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Force implication: A new approach to human reasoning

Ch. Dujet*, N. Vincent

INSA Lyon, Centre de Mathematiques, 20, Avenue Albert Einstein, F-69621 Villeurbanne Cedex, France

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Abstract

As it is well known the logical "material implication" is not natural from the point of view of human reasoning; besides, in fuzzy logic, it is possible to classify the ply operators refering either to the boolean implication, or to the boolean conjunction (Mamdani's operator). Combining the aim to modelize human reasoning in a more natural way (subsequently in the spirit of Mamdani) with the necessity to get an implication (that is to say a non-symmetric operator), a new ply operator is introduced here, named the force implication.

Then conditional objects and generalized modus ponens are evoked, through their links with the force implication.

Keywords: Implication; Fuzzy logic; Conditional objects; Force implication; Generalized modus ponens

0. Introduction

In the following, the logical notions of implication and modus ponens are investigated in a fuzzy context.

A new implication is introduced (Section 1); the motivation of this work is to try to modelize human sentences such as "proposition A leads to proposition B" for which, generally, it does not make sense to say that "A leads to B" is true when the antecedent A is not satisfied.

From this point of view, Mamdani's ply operator is relevant, but because of its symmetry with respect to the antecedent and to the consequent, it is not seen as an implication. Therefore, what is needed is some kind of non-symmetric generalization of Mamdani's operator, which is achieved by means of our force implication.

Some properties are studied, and examples are given, leading to compare the use of the force implication with other classical ply operators such as Mamdani and Lukasiewicz ones (Section 2).

In Section 3 possible links of force implication with conditional logic are shown, then a new generalized modus ponens based on conditional logic is proposed.

* Corresponding author.

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1. Force implication

Working in the frame work of fuzzy logic, our concern is namely the use of rules of the type

"if X is A then Y is B"

for which X and Y denote linguistic variables, taking values in universes $\mathscr{X}$ and $\mathscr{Y}$, respectively (most often, $\mathscr{X}$ and $\mathscr{Y}$ are numerical ordinal scales); moreover, the linguistic variables are described by means of attributes (or labels).

For example, the linguistic variable "temperature" may take numerical values on some ordinal scale, and may be also described as "high", "very high", "cool", etc.

In control problems, the labels more generally encountered are: "large positive", "small positive", "zero", "small negative", "strong negative", etc.

Let us denote by $\mathscr{L}(X)$ the set of attributes associated to X. Then each attribute A of X $(A \in \mathscr{L}(X))$ is represented by a fuzzy set $\tilde{A}$ of the universe $\mathscr{X}$ of X. The membership function of some fuzzy set $\tilde{A}$ of X is denoted by μ_A.

Back to our previous example with temperature as the linguistic variable, some elements of $\mathscr{L}(X)$ are sketched in Fig. 1.

1.1. About ply operators: a classification

Working from a semantical point of view, the problem to be solved is to estimate to which degree an inference rule "if X is A then Y is B is true", knowing that couple (X, Y) takes some precise numerical value (x, y) in the universe $(\mathscr{X} \times \mathscr{Y})$.

According to fuzzy logic, an interpretation of atomic sentences "X is A" is determined by a truth qualification function denoted by Truth(X is A) and defined as follows:

$$\text{Truth}(X \text{ is } A) = \mu_A.$$

More precisely, if X takes some precise numerical value x, then X is A is true to the degree $\mu_A(x)$.
This fact may be rewritten as

$$\text{Truth}(X \text{ is } A \mid X = x) = \mu_A(x),$$

which may be interpreted as a conditional truth value (see Section 2).

Then different ply operators may be used, in order to get the truth qualification function of an if–then rule.

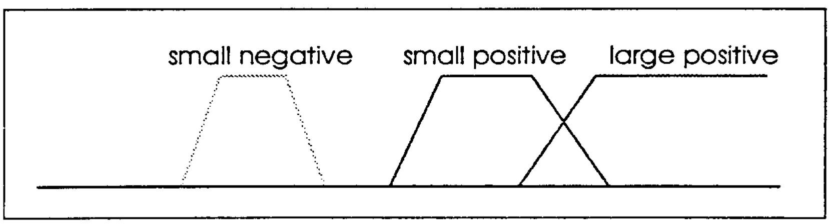

Fig. 1. Examples of membership functions associated to labels.

Let recall [12] that a ply operator should have the desired following properties:
- it depends only on the truth value of the antecedent and of the consequent,
- it is transitive,
- it is non-symmetric.

In fuzzy logic, the truth qualification function may be seen as a fuzzy relation, that is to say

$$\text{Truth }[X \text{ is } A \to Y \text{ is } B]: \mathscr{X} \times \mathscr{Y} \to [0, 1].$$

To choose the right ply operator in a given realization is mainly depending on the context, many authors do agree about this point.

Most of ply operators (in a fuzzy context) are defined following the reference boolean scheme, meaning that implication $a \to b$ is semantically equivalent to $\neg a \vee b$ (disjunction of b and negation of a).

A generalization follows straightforward by replacing $\neg$ by any fuzzy negation operator and $\vee$ by any disjunction operator, using any T-conorm.

Keeping in mind the will to build up a new ply operator, in the spirit of Mamdani's one, but fulfilling the three properties recalled above, a classification of the existing methods is proposed, which is not referring to the usual criteria of classification as proposed in [1,6].

They are compelled to put aside Mamdani and Larsen's methods. Though these last methods are most often bringing up very good results into real situations, they are sometimes contested because of their lack of logical foundations. See for instance [8] for a discussion of Mamdani's point of view.

The classification adopted here consists in discriminating two families of ply operators, according to the fact that a given operator is either an extension of the *boolean implication*, or an extension of the *boolean conjunction*:

(1) *The first family* (offering numerous examples, Lukasiewicz, Gödel, ...) is including all functions which are some extension of the boolean implication, meaning all functions compatible, when restricted to $\{0, 1\}$, with the classical following truth table:

$a \to b$

$a \backslash b$	0	1
0	1	1
1	0	1

(2) *The second family* (e.g. Mamdani, Larsen, ...) brings together all operators which are extensions of the boolean conjunction, thus compatible with the following truth table:

$a \wedge b$

$a \backslash b$	0	1
0	0	0
1	0	1

A question arises: Is the generalized boolean scheme more suited, for example, in control problems? It may be argued that, assuming the truth value of the antecedent to be 0 and in the same time, the truth value of consequent to be 1, it does not seem reasonable to believe that a realization of the premise leads to some realization of the consequent. In control, the use of an if–then rule rather seems to convey a kind of implication to be understood as usually used in natural language, that is to say, as "*a forces b*" (besides, an if–then rule in control reflects, in a formal way, the knowledge of a human expert speaking in natural language).

Under this consideration, Mamdani's ply operator, which belongs to the second family of the above classification, appears more appealing, because more approaching the intuitive idea of "a forces b". But Mamdani's operator is not a logical implication.

1.2. Force implication: definition, properties

Definition. The force implication operator is defined by

$$a \to b = a(1 - |a - b|) \quad (a, b \text{ in } [0,1]).$$

Interpretation in the fuzzy context: Denoting by $\mu_{A \to B}$ the truth qualification function of the if–then rule R: "if x is A, then y is B", we get

$$\mu_{A \to B}(x, y) = \mu_A(x) \cdot [1 - |\mu_A(x) - \mu_B(y)|] \quad \forall (x, y) \in \mathscr{X} \times \mathscr{Y}.$$

Obviously, force implication is a generalization of the boolean conjunction, because the truth table

$a \backslash b$	0	1
0	0	0
1	0	1

still holds, but this operator is not a symmetrical one, which is satisfying for an implication! We see in Fig. 2, the graph of $\mu_{A \to B}$.

Elementary properties

Let f denote the first function proposed $f(a,b) = a\ (1 - |a - b|)$. Then we have:

(P1) $f(a,a) = a\ \forall a \in [0,1]$. This property reflects the reflexivity (or idempotency) of the force implication, meaning that "A forces A" to a degree which equates the degree of realization A (and cannot be greater). From our point of view, this is more satisfying than the use of a function with $f(a,a) = 1$, as is commonly seen.

(P2) $f(1,b) = b\ \forall b \in [0,1]$.

(P3) $f(0,b) = 0\ \forall b \in [0,1]$.

(P4) $f(a,b) \leqslant a\ \forall (a,b) \in [0,1]^2$. This last property expresses that "A forces B" to a degree which cannot exceed the degree of truth (or realization) of the antecedent A. It fulfills one of our requirement to build up this new operator, namely to reinforce the role of the antecedent.

(P5) If $f(a,b) = \alpha$ and $f(a \wedge a', b) = \beta$, then $\forall a'$, β is not comparable to α. It may be greater, it may be smaller (non-monotonicity).

1.3. Possible generalizations

(a) Possible generalizations are to be obtained by using various distances defined on the scale $[0,1]$, and substituting to the term $|a - b|$ in definition of force implication, any distance $d(a,b)$.

Similarly, the product may be replaced by a T-norm. Therefore, a more general definition may be given as

$$a \to b = T(a, 1 - d(a,b)).$$

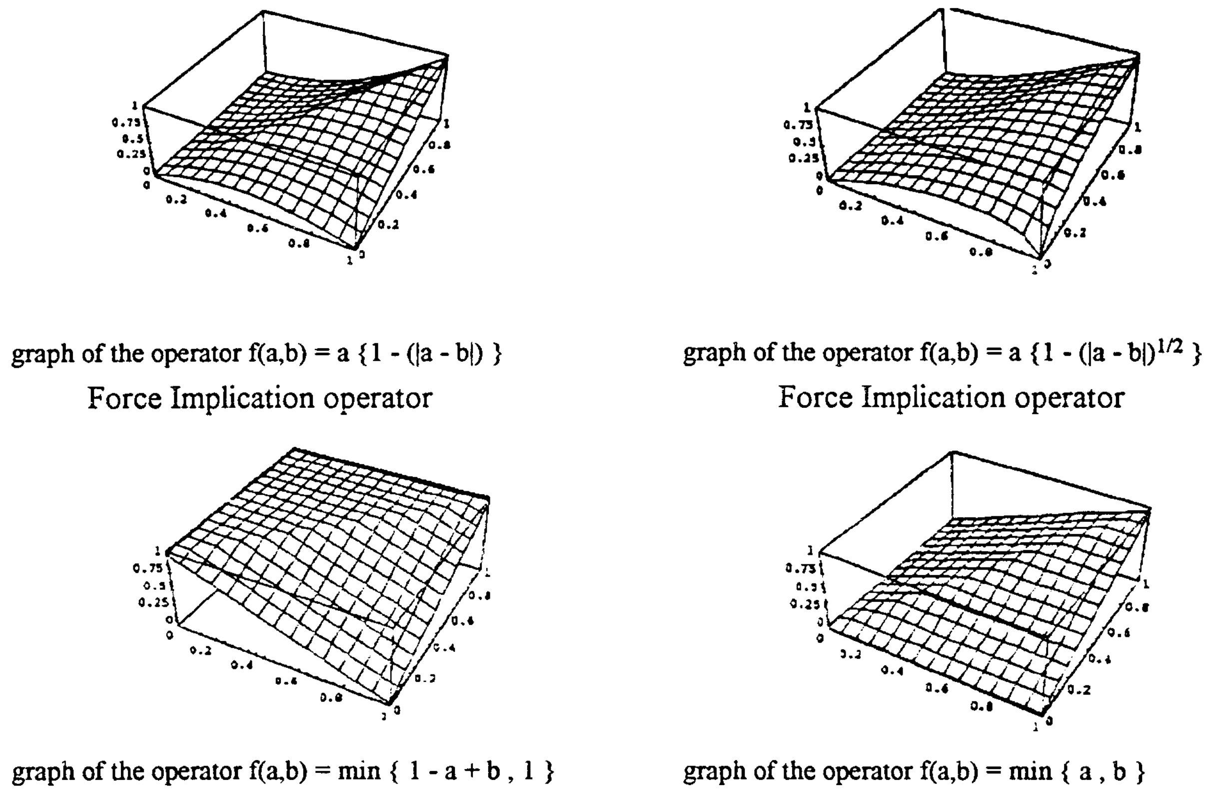

Fig. 2. Various ply-operators: (a) Graph of $f(a,b) = a\{1 - (|a-b|)\}$ (force implication operator); (b) graph of $f(a,b) = a\{1 - (|a-b|)^{1/2}\}$ (force implication operator); (c) graph of $f(a,b) = \min\{1 - a + b, 1\}$ (Lukasiewicz operator); (d) graph of $f(a,b) = \min\{a, b\}$ (Mamdani operator).

(b) Let work in the lattice of truth degrees in [0, 1] (or any Heyting algebra) equipped with an indistinguishability relation, denoted by I.

It is worth to notice that defining

$$I(a,b) = 1 - |a - b|$$

is a particular case of indistinguishability relation. Therefore, another possible generalization of force implication operator is

$$a \to b = T(a, I(a,b))$$

where T is a T-norm and I an indistinguishability relation.

Remark. The force implication takes into account degrees of realization of both members of the given if–then rule.

For example, to get $f(0.6; 0.8) = 0.48$ turns out to be some indication that, perhaps, it is not the use of the involved rule which effectively induces the result.

2. Tests and comparisons

2.1. Graphs of ply operators

In Fig. 2, we compare graphics representation of $\mu_{A \to B}$ using another distance and another classical ply-operator.

2.2. Illustration

After this theoretical work, it seems very important to legitimate the introduced concept of force implication by testing its efficiency at least on one application, through comparison with two usual methods, one based on Mamdani's operator, the other one based on Lukasiewicz one.

The example that looks, in our opinion as the most appropriate to achieve this aim, due to the eminent place it takes in research, is the realization and command of robots. This command has to offer, with the most simplicity, a maximum of smoothness allied with the less possible jerks, in the choice of the path. That is why we consider the orientation of the robot's wheels by analogy to that of a car's wheels, which is faced to a sinuous road.

The problem will be then to manage the motion of a moving car.

In the proposed example, the position of the car is defined by means of a sole parameter, the measure of angle α. This angle α is the angle of the car axis with the tangent line to the left border of the road, at the point P, where the border of the road crosses the orthogonal line to the car axis, on the left front wheel center.

Thus, the measure of this angle is varying depending on the sinuosities of the road and on the car orientation (see the sketch in Fig. 3).

The command of the car is actually linked to the position that it is wanted to give to the wheels. Subsequently, two variables are to be considered:

A: the angle of the car with respect to the road.

B: the orientation of the wheels.

Each of these two variables may be assigned with the following three attributes:

A	B
Negative	On the left of the axis
Zero	Straight in the axis
Positive	On the right of the axis

In this elementary scholar example, the three following methods were used and compared:

(1) Mamdani's method,

(2) Lukasiewicz's method,

(3) Force implication method.

In Fig. 4 are shown the curves allowing to visualize, for each location of the car, the advised position of the wheels. The horizontal axis is used for the position of the car with respect to the road, and the vertical axis is used for the wheels orientation.

If we overlay these graphs, in order to ease their comparison, we get the graph in Fig. 5.

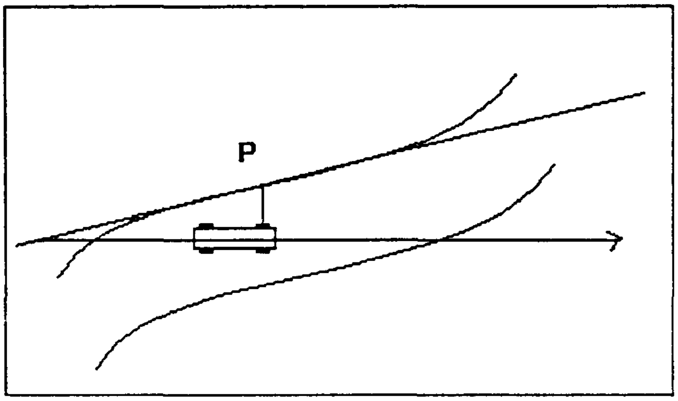

Fig. 3. Car position.

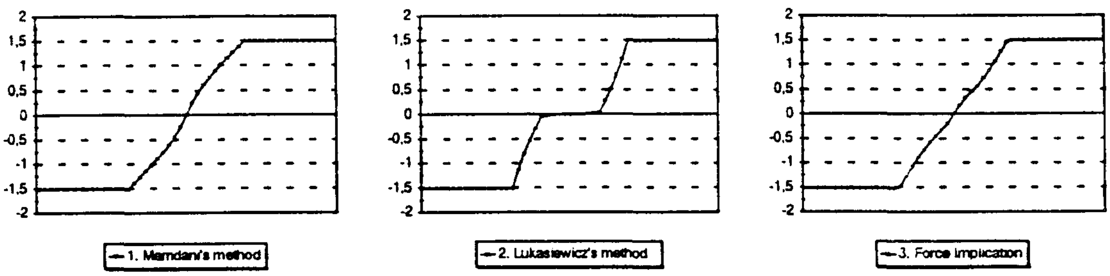

Fig. 4. Graphs showing the results obtained with three ply-operators: (a) Mamdani's method; (b) Lukasiewicz's method; (c) Force implication method.

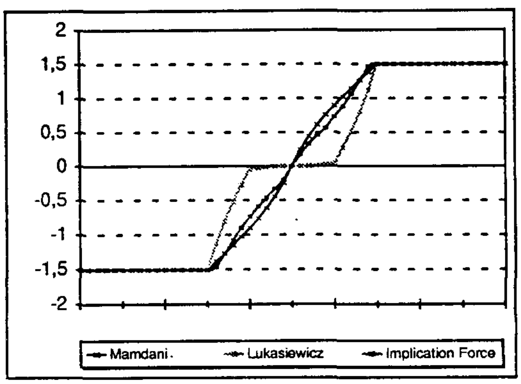

Fig. 5. Comparison between the three methods.

2.3. Concluding remarks

In Lukasiewicz's method, when the car is close to a "straight line" (along the road axis), the steer wheel is kept unmoved, the driver is waiting long enough a time, before starting to deviate, he makes a turn to the last moment, and steering-wheel-jerk occurs only when the car is not very far from the road axis.

In Mamdani's method, there is anticipation of the motion, so jumps will happen, even on an almost straight road.

In the force implication method that we proposed, the jumps are minimized in all cases; the car will do the ideal path, for which the jumps will be absorbed in a maximal way on sharp turns as well as on about straight roads.

3. Conditional truth value and modus ponens

Fuzzy logic deals, as already said, essentially with atomic propositions such as X is A, where X stands for a so-called linguistic variable and is labelled by some attribute A which reflects some particular property or quality of X.

Then such atomic propositions are assigned (what is called previously a truth qualification function); a truth qualification function is a fuzzy subset of the lattice $[0, 1]$; the assignment of such a function to elementary proposition is not a standardized one, because of the problem of building up these functions which is related to some human experts.

Therefore, at the very beginning of any study about fuzzy logic is encountered the problem of facing empirical groundings [7], which leads to the well-known controversy about fuzzy logic itself.

Our purpose here is anyway to subdume this discussion and stay at some kind of naïve point of view, partly because from this empirical basis, it is possible to get a clear framework for fuzzy logic.

Given what will be called an interpretation I, that is to say a mapping from the set of atomic formulae of our language into the set of fuzzy subsets of $[0, 1]$, it is always possible to extend I to compound formulae as usually.

But the analogy with classical logics (including modal logics) stops here; it strikes us that a new level appears in the semantical approach of fuzzy logic, which puts fuzzy logic both in the new trend of conditional logic and in Kripke semantics. Actually, once given an interpretation (on our language), what is needed is to assign a grade of truth (belonging to $[0, 1]$) to an atomic formula "X is A", given an information on X (at some particular instant, in some particular or possible world, for a given realization of the event "X is A" and so on ...). Thus it is clear that the final assignment of a grade of truth to some atomic formula is depending on the context of possible realizations and also on the information pertaining to these realizations.

In order to define clearly this second level assignment, a new kind of propositions is needed, which will be called a conditional formula. (Remark that it is in the spirit of conditional logic, but not exactly as defined by Dubois and Prades [2]; or Nguyen [10]).

Definition. An *atomic conditional formula* is any formula built by means of two atomic formulae refering to the same linguistic variable, and meaning

X is A given X is A'

and denoted by "X is $A \mid X$ is A'".

Next step (first level) is to give a truth degree for a conditional formula, through a given interpretation.

To achieve this purpose, it may be considered two different points of view; stemming either from measure or logic.

3.1. Truth degree of a conditional formula via measure

The degree of truth of proposition "X is $A \mid X$ is A'" may be considered as the measure of the matching degree between proposition "X is A" and proposition "X is A'". This is particularly appropriate to the case of some attribute A' derived from A by means of a modifyer as can be seen in [1].

Example: Temperature is hot; temperature is very hot.

Two numerical definitions for measuring this degree named "conditional truth value" are suggested (this conditional truth value will be denoted by [[ ]]):

$$[[X \text{ is } A \mid X \text{ is } A']] = \sup_{t \in \operatorname{supp}(A) \cap \operatorname{supp}(A')} \{\mu_A(t) \wedge \mu_{A'}(t)\}, \tag{1}$$

$$\text{when } \mathring{A}' \neq \emptyset \text{ and } \mathring{A} \neq \emptyset, \quad [[X \text{ is } A \mid X \text{ is } A']] = \frac{1}{\int_{\operatorname{supp}(A)} \mu_A(t)\,\mathrm{d}t} \int_{\operatorname{supp}(A) \cap \operatorname{supp}(A')} \mu_A(t) \wedge \mu_{A'}(t)\,\mathrm{d}t, \tag{2}$$

$$\text{if } \mathring{A}' = \emptyset, \quad [[X \text{ is } A \mid X \text{ is } A']] = \langle \delta_{\operatorname{supp}(A')}, \mu_A \wedge \mu_{A'} \rangle.$$

Proposition. *When the atomic proposition "X is A'" expresses a precise numerical occurrence x of the linguistic variable X ($X = x$), then the conditional truth degree of $[X$ is $A \mid X$ is $A']$ is equal to $\mu_A(x)$ using either formula* (1) *or* (2).

A classical interpretation of the degree of truth (or degree of realization of event "X is A") of proposition "X is A" is thus recovered, meaning that given an interpretation (equivalently given the membership function), the degree of truth of proposition "X is A" is naturally given by the degree of membershipness of the variable X to the fuzzy subset $\tilde{A}$, knowing that X takes precise value x.

3.2. Truth qualification function of a conditional formula via logic

Another point of view may be adopted, in order to estimate to which extent a given proposition "X is A" is true, with respect to a known information on X, "X is A'". It is needed here to extend the given interpretation of our set of propositions to conditional propositions.

In our opinion (and to be found also from authors dealing with probabilistic logic see [11]), such a proposition may be understood as kind of logical implication, in which "X is A'" becomes the antecedent and "X is A" the consequent. But it seems clear enough, in order not to be disputed when arguing this point of view, that such a conditional proposition holds (is true) when the antecedent holds; it does not make sense to take into some account the proposition "X is $A \mid X$ is A'" if it is known that "X is A'" has no reality.

Therefore, only specifical logical implication are to be considered, namely what was called in the first section force implication (but including also Mamdani's operator).

So it is possible to extend a given interpretation of our language to conditional formulae.

Truth qualification function $[X$ is $A \mid X$ is $A'] = \text{Truth}[X$ is A' *forces* X is $A]$.

Using Mamdani's operator, we get

$$\text{Truth}[X \text{ is } A \mid X \text{ is } A'] = \mu_A \wedge \mu_{A'}. \tag{3}$$

Using force implication,

$$\text{Truth}[X \text{ is } A \mid X \text{ is } A'] = \mu_{A'}(1 - |\mu_A - \mu_{A'}|). \tag{4}$$

Proposition. *In the case of a precise (numerical) occurrence of the variable X, the truth qualification function of conditional proposition "X is A | X is A′", when "X is A′" stands for X takes precise numerical value x_0, coincides with the characteristic boolean function $\chi_{\{x_0\}}$ deriving this truth function either from formula* (3) *or* (4).

Proof. "X is A'" is interpreted by

$$\mu_{A'}(x) = 0 \quad \text{if } x \neq x_0, \qquad \mu_{A'}(x_0) = 1.$$

It follows straight away that

$$\mu_A \wedge \mu_{A'}(x) = 0 \quad \text{if } x \neq x_0, \qquad \mu_A \wedge \mu_{A'}(x_0) = \mu_A(x_0)$$

and

$$(\mu_{A'} \cdot (1 - |\mu_A - \mu_{A'}|))(x) = 0 \quad \text{if } x \neq x_0, \qquad (\mu_{A'} \cdot (1 - |\mu_A - \mu_{A'}|))(x_0) = \mu_A(x_0).$$

Therefore, at a second level, it is possible to get a degree of truth of proposition "X is $A \mid X = x_0$" taking the supremum of the truth qualification function, that is to say $\mu_A(x_0)$. □

3.3. Application to modus ponens

In [14], Zadeh proposed what is called a generalized modus ponens, meaning that the inference rule of modus ponens

$$\begin{array}{l} A \\ A \to B \\ \hline \quad B \end{array}$$

is given a slightly modified version in order to take into account the fact actually occuring, let us say A', instead of A, following the scheme:

$$\begin{array}{l} A' \\ A \to B \\ \hline \quad B' \end{array}$$

Using conditional formulae, it seems natural to propose rather for the generalized modus ponens rule.

$$\begin{array}{l} A \mid A' \\ A \to B \\ \hline \quad B' \end{array}$$

where the premise is now "X is A knowing X is A'", instead of "X is A" (which is the premise of the given rule), and instead of "X is A'", which is the given occurrence of the premise.

So, we propose to define the truth qualification function of modus ponens as

$$\mathrm{TQF}\left(\frac{A, A \to B}{B}\right) = G([[A \mid A']], [[A \to B]]),$$

where G is a 2-valued function to be defined, as a T-norm for example (according to the fact that the inference modus ponens rule means that the conjunction of A and $A \to B$ entails B), and $[[A \mid A']]$ has to be calculated by means of the above force implication

$$[[A \mid A']] = [[A' \to A]].$$

It was already shown in [13] that dealing with this modus ponens allows to avoid the incoherence in aggregating the rules (for control problems for instance), incoherence due to the use of either a conjunction operator, or a disjunction operator, depending on the choice of the ply operator to compute $A \to B$.

Due to the importance given both to A and A' in the premise of this modus ponens rule, by means of the conditional formulae, whatever should be the ply operator chosen in order to compute the truth degree of $A \to B$ (Mamdani or Lukasievicz or anyone else), aggregation of rules is achieved here in a sole way: a disjunctive operator (as for Mamdani's method, but conversely to Lukasievicz's method).

In a forthcoming paper, the modus ponens (based on conditional formula and force implication) will be extensively studied, and will be shown as bringing a unified framework.

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