Integration Versus Trend Stationary in Time Series

Econometrica, Vol. 60, No. 2, (March, 1992), 423–433

NOTES AND COMMENTS

INTEGRATION VERSUS TREND STATIONARITY IN TIME SERIES

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1. INTRODUCTION

A WELL-KNOWN APPROACH to modeling macroeconomic time series is to assume that the natural logarithm of the series can be represented by the sum of a deterministic time trend and a stochastic term. The trend need not literally be part of the data generation process, but may be viewed as a substitute for a complicated and unknown function of population, capital accumulation, technical progress, etc. Within this approach there are two competing models; in the *trend-stationary* specification the stochastic term follows a stationary process, while in the *integrated* specification the stochastic term follows a random walk. The essential difference between the models is the nature of the process driving the stochastic component, not whether the series is trended. The conclusion of this study is that it is difficult to discriminate between the two models using classical testing methods. This is the consequence of low power: the powers of integration tests against plausible trend-stationary alternatives can be quite low, as can the powers of trend-stationarity tests against integrated alternatives. Our analysis thus suggests that it is premature to accept the integration hypothesis as a stylized fact of macroeconomic time series.

The leading case we examine is a model with linear trend and iid normal innovations, which we use to study the power of integration and trend-stationarity tests. This strategy is motivated by the idea that the study should begin with the case which is most favorable to high power; the presumption is that the finite sample powers of tests designed for this case are superior to the powers of tests designed for models with more general innovation sequences.

2. LEADING CASE

Let the time series $\{y_t\}$ be the stochastic process generated by the linear model

$$y_t = \alpha_0 + \alpha_1 t + x_t \qquad (t = 0, 1, \ldots), \tag{2.1}$$

and the first-order autoregressive (AR) process

$$x_t = \beta x_{t-1} + u_t \qquad (t = 1, 2, \ldots). \tag{2.2}$$

It is assumed that the innovation sequence $\{u_t\}$ is iid $N(0, \sigma^2)$, and x_0 is an unknown constant. Thus model (2.1)–(2.2) can be interpreted as a random walk about a linear trend when $\beta = 1$ and an asymptotically stationary AR(1) process about a linear trend when $|\beta| < 1$. In either case, the standardized initial displacement plays an important role below, and will be denoted by $x_0^* \equiv x_0/\sigma = (y_0 - \alpha_0)/\sigma$. This parameter measures the distance (in units of innovation standard deviations) between the initial value y_0 and the trend line.

[1]We thank John Kennan, Doug McManus, Forrest Nelson, Joon Park, Peter Schmidt, two referees, and associate and co-editors for helpful comments. This is a revision of Working Papers 88-27 (December, 1988) and 89-31 (December, 1989), which carried similar titles.

Substituting (2.2) into (2.1) and rearranging gives

$$(2.3) \qquad y_t = \gamma + \delta t + \beta y_{t-1} + u_t \qquad (t = 1, 2, \ldots),$$

where $\gamma = [\alpha_0(1-\beta) + \alpha_1\beta]$ and $\delta = \alpha_1(1-\beta)$. Equation (2.3) is a rearrangement of the quasi-first-difference transform of (2.1). The coefficients of interest are α_0, α_1, and β; equation (2.3) is viewed as the reduced form of (2.1)–(2.2), and the coefficients γ and δ are treated as reduced-form parameters. This approach was presented in Dickey (1984), and was followed by Bhargava (1986), Park and Choi (1988), and Schmidt and Phillips (1989).

A. Unit Root Tests

The integration (unit root) hypothesis is

$$(2.4) \qquad H: \beta = 1.$$

Dickey (1976), Fuller (1976), and Dickey and Fuller (1981) introduced tests of (2.4) based on statistics obtained from applying ordinary least squares (OLS) to (2.3). For a sample of size T the reduced form (2.3) can be written as

$$(2.5) \qquad y = \gamma\iota + \delta\tau + \beta y_{-1} + u,$$

where $y = (y_1, y_2, \ldots, y_T)'$, $y_{-1} = (y_0, y_1, \ldots, y_{T-1})'$, $\iota = (1, 1, \ldots, 1)'$, $\tau = (1, 2, \ldots, T)'$, and $u = (u_1, u_2, \ldots, u_T)'$, with $u \sim N(0, \sigma^2 I)$. Writing (2.5) in matrix notation, we have

$$(2.6) \qquad y = W\pi + u,$$

where $W = (\iota, \tau, y_{-1})$ and $\pi = (\gamma, \delta, \beta)'$. The OLS estimator of π is $\hat{\pi} = (\hat{\gamma}, \hat{\delta}, \hat{\beta})' = (W'W)^{-1}W'y$. Let $s^2 = (T-3)^{-1}(y - W\hat{\pi})'(y - W\hat{\pi})$, and let G_i denote the ith diagonal element of $(W'W)^{-1}$. The test statistics proposed by Dickey and Fuller for testing H: $\beta = 1$ are[2]

$$(2.7) \qquad K(1) = T(\hat{\beta} - 1),$$

the conventional regression t statistic,

$$(2.8) \qquad t(1) = (\hat{\beta} - 1)/(s^2 G_3)^{1/2},$$

and the F statistic for testing the joint hypothesis H^*: $\delta = 0$, $\beta = 1$,

$$(2.9) \qquad F(0,1) = (Q'\hat{\pi} - c)'\left[s^2 Q'(W'W)^{-1}Q\right]^{-1}(Q'\hat{\pi} - c)\Big/2;$$

the joint hypothesis can be written as $Q'\pi = c$ where Q' is a 2×3 matrix whose first column is the zero vector and whose second and third columns form an identity matrix, and $c = (0, 1)'$. In Dickey and Fuller (1981), $K(1)$, $t(1)$, and $F(0,1)$ correspond to $T(\rho_\tau - 1)$, τ_τ, and Φ_3.

The $t(1)$ statistic (2.8) is the Wald statistic which employs the Hessian-based estimator of the standard error, and the $F(0,1)$ statistic is a transform of the likelihood ratio statistic. Note that $\beta = 1$ implies that the reduced-form coefficient $\delta = 0$, and hence the unit root hypothesis (2.4) implies the joint hypothesis H^*.

Under the unit root hypothesis, the Dickey-Fuller statistics have nonstandard distributions: $K(1)$ and $t(1)$ are not asymptotically distributed as standard normal, and the asymptotic distribution of $F(0,1)$ is not proportional to a chi-square distribution. Dickey (1976) and Fuller (1976, Table 8.5.2) give Monte Carlo critical values for the $K(1)$ and $t(1)$ tests; our Table Ia provides *exact* critical values for $K(1)$; and Dickey and Fuller

[2]Normalization by T rather than $\sqrt{T}$ is necessitated by the nonstationarity of $\{y_t\}$ under the null.

(1981, Table VI) provide critical values for $F(0,1)$. Under the unit root hypothesis, the distributions of $K(1)$, $t(1)$, and $F(0,1)$ do not depend upon "nuisance parameters;" i.e., under (2.4) the Dickey-Fuller tests are *similar* with respect to α_0, α_1, and σ. This property is established as a consequence of a more general result in Section 3 on the invariance of the distributions of the test statistics under the null *and* the alternative.[3] Similarity is achieved in these tests by adding time as an extraneous regressor in (2.3) (recall that under (2.4), $\delta = 0$).

B. Tests for Trend Stationarity

The trend-stationarity hypothesis is

$$H: \beta = \beta_0, \quad |\beta_0| < 1. \tag{2.10}$$

For example, Phillips and Perron (1988) consider testing the unit root hypothesis against the alternative $\beta_0 = 0.85$. As in the unit root case, tests of (2.10) can be similar, though direct translation of unit root tests to the trend-stationary case yields tests which are nonsimilar with respect to α_0 and σ. We introduce similar tests to overcome the low power of the nonsimilar tests.

Nonsimilar tests. To construct a size α nonsimilar test, one finds the critical value for which the rejection probability does not exceed α for any value of the nuisance parameters. Since the size is the maximum rejection probability across alternative nuisance parameter values, the actual rejection probability can be substantially smaller; this can cause low power.

The upper-tail nonsimilar tests of the trend-stationarity hypothesis against the alternative H: $\beta = 1$ are based on the statistic[4]

$$S(\beta_0) = T^{1/2}(\hat{\beta} - \beta_0)/(1 - \beta_0^2)^{1/2}, \tag{2.11}$$

which is the analogue of $K(1)$, and the conventional t statistic

$$t(\beta_0) = (\hat{\beta} - \beta_0)/(s^2 G_3)^{1/2}. \tag{2.12}$$

Nonsimilar tests can be based either on small-σ or large-T asymptotic theory. Under (2.10), small σ-asymptotic theory (Nankervis and Savin (1985, 1987)) implies that as x_0^* increases in absolute value the distribution of $t(\beta_0)$ converges (with T fixed) to Student's t with $T-3$ degrees of freedom; hence the critical value of an upper-tail nonsimilar test based on $t(\beta_0)$ is the $1-\alpha$ quantile of the t distribution with $T-3$ degrees of freedom. For the sample sizes to be considered below, this is practically the large-T asymptotic critical value (from the $N(0,1)$ distribution). However, our calculations indicate that for relevant values of x_0^*, this critical value produces a test with a size which is much too small even for samples of $T = 100$.

Critical values for the S test are also problematic. The upper 0.05 critical value of the S distribution first increases, then decreases, as x_0^* increases, and in the limit the distribution is degenerate. For large T the critical value of a size-α test may be approximated by the $1-\alpha$ quantile of the standard normal. However, our calculations indicate that this approximation produces a test with a size which is too small even for samples of $T = 100$: the actual size of the nominal 5% test is less than 0.01 for $0 \leqslant x_0^* \leqslant 10$.

[3]The similarity of $K(1)$ and $t(1)$ with respect to γ was established by Dickey (1976), and the similarity of $F(0,1)$ by Dickey and Fuller (1981). That these results could be used to establish similarity with respect to α_1 apparently has not previously been noted (under the null that $\beta = 1$, only α_1 influences γ).

[4]"S" for "stationarity" (the null).

TABLE Ia[a]

EXACT QUANTILES OF $K(1)$

T	.01	.025	.05	.1	$K(1)$ .5	.9	.95	.975	.99
50	−25.23	−21.97	−19.34	−16.53	−8.629	−3.643	−2.555	−1.667	−0.687
100	−27.17	−23.43	−20.47	−17.35	−8.859	−3.705	−2.615	−1.736	−0.774
200	−28.23	−24.22	−21.08	−17.79	−8.979	−3.736	−2.644	−1.768	−0.814

Table Ib

EXACT QUANTILES OF $S_A(\beta)$ TEST OF TREND-STATIONARITY FOR $T=100$: AUGMENTED REGRESSION MODEL WITH WHITE NOISE ERRORS

β	.01	.025	.05	.1	$s_A(\beta)$ .5	.9	.95	.975	.99
.80	−4.62	−3.95	−3.41	−2.81	−1.01	0.353	0.667	0.916	1.181
.85	−5.28	−4.53	−3.91	−3.25	−1.26	0.184	0.506	0.758	1.024
.90	−6.50	−5.58	−4.85	−4.05	−1.73	−0.116	0.232	0.502	0.787
.95	−9.53	−8.25	−7.23	−6.13	−2.97	−0.862	−0.417	−0.068	0.308

Table Ic

QUANTILES OF $t_A(\beta)$ TEST OF TREND-STATIONARITY FOR $T=100$: AUGMENTED REGRESSION MODEL WITH WHITE NOISE ERRORS

β	.01	.025	.05	.1	$t_A(\beta)$ .5	.9	.95	.975	.99
.80	−3.23	−2.84	−2.52	−2.16	−0.88	0.38	0.75	1.05	1.40
.85	−3.37	−2.99	−2.68	−2.32	−1.05	0.20	0.56	0.84	1.21
.90	−3.60	−3.21	−2.90	−2.55	−1.32	−0.10	0.24	0.53	0.87
.95	−3.94	−3.57	−3.28	−2.94	−1.78	−0.67	−0.34	−0.07	0.25
s.e.	0.04	0.03	0.02	0.01	0.01	0.02	0.02	0.03	0.05

Estimates obtained from 20,000 replications.

[a] The exact quantiles and powers were calculated by numerical integration using a Fortran version of Davies (1980) where the accuracy was set at 0.5×10^4. The other quantiles and powers were calculated by the crude Monte Carlo method; see Nankervis and Savin (1987). Monte Carlo power calculations were significantly accelerated by (i) reusing random numbers across cells; (ii) calculating the statistics expressed in terms of sums of squares and cross-products of My_{-1} and Mu with $M=I-Z(Z'Z)^{-1}Z'$, $Z=(\iota\tau^*)$, and where τ^* is the demeaned time trend. We used the fact that $My_{-1}=x_0MCd+MCu$ where $d=(1,0,\dots,0)$ and $C=L(I-\beta L)^{-1}$ so that terms involving MCd were calculated once only for each value of β and terms involving MCu were calculated once in each replication for every value of β. Each extra row of the power tables then just required a few scalar operations and so was virtually costless.

Similar tests. The probability that a similar test rejects under the null is equal to the size α for all values of the nuisance parameters. Recall that under the unit root hypothesis, $K(1)$, $t(1)$, and $F(0,1)$ are similar with respect to γ when time is included as an extraneous regressor in (2.3). Similar tests of trend-stationarity can also be obtained by introducing the appropriate extraneous regressor in (2.3).

Consider the augmented reduced-form model

$$(2.13)\qquad y=\gamma\iota+\delta\tau+\xi r+\beta y_{-1}+u,$$

where r is the extraneous regressor ($\xi=0$). The vector r is defined by $r=C_0\iota$, where $C_0=L(I-\beta_0L)^{-1}$ and L is a lag matrix (a $T\times T$ matrix with a principal subdiagonal of ones and zeros elsewhere). Thus the typical element of r is $r_t=\sum_{k=0}^{t-1}\beta_0^k=(1-\beta_0)^{-1}\times(1-\beta_0^t)$. (If a constant is included in the regression, the regressor β_0^t is sufficient.)

Statistics for constructing similar tests of H: $\beta = \beta_0$ are obtained by applying OLS to the augmented model (2.13): the test statistic based on the OLS estimator of β in (2.13) is denoted by $S_A(\beta_0)$, and the conventional t statistic for β in (2.13) is denoted by $t_A(\beta_0)$.[5] The statistics $S_A(\beta_0)$ and $t_A(\beta_0)$ yield similar tests. (See DeJong, Nankervis, Savin, and Whiteman (1988) for details of constructing similar tests in AR(1) models with arbitrary exogenous regressors.) To conduct a similar test it is sufficient to know the quantiles of the $S_A(\beta_0)$ and $t_A(\beta_0)$ statistics under the null, with $\alpha_0 = \alpha_1 = 0$ ($\gamma = \delta = 0$). These quantiles are given in Table Ib and Table Ic for $T = 100$.

3. INVARIANCE AND SYMMETRY

The model (2.1)–(2.2) is a special case of a linear regression with exogeneous regressors and a first-order AR error process. For stationary error processes, it is well known that under certain regularity conditions the asymptotic distribution of the maximum likelihood (ML) estimator of the regression coefficients is independent of the asymptotic distribution of the ML estimator of the AR parameter; for example, see Dhrymes (1981, pp. 77–96). Moreover, for (2.1)–(2.2), the asymptotic variance of $\hat{\beta}$ depends on σ^2 and β, but *not* α_0 and α_1. Thus the asymptotic distribution of the ML estimator of β is independent of α_0 and α_1. This asymptotic result is appealing since intuitively the distribution of an estimator of β should not depend upon the intercept and slope of the trend line. However, a commonly-held belief is that the finite sample distribution of the ML estimator of β depends on both α_0 and α_1. Note that the ML estimator of β in (2.1)–(2.2) is the same as the OLS estimator of β in the reduced form (2.3).

On the basis of a Monte Carlo experiment conducted under a slightly different setup, Dickey (1984) reported that empirical powers of $K(1)$, $t(1)$, and $F(0,1)$ do not depend on α_0 and α_1. Similarly, Schmidt and Phillips (1989) noted that the empirical powers of the $K(1)$ and $t(1)$ tests appear not to depend upon the value of α_1, and conjectured that the finite-sample distributions of $K(1)$ and $t(1)$ do not depend upon the value of α_1 *whatever* the value of β. We now prove a generalization of these conjectures and discuss its implications.

THEOREM (Invariance): *The statistics $K(1)$, $t(\beta_0)$, $S(\beta_0)$, $t_A(\beta_0)$, $S_A(\beta_0)$, and $F(0,1)$ depend only on β and x_0^* for fixed T, where $x_0^* = (y_0 - \alpha_0)/\sigma$. When $\beta = 1$, the statistics are invariant with respect to x_0^*. When $\beta = \beta_0$, $t_A(\beta_0)$ and $S_A(\beta_0)$ are invariant with respect to x_0^*. This also holds for the higher-order trends case.*

PROOF: We prove the theorem for the linear trend case. Generalization to the higher order case is straightforward. Substitute (2.2) into (2.1) to obtain

$$y_t = \alpha_0 + \alpha_1 t + \beta x_{t-1} + u_t, \tag{3.1}$$

and then recursively substitute for x_{t-1} using (2.2) to get

$$y_t = \alpha_0 + \alpha_1 t + \beta\left[\beta^{t-1}x_0 + \sum_{k=0}^{t-1} \beta^{t-j+1}u_j\right] + u_t. \tag{3.2}$$

Finally, dividing both sides by the constant σ yields

$$y_t/\sigma = \alpha_0/\sigma + (\alpha_1/\sigma)t + \beta\left[\beta^{t-1}x_0^* + \sum_{k=0}^{t-1} \beta^{t-j+1}\varepsilon_j\right] + \varepsilon_t, \tag{3.3}$$

[5] "A" for the "augmented" regression from which the statistics are computed.

where $\varepsilon_t = u_t/\sigma \sim N(0,1)$. When $\beta = 1$,

$$(3.4) \qquad y_t/\sigma = \alpha_0/\sigma + x_0^* + (\alpha_1/\sigma)t + \left[\sum_{k=0}^{t-1} \varepsilon_j\right] + \varepsilon_t .$$

Observe that regression equation (2.3) is equivalent to (3.3). Applying partitioned regression, the least squares estimator of β and the associated standard error can be thought of as being computed using the residuals from two detrending regressions. The regressand is the residual from the projection of y_t/σ against a constant and trend, which does not depend on α_0 and α_t by the least squares orthogonality conditions. The regressor is the residual from the projection of the bracketed expression in (3.3) against a constant and trend, which is independent of α_0 and α_1 by construction. Thus the least squares estimator of β and the associated standard error can depend only on the properties of the regressor and $\{\varepsilon_t\}$, which implies that $K(1)$, $S(\beta_0)$, and $t(\beta_0)$ depend only on β and x_0^*. These results are not affected if β_0^{t-1} is added as a regressor, so $t_A(\beta_0)$ and $S_A(\beta_0)$ also depend only on β and x_0^*. Further, when $\beta = \beta_0$, adding the regressor β_0^{t-1} purges the bracketed variable of its dependence on x_0^*, which implies that $t_A(\beta_0)$ and $S_A(\beta_0)$ are invariant to x_0^* when $\beta = \beta_0$.

Next consider testing H^*: $\delta = 0$, $\beta = 1$ using $F(0,1)$. This statistic can be written $F = [(T-3)/2][SSR/SSU - 1]$ where SSR and SSU are the restricted and unrestricted sums of squared residuals. From above, SSU depends only on β and x_0^*; SSR is obtained from the regression of $y_t - y_{t-1}$ on a constant. But using (3.1) and (2.2),

$$(3.5) \qquad (y_t - y_{t-1})/\sigma = \alpha_1/\sigma + (\beta - 1)x_{t-1}/\sigma + \varepsilon_t .$$

Thus SSR depends only on the properties of $\{x_{t-1}\}$ and $\{\varepsilon_t\}$—only on β and x_0^*.

Finally, note that when $\beta = 1$, the bracketed variable in (3.4) does not depend on x_0^*. Thus again applying the partitioned regression argument, it follows that the statistics are invariant to x_0^* when $\beta = 1$. *Q.E.D.*

Corollary (Symmetry): *If $\{u_t\}$ is symmetrically distributed about zero, then the distributions of $K(1)$, $t(\beta_0)$, $t_A(\beta_0)$, $S(\beta_0)$, and $F(0,1)$ depend only upon β and $|x_0^*|$ for fixed T. This also holds for the higher-order trends case.*

Proof: If $\{u_t\}$ is symmetrically distributed about zero, the same holds for $\{\varepsilon_t\}$. Hence ε and $-\varepsilon$ have the same distribution, and any function $f(x_0^*, \varepsilon)$ has the same distribution as $f(x_0^*, -\varepsilon)$. Moreover, the statistics $(\hat{\beta} - \beta)$ and $t(\beta_0)$ depend upon x_0^* only through the bracketed term in (3.3). But evaluation of this term at $(-x_0^*, -\varepsilon)$ merely changes the sign of the regressor. Thus $(\hat{\beta} - \beta)$ evaluated at (x_0^*, ε) is equal to $(\hat{\beta} - \beta)$ evaluated at $(-x_0^*, -\varepsilon)$, and the symmetry result for $K(1)$ and $S(\beta_0)$ is established. The symmetry results for $t(\beta_0)$, $t_A(\beta_0)$, $S_A(\beta_0)$, $F(0,1)$, and higher order trends can be established similarly. *Q.E.D.*

The theorem extends a result of Evans and Savin (1981): they considered a special case of model (2.1)–(2.2), where it is known that $\alpha_0 = \alpha_1 = 0$, and hence $y_t = x_t$; they showed that the power of the $K(1)$ test depends only upon β and x_0^*. The essential difference between the Evans-Savin study and the present study is that here α_0 and α_1 are unknown, hence x_t is not observable. While Dickey (1984) conjectured invariance to α_0 and α_1, he did not discuss dependence of the test statistics on x_0^*; he had in fact fixed x_0^*. Our theorem establishes that for all of the tests, powers do not depend upon the time trend coefficient α_1—the average exponential growth rate in the series if y is measured in logarithms. A practical consequence of the theorem is that the presentation of the power tables can be very substantially simplified—only two-dimensional tables (x_0^* and β) are needed.[6]

Another approach to modeling trend stationarity is to assume that $\{x_t\}$ is a strictly stationary process when $|\beta| < 1.0$. In this case x_0^* is a random variable which is distributed as $N(0,(1-\beta^2)^{-1})$; hence the powers of the Dickey-Fuller tests depend only upon β. In fact, Dickey and Fuller (1981) also report the powers of their tests under this random start-up assumption.

4. POWERS

A. Unit Root Tests

The powers of 0.05-size Dickey-Fuller tests are reported in Table II for $T = 100$. The powers of the $K(1)$ and $t(1)$ tests are for a one-sided test of H: $\beta = 1$, where the alternatives are $\beta < 1$; the powers of the $F(0,1)$ test are for a two-sided test of H: $\delta = 0$, $\beta = 1$. Three features of the power tables are of interest. First, for a given value of $\beta \geqslant 0.8$, the power of the $K(1)$ test *decreases* as $|x_0^*|$ increases, while for $\beta = 0.75$, the power *increases* as $|x_0^*|$ increases. The reason for this is that $K(1)$ can be written as

$$K(1) = T\left[(\hat{\beta} - \beta) + (\beta - 1)\right] = T\left[\left(c(\hat{\beta} - \beta)/c\right) + (\beta - 1)\right]$$

where the normalization factor $c = (w'Mw)^{1/2}$, with $M = I - Z(Z'Z)^{-1}Z'$, $Z = (\iota\tau)$, and $w = (I - \beta L)^{-1}x_0^* d$, $d = (1,0,\ldots,0)'$. As $|x_0^*|$ increases, $c(\hat{\beta} - \beta)$ converges to $N(0,1)$ and c diverges to infinity, hence the first term in square brackets converges to 0. Thus as $|x_0^*|$ increases, for fixed T, the distribution of $K(1)$ converges in probability to $T(\beta - 1)$.[7] Note that for $T = 100$ the critical value of the $\alpha = 0.05$ size $K(1)$ test is -20.47, and that $100(\beta - 1) = -20.47$ implies $\beta = 0.7953$. Hence for $T = 100$, as $|x_0^*|$ increases, the power of $K(1)$ goes to 0 when $\beta > 0.7953$ and to unity when $\beta < 0.7953$.

Second, the tabled powers of the $t(1)$ and $F(0,1)$ tests do not decrease with $|x_0^*|$. Although it is not apparent from Table II, when β is very close to unity the power of the $t(1)$ test first declines and then increases. For example, at $\beta = 0.99$ the minimum power of the $t(1)$ test is 0.02 and this occurs at about $|x_0^*| = 350$. Hence, like $K(1)$, the $t(1)$ test is in fact biased.

Third, the power surfaces of the tests cross. For example, for $\beta = 0.85$ the $K(1)$ test has the highest power among all the Dickey-Fuller tests when $|x_0^*| \leqslant 3$; the $t(1)$ test has higher power than the $K(1)$ test when $|x_0^*| \geqslant 4$; and the $F(0,1)$ test has the highest power among all the Dickey-Fuller tests when $|x_0^*| \geqslant 8$.

Schmidt and Phillips (1989) report powers of the LM tests for $\beta \geqslant 0.8$; these tests have the property that for a given value of β the powers decline as $|x_0^*|$ increases. As is expected, LM tests have more power than the $K(1)$ and $t(1)$ tests when β is close to unity and $|x_0^*|$ is small, say $\beta > 0.95$, and $|x_0^*| < 1.0$. By contrast, at $\beta = 0.9$ and $|x_0^*| = 5$ the powers of the Dickey-Fuller tests are larger.

Given that the dependence of the powers on x_0^* is not significant until $|x_0^*| > 2$, it is useful to recognize that such values are quite likely. When $\beta < 1$, in treating x_0^* as fixed we are conditioning on a drawing from a Gaussian distribution with mean zero and variance $(1 - \beta^2)^{-1}$. Thus with $\beta = 0.85$, such a drawing is one standard deviation away from its mean at 1.9, two standard deviations at 3.8, three standard deviations at 5.7.

[6]Schmidt and Phillips (1989) show that the distributions of related LM test statistics depend only upon β and x_0^*, and note that the empirical powers of the LM tests depend only upon the absolute value of x_0^*.

[7]This feature of the limiting distribution of $K(1)$ is confirmed by the small-σ asymptotic theory in Evans and Savin (1984).

TABLE II

POWERS OF UNIT ROOT TESTS FOR $T = 100$

Exact powers of the $K(1)$ test, critical value = -20.47

$\lvert x_0^* \rvert$	β 0.75	0.80	0.85	0.90	0.95	0.99	1.00
0.00	0.92	0.75	0.49	0.24	0.10	0.05	0.05
1.00	0.92	0.75	0.49	0.24	0.10	0.05	0.05
2.00	0.92	0.75	0.48	0.24	0.10	0.05	0.05
3.00	0.92	0.74	0.47	0.23	0.10	0.05	0.05
4.00	0.92	0.73	0.45	0.21	0.09	0.05	0.05
5.00	0.92	0.72	0.43	0.20	0.09	0.05	0.05
6.00	0.92	0.72	0.41	0.18	0.08	0.05	0.05
7.00	0.92	0.71	0.39	0.16	0.08	0.05	0.05
8.00	0.92	0.69	0.37	0.14	0.07	0.05	0.05
9.00	0.93	0.68	0.34	0.13	0.06	0.05	0.05
10.00	0.93	0.67	0.31	0.11	0.06	0.05	0.05

Powers of the $t(1)$ test, critical value = -3.45

$\lvert x_0^* \rvert$	β 0.75	0.80	0.85	0.90	0.95	0.99	1.00
0.00	0.86	0.65	0.39	0.19	0.08	0.05	0.05
1.00	0.87	0.66	0.40	0.19	0.08	0.05	0.05
2.00	0.88	0.68	0.41	0.19	0.08	0.05	0.05
3.00	0.90	0.71	0.43	0.20	0.08	0.05	0.05
4.00	0.93	0.75	0.46	0.21	0.08	0.05	0.05
5.00	0.95	0.79	0.50	0.22	0.08	0.05	0.05
6.00	0.97	0.84	0.54	0.24	0.09	0.05	0.05
7.00	0.98	0.89	0.60	0.26	0.09	0.05	0.05
8.00	0.99	0.92	0.66	0.29	0.09	0.05	0.05
9.00	1.00	0.95	0.72	0.32	0.09	0.05	0.05
10.00	1.00	0.97	0.78	0.35	0.10	0.05	0.05

Powers of the $F(0, 1)$ test, critical value = 6.49

$\lvert x_0^* \rvert$	β 0.75	0.80	0.85	0.90	0.95	0.99	1.00
0.00	0.80	0.56	0.31	0.14	0.06	0.05	0.05
1.00	0.80	0.57	0.32	0.14	0.06	0.05	0.05
2.00	0.83	0.60	0.34	0.15	0.07	0.05	0.05
3.00	0.86	0.64	0.37	0.16	0.07	0.05	0.05
4.00	0.90	0.70	0.41	0.18	0.07	0.05	0.05
5.00	0.94	0.77	0.46	0.20	0.07	0.05	0.05
6.00	0.97	0.83	0.53	0.23	0.08	0.05	0.05
7.00	0.99	0.90	0.61	0.27	0.08	0.05	0.05
8.00	1.00	0.94	0.70	0.32	0.09	0.05	0.05
9.00	1.00	0.97	0.79	0.37	0.10	0.05	0.05
10.00	1.00	0.99	0.86	0.45	0.11	0.05	0.05

Estimates based on 20,000 replications.

From the power tables, as x_0^* ranges from 0 to 5.7, the power of $K(1)$ against $\beta = 0.85$ falls from 0.49 to 0.41, the power of $t(1)$ rises from 0.39 to about 0.54, and the power of $F(0, 1)$ rises from 0.31 to over 0.5. Thus likely variation in x_0^* appears to be important for power of the unit root tests.

Finally, for all tests the powers when $T = 50$ (not reported) are substantially lower than those at $T = 100$. For example, when $x_0^* = 0$ and $\beta = 0.8$ the power of the $\alpha = 0.05$ size $F(0, 1)$ test is 0.16 and the power of the $\alpha = 0.05$ size lower-tail $t(1)$ test is 0.20. In

TABLE III

POWERS OF TESTS OF H_0: $\beta = 0.85$ FOR $T = 100$

Powers of the $t_A(0.85)$ test, critical value = 0.56

$\lvert x_0^* \rvert$	β 0.85	0.90	0.95	0.99	1.00
0.00	0.05	0.20	0.44	0.60	0.61
1.00	0.05	0.20	0.44	0.60	0.61
2.00	0.05	0.20	0.45	0.60	0.61
3.00	0.05	0.20	0.45	0.60	0.61
4.00	0.05	0.20	0.45	0.60	0.61
5.00	0.05	0.20	0.45	0.60	0.61
6.00	0.05	0.20	0.46	0.60	0.61
7.00	0.05	0.20	0.46	0.60	0.61
8.00	0.05	0.20	0.47	0.60	0.61
9.00	0.05	0.20	0.47	0.60	0.61
10.00	0.05	0.20	0.48	0.60	0.61

Estimates based on 20,000 replications

Exact Powers of the $S_A(0.85)$ test, critical value = 0.506

$\lvert x_0^* \rvert$	β 0.85	0.90	0.95	0.99	1.00
0.00	0.05	0.20	0.45	0.60	0.61
1.00	0.05	0.20	0.45	0.60	0.61
2.00	0.05	0.20	0.45	0.60	0.61
3.00	0.05	0.20	0.45	0.60	0.61
4.00	0.05	0.20	0.46	0.60	0.61
5.00	0.05	0.21	0.46	0.60	0.61
6.00	0.05	0.21	0.46	0.60	0.61
7.00	0.05	0.21	0.47	0.60	0.61
8.00	0.05	0.21	0.47	0.60	0.61
9.00	0.05	0.21	0.48	0.60	0.61
10.00	0.05	0.21	0.48	0.60	0.61

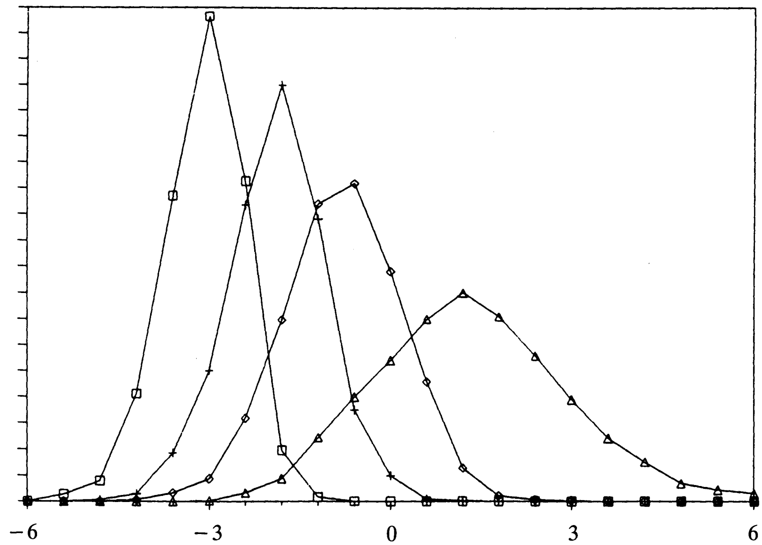
general, the powers of the integration tests are so low when $T = 50$ that they are not worth reporting in detail. In light of the low powers we do not recommend performing the integration tests for $T < 100$.

B. Trend-Stationarity Tests

The powers of 0.05 one-sided similar tests of H: $\beta = 0.85$ for $T = 100$ are given in Table III. Several features of the tables stand out. For example, the powers of the two tests are nearly the same, and increase with β. Also, for the tabled values, the powers of the tests are nearly invariant with respect to x_0^*. For the tabled values of x_0^*, the exact power of detecting a unit root with the $S_A(.85)$ test is 0.61; the empirical power of the $t_A(.85)$ test at $\beta = 1$ is 0.61. These results indicate that the trend-stationarity tests have somewhat better than a 50% chance of detecting a unit root for plausible values of x_0^*.

C. Comparison

Our analysis has revealed that both types of tests are beset by lower power at the alternatives of interest, with the performance of the trend-stationarity tests somewhat better than that of the integration tests. Figure 1 illustrates this situation. The tests examined are the $\alpha = 0.05$ lower-tail $t(1)$ test and the $\alpha = 0.05$ upper-tail $t_A(.85)$ test. The powers of the tests are illustrated for $T = 100$ using four distributions: the distribu-



□ $t(1 \mid .85)$ + $t(1 \mid 1)$ ◇ $ta(.85 \mid .85)$ △ $ta(.85 \mid 1)$

FIGURE 1.—Empirical distributions of $t(\cdot \mid \cdot)$ statistics.

tions of $t_A(.85)$ when $(|x_0^*|, \beta) = (2, 0.85)$ and $(0, 1)$; and the distributions of $t(1)$ when $(|x_0^*|, \beta) = (0, 1)$ and $(2, 0.85)$. Denote the first two distributions by $t_A(.85|.85)$ and $t_A(.85|1)$ and the second two by $t(1|1)$ and $t(1|.85)$: $t_A(.85|.85)$ and $t(1|1)$ are the distributions of $t_A(.85)$ and $t(1)$ under the respective nulls, and $t_A(.85|1)$ and $t(1|.85)$ are the distributions under the alternatives.

The low power—about 0.4—of the $t(1)$ test is shown by the overlap of the $t(1|1)$ and $t(1|.85)$ distributions. The power—about 0.6—of the $t_A(.85)$ test is due to the difference in the shapes of the $t_A(.85|.85)$ and $t_A(.85|1)$ distributions. These power comparisons emphasize the importance of considering the nature of the alternative in hypothesis testing. In particular, the powers of unit root and stationarity tests may not be high enough to *settle* the integration vs. trend-stationarity issue in many practically relevant situations.

6. CONCLUSIONS

There are three major findings in this paper. First, unit root tests have low power against plausible trend-stationary alternatives. Second, tests of an empirically plausible trend-stationarity hypothesis have moderate power against the unit root alternative. Third, there are many cases in which neither test will reject. This suggests that inferences based exclusively on tests for integration may be fragile.

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Manuscript received February, 1989; final revision received June, 1991.

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