	8.41 ± 0.45	8.03 ± 0.29
$\lambda \pm \sigma$ [W m^{-1} K^{-1}]	0.0288 ± 0.0007	0.0353 ± 0.0051
$\rho \pm \sigma$ [kg m^{-3}]	37 ± 2	37 ± 2
$c \pm \sigma$ [J kg^{-1} K^{-1}]	926 ± 123	1189 ± 281

CONCLUSIONS

Despite its ability to accurately account for the physical structure of a system, the complex Blackwell model is not reliable for calculating thermal properties, especially when experimental testing is conducted on a simplified test rig. A simpler, faster model often gives better results.

REFERENCES

1. H. S. Carslaw and J. C. Jaeger, *Conduction of Heat in Solids* (2nd Edn), Chap. 10. Oxford University Press, London (1959).
2. G. Grazzini, F. Gori and T. F. Irvine, Analisi dei metodi di calcolo della conducibilità termica effectiva nella misura con sonda termica, *Proceedings, 43rd ATI Congress*, Ancona, Italy, pp. 159–168 (1988).
3. J. H. Blackwell, A transient-flow method for determination of thermal constants of insulating materials in bulk. Part I: theory, *J. Appl. Phys.* **25**, 137–144 (1954).
4. B. Häkansson, P. Andersson and G. Bäckström, Improved hot-wire procedure for thermophysical measurements under pressure, *Rev. Sci. Instrument.* **59**, 2269–2275 (1988).
5. S. Pettersson, Theoretical considerations on the hot-wire method applied to solids under pressure, *Rev. Sci. Instrument.* **61**, 1308–1313 (1990).
6. W. R. Davis, F. Moore and A. M. Downs, The hot-wire method for the determination of thermal conductivity: castables and modifications to the standard method, *Trans. Brit. Ceramic Soc.* **79**, 158–166 (1980).
7. J. Laurent, Evaluation des parametres thermiques d'un milieu poreux: optimisation d'outils de mesure *in situ*, *Int. J. Heat Mass Transfer* **32**, 1247–1259 (1989).

8. M. Abramowitz and A. I. Stegun, *Handbook of Mathematical Functions*. Dover, New York (1972).
9. A. E. Wechsler, Development of thermal conductivity probes for soils and insulations, CRREL Technical Report no. 182 (1966).
10. G. Grazzini and C. Balocco, Determination of thermal parameters of poor conductors by transient techniques, *Int. J. Energy Res.* **19**, 37–44 (1995).
11. ASHRAE, *Handbook Fundamentals*. American Society of Heating Refrigerating and Air Conditioning Engineers, Atlanta, GA (1989).
12. M. J. Box, D. Davies and W. H. Swann, *Non-Linear Optimisation Techniques*, ICI Monograph no. 5. Oliver and Boyd, London (1969).

APPENDIX A
EVALUATING TEMPERATURE VARIATIONS

In order to solve equation (14), variable z defined as

$$z = \frac{r_b^2 c^2}{4a} \tag{A1}$$

is introduced. According to Blackwell [3] and Carslaw and Jaeger [1], an approximation of the order gr_b^4 can be used for $gr_b K_1(gr_b)$

$$gr_b K_1(gr_b) = 1 + \frac{(gr_b)^2}{z}\left[\ln\left(\frac{gr_b c}{z}\right) - \frac{1}{z}\right] + O((gr_b)^4)$$
$$\cong 1 + \frac{pr_b^2}{4a}[\ln(pz) - 1]. \tag{A2}$$

The following simplification can be applied:

$$\Delta = K_1(gr_b)\left[R + \frac{K_0(gr_b)}{gr_b K_1(gr_b)}\right] \cong \frac{K_1(gr_b)}{z}\left\{zR - \ln(pz) - \frac{pr_b^2}{4a}[z\ln(pz) - z - \ln^2(pz)]\right\} \tag{A3}$$

in which the following expression is used:

$$\frac{K_0(gr_b)}{gr_b K_1(gr_b)} = -\frac{1}{2}\left\{\ln(pz) + \frac{pr_b^2}{4a}[z\ln(pz) - z - \ln^2(pz)]\right\} + O((gr_b)^4). \tag{A4}$$

The following expression can thus be used:

$$p^2 \alpha r_b g r_b \Delta \cong \frac{p^2 r_b \alpha}{z}\left[1 + \frac{pr_b^2}{4a}(\ln(pz) - 1)\right] \times \left[2R - \ln(pz) - \frac{pr_b^2}{4a}(z\ln(pz) - z - \ln^2(pz))\right]$$
$$\cong p^2 \alpha r_b R - \frac{p^2 \alpha r_b \ln(pz)}{z}. \tag{A5}$$

The last simplification allowed is of the order of $O(z^4)$:

$$2\lambda p g r_b K_1(gr_b) \approx 2\lambda p + 2\lambda p^2 r_b^2(\ln(pz) - 1)/4a. \tag{A6}$$

Consequently, equation (14) becomes:

$$\Theta(r,p) = \frac{2q' r_b K_0(gr)}{2\lambda p\left[1 + \frac{pr_b^2}{4a}\ln(pz) - \frac{pr_b^2}{4a}\frac{pr_b\alpha}{4\lambda}\ln(pz) + \frac{pr_b \alpha R}{2\lambda}\right]}$$
$$\cong \frac{q' r_b}{\lambda}\left[\frac{K_0(gr)}{p} + \left(\frac{p\alpha}{4\lambda} - \frac{r_b^2}{4a}\right)K_0(gr)\ln(pz) + \left(\frac{r_b^2}{4a} - \frac{\alpha r_b R}{2\lambda}\right)K_0(gr)\right]. \tag{A7}$$

Now, with Laplace antitransformers

$$\left(\frac{r_b^2}{4a} - \frac{\alpha r_b R}{2\lambda}\right)K_0(gr) = \left(\frac{r_b^2}{4a} - \frac{\alpha r_b R}{2\lambda}\right)K_0\left(\frac{r}{\sqrt{a}}\sqrt{p}\right)$$
$$\to \left(\frac{r_b^2}{4a} - \frac{\alpha r_b R}{2\lambda}\right)\frac{1}{2t}\exp\left(-\frac{r^2}{4at}\right) \tag{A8}$$

and

$$K_0(gr)\ln(pz) = K_0\left(\frac{r}{\sqrt{a}}\sqrt{p}\right)\ln(pz)$$
$$\to \int_0^t \left(\frac{-1}{t-u}\right)\frac{\exp(-r^2/4au)}{2u}\,du$$
$$= \frac{1}{2}\int_0^t \frac{\exp(-r^2/4au)}{u(u-t)}\,du \xrightarrow{z=(t/u)-t}$$
$$\int_0^t \frac{\exp(r^2 z/4at)}{z}\,du = \frac{1}{2t}\exp\left(-\frac{r^2}{4at}\right)\ln\left(\frac{4at}{r^2}\right). \tag{A9}$$

Lastly, the convolution theorem gives

$$\frac{K_0(gr)}{p} = \frac{K_0(r\sqrt{p}/\sqrt{a})}{p} \to -\frac{1}{2}Ei\left(-\frac{r^2}{4at}\right). \tag{A10}$$

APPENDIX B

The complex method proposed by Box *et al.* [12] starts out by randomly and sequentially generating a set of P trial points in the space of the independent variables and evaluating the function at each vertex. Each newly generated point is tested for feasibility, and if unfeasible, is retracted toward the centroid of the previously generated points until it becomes feasible. Given this set of points, the objective function is evaluated at each point, and the point corresponding to the lowest value is rejected. A new point is generated by reflecting the rejected one at a certain distance through the centroid of the remaining points. Thus, if x^R is the rejected point and $\bar{x}$ is the centroid of the remaining points, then the new point will be calculated as

$$x^m = \bar{x} + \gamma(\bar{x} - x^R), \tag{B1}$$

where the size parameter γ determines the distance of the reflection. The objective function and the constraints are evaluated at the new point. Three alternatives are possible:

(1) The new point is feasible and its function value is not the highest of the set of points. In this case, the point with the highest value is selected and the procedure continues with a reflection.
(2) The new point is feasible and its function value is the highest of the current set of points. Rather than back-reflecting again (which would cause cycling), the point is retracted by half the distance to the previously calculated centroid.
(3) The new point is unfeasible. The point is retracted by half the distance to the previously calculated centroid.

The search ends when the pattern of points has shrunk so that the points are sufficiently close together and/or the difference between the function values at the points becomes small enough.