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Damage spreading and Lyapunov exponents in cellular automata

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Using the concept of the Boolean derivative we study local damage spreading for one-dimensional elementary cellular automata and define their maximal Lyapunov exponent. A random matrix approximation describes quite well the behavior of "chaotic" cellular automata and predicts a directed percolation-type phase transition. After the introduction of a small amount of noise elementary cellular automata reveal the same type of transition.

1. Introduction

The behavior of the distance between two configurations submitted to the same dynamics (damage spreading) is considered to be a good tool to investigate the ergodic properties of the dynamics of discrete statistical models [1]. Although the relation between these properties and "chaotic" behavior is still unclear, there is an intuitive connection between "chaos" and damage spreading on one side, and between a periodic attractor and damage collapsing on the other. For continuous dynamical systems a positive maximal Lyapunov exponent (MLE) implies chaotic motion. The MLE is roughly defined as the rate of the exponential divergence of the distance between two initially close trajectories in the limit of long times and vanishing initial distances. In what follows we show how Boolean derivatives may be used to define the MLE of a cellular automaton.

A Boolean one-dimensional cellular automaton (CA) is a discrete dynamical system defined on a lattice. The state of the system is represented by a configuration $\boldsymbol{x}=(x_1, ..., x_i, ..., x_L)$ of Boolean variables, where L is the size of the lattice. We always use periodic boundary conditions ($x_{i+L}=x_i$). The time evolution of the system is given by a Boolean function $\mathbf{F}$,

$$\boldsymbol{x}(t+1)=\mathbf{F}(\boldsymbol{x}(t)) , \qquad (1)$$

which is in turn defined locally by a uniform rule f,

$$x_i(t+1)=f(x_{i-r}(t), ..., x_i(t), ..., x_{i+r}(t)) , \qquad (2)$$

where r is the range of the function f. There are $2^{2(2r+1)}$ different CA of range r. In what follows we restrict our study to elementary CA for which $r=1$ and use Wolfram's labeling convention [2].

In the context of CA where time, space and dynamical variables are discrete, we cannot extend directly the definition of Lyapunov exponents [3,4]. Due to the finite interaction range r and to the finite number of states of the variables x_i, the distance between two initially close configurations can increase at most linearly for long times.

Instead of looking at the long time behavior of the distance between two configurations, we can use some hints from the theory of continuous dynamical systems and study the local stability of a single trajectory with respect to a small perturbation (a damage or defect in the configuration). This defect can be readily recovered, or it can freeze being replicated without change, or finally it can propagate increasing the distance between the configurations.

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In section 2, we show that the distance between two configurations after the introduction of a defect is given by the Boolean Jacobian matrix of the evolution function $\mathbf{F}$ [5,6]. While in the actual evolution of the automaton the defects can interact and annihilate themselves, we are interested in the stability of a single trajectory, and we restrict to the case of non-interacting defects which is equivalent to considering a product of Boolean Jacobian matrices on a trajectory. For the elementary CA it is a Jacobi matrix with elements equal to zero or one on the three main diagonals. This in turn suggests a relation with the product of random matrices of the same type. The MLE of the product of these random matrices shows a transition related to that of directed percolation [7,8].

As reported in section 3, the results of simulations of "chaotic" CA (whose space–time patterns, starting from a random configuration, are disordered and aperiodic) agree quite well with the predictions of the random matrix approximation. Adding a small amount of noise to the evolution of the automaton, our approach reveals the existence of a transition in the space of CA from a "frozen phase" where damage does not spread, to a phase where damage spreads locally with a positive MLE close to the one given by the product of random matrices.

2. Boolean derivatives, defects and random matrices

We are interested in the local stability with respect to a small perturbation, of the time evolution (1) of the configuration $\boldsymbol{x}$. Let us denote by $\boldsymbol{z}^{(i)}$ a defect at site i as the configuration with elements $z_j^i=\delta_{i,j}$, $j=1, ..., L$, and $\delta_{i,j}$ the usual Kronecker symbol. The configuration $\boldsymbol{y}(t)=\boldsymbol{x}(t)\oplus\boldsymbol{z}^{(i)}$ differs from $\boldsymbol{x}(t)$ only at site i (the XOR operation $\oplus$ is performed site by site). Depending on $\mathbf{F}$ and on the configuration $\boldsymbol{x}$, the defect $\boldsymbol{z}^{(i)}$ can originate in one time step up to three defects in sites $i-1$, i and $i+1$. Then

$$\boldsymbol{x}(t+1)\oplus\boldsymbol{y}(t+1)$$
$$=F'_{i,i-1}\wedge\boldsymbol{z}^{(i-1)}\oplus F'_{i,i}\wedge\boldsymbol{z}^{(i)}\oplus F'_{i,i+1}\wedge\boldsymbol{z}^{(i+1)} ,$$

where $F'_{i,j}=0$, 1 and the AND operation $\wedge$ is performed between the number $F'_{i,j}$ and each element of the defect $\boldsymbol{z}^{(j)}$. The quantities

$$F'_{i,j}=\frac{\partial x_i(t+1)}{\partial x_j(t)}$$

are the elements of the Boolean Jacobian matrix $\mathbf{F}'$ of $\mathbf{F}$. These are defined in terms of the Boolean derivative of the local evolution rule f of eq. (2); for instance

$$\frac{\partial x_i(t+1)}{\partial x_{i+1}(t)}$$
$$=f(x_{i-1}, x_i, x_{i+1})\oplus f(x_{i-1}, x_i, x_{i+1}\oplus 1) .$$

Since f has range 1, $\partial x_i(t+1)/\partial x_j(t)$ vanishes if $|i-j|>1$, and $\mathbf{F}'$ is a Jacobi matrix. If the local evolution rule is expressed in terms of AND and XOR operations (ring sum expansion), the Boolean derivatives extract the linear part of f.

We are interested in the limit of a small initial perturbation to a given trajectory. This limit corresponds in discrete dynamics to the presence of only one point defect. If, during the evolution, m defects appear, we consider m replicas of the system and assign one of the defects to each one. We indicate with $N_i(t)$ the number of replicas carrying the defect $\boldsymbol{z}^{(i)}$ at time t. If for instance, we start at time zero with only one defect at some site i ($N_i(0)=1$), and the rule allows the spreading of the defects to the sites of the neighborhood at each time step; at $t=1$, $N_{i-1}=N_i=N_{i+1}=1$; at $t=2$, $N_{i-2}=N_{i+2}=1$, $N_{i-1}=N_{i+1}=2$, $N_i=3$, etc.

The time evolution of the number of defects at site i is given by

$$N_i(t+1)=\sum_j F'_{i,j}(t)N_j(t) ,$$

or, in matrix form,

$$\boldsymbol{N}(t+1)=\mathbf{F}'\boldsymbol{N}(t) , \tag{3}$$

where the elements of $\mathbf{F}'$ are not interpreted as integer numbers. It is worth noting that $N_i(t)$ is also the number of paths in defect space that reach the site i at time t starting from any defect at time $t=0$.

We define the finite-time MLE $\lambda(T)$ of the map (3) as

$$\lambda(T)=\frac{1}{T}\sum_{t=1}^{T}\log\eta(t) ,$$

where the local expansion rate of defects η is defined as

$$\eta(t) = |N(t+1)|/|N(t)| ,$$

and $|N| = \sum_i N_i$. In the following $\lambda(\infty)$ will be denoted simply by λ. This definition is meaningful because the number $|N(t)|$ can diverge exponentially.

If $\lambda < 0$ the number of defects (the damage) decreases exponentially to zero, while if $\lambda > 0$ the damage spreads. Let us give some simple examples. Rule 0 that maps all the configurations to the configuration $\{0\}^L$, has $\lambda = -\infty$ because the Jacobian is zero. The "chaotic" rule 150 has $\lambda = \log 3$, because all its Boolean derivatives are equal to one. A marginal case is rule 204, for which $\mathbf{F}'$ is the identity matrix and $\lambda = 0$. The derivatives of the 88 "minimal" elementary CA may be found in ref. [5].

In the spreading case, a reasonable approximation to the dynamics of defects (3) consists in substituting the deterministic matrix $\mathbf{F}'$ with a random matrix of the same form. We therefore consider the product of random tridiagonal matrices $\mathbf{M}(p)$ having a fraction p of elements on the three principal diagonals equal to one. The quantity p is naturally interpreted as the geometric mean μ of the derivative on the CA configuration for large T, i.e.,

$$\mu(T) = \left(\prod_{t=1}^{T} \tilde{\mu}(t)\right)^{1/T}$$

and

$$\tilde{\mu}(t) = \frac{1}{3L} \sum_{i=1}^{L} \sum_{k=-1}^{1} F'_{i,i+k} .$$

The evolution of the number of defects in the random matrix approximation defines a directed bond percolation problem with control parameter p, assuming that a site at location i at time t is "wet" if $N_i(t) > 0$. Observe that $N_i(t)$ gives the number of directed paths that reach site i at time t inside the percolating cluster. Therefore we expect a second-order phase transition at $p = p_c$ with order parameter the density of wet sites $\rho(t)$.

We have first localized the percolation threshold at $p_c = 0.441(1)$ (where the number in parenthesis is the error on the last significant digit) by looking at the asymptotic behavior of the order parameter. Then, starting with an initial condition where all the sites are wet, we have verified that $\rho(t) \sim t^{-\beta/\nu_\parallel}$ at p_c, with $\beta/\nu_\parallel = 0.155(3)$ the usual exponents of directed percolation.

The results of the random matrix approximation are reported in fig. 1, where the curve shows the dependence of the MLE λ for the product of random matrices $\mathbf{M}(p)$ as a function of p. For $p < p_c$ $\lambda = -\infty$. At p_c, for sufficiently large T,

$$\lambda(T) = \lambda + aT^{-\chi} ,$$

with $\lambda = 0.237(2)$, $\chi = 0.68(4)$ and $a > 0$. This shows that at the critical point the number of walks on the percolation cluster grows exponentially with time, with an effective coordination $\exp(\lambda)$. The exponent χ, which is not usually defined in percolation, might be related to the critical exponents for directed walks [9]. The data at the percolation threshold were obtained by letting a $10^4 \times 10^4$ random tridiagonal matrix evolve during 4000 time steps for 30 realizations.

We obtain a mean field approximation replacing $\mathbf{M}$ with a constant tridiagonal matrix with elements equal to p. The corresponding MLE is $\lambda = \log 3p$ which

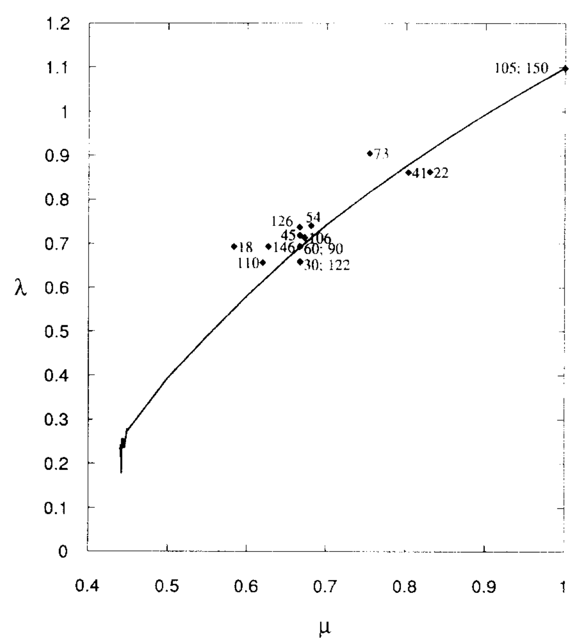

Fig. 1. The curve shows the MLE λ of a random tridiagonal matrix as a function of p. The diamonds show the asymptotic value of λ for "chaotic" CA. Results for the CA were obtained with $T = 5000$, $L = 512$ and $\alpha_0 = 0.5$.

is positive for $p \geqslant \frac{1}{3}$. Our numerical simulations agree well with this mean field approach for $p \geqslant p_c$, with a maximum deviation of 18% at $p = p_c$.

In the numerical calculation of the Lyapunov exponent λ one needs to renormalize $N(t)$ [10]. This is impossible if N is defined over the integers. However, since the Lyapunov exponents are independent of the choice of the initial vector and of the norm in the ergodic case [11], we let $\mathbf{F}'$ (or $\mathbf{M}$) act on some abstract "tangent" space in $\mathbb{R}^L$, using the usual Euclidean norm. Applying standard methods one obtains the Lyapunov exponents related to the exponential divergence of the norm of the product of $\mathbf{F}'$.

3. Elementary cellular automata

We computed the mean number of ones $\mu(T)$ in the Jacobian matrix and the finite-time MLE $\lambda(T)$ for all the 88 "minimal" elementary CA for $L=256$ and $L=512$ and $5000 \leqslant T \leqslant 15000$ starting from random initial configurations with fixed fraction α_0 of live sites, $\alpha_0 = L^{-1} \sum x_i(0)$. The quantities $\mu(T)$ and $\lambda(T)$ are generally already asymptotic for $T \sim 5000$; moreover they show a very weak dependence on α_0 for $0.2 \leqslant \alpha_0 \leqslant 0.8$ (only rules 6, 25, 38, 73, 134 and 154 vary between 10% and 20%).

We note that

(i) CA with constant $\mathbf{F}'$ independent of the configuration (rules 0, 15, 51, 60, 90, 105, 150, 170 and 204) have $\lambda = \log 3\mu$ with $\mu = 0, \frac{1}{3}, \frac{2}{3}$ or 1.

(ii) CA for which all configurations are mapped to a homogeneous state (rules 0, 8, 32, 40, 128, 136, 160 and 168) have $\lambda = -\infty$. The control parameter μ is zero. These are class 1 CA in Wolfram's classification [3].

(iii) "Chaotic" class 3 CA with nonconstant $\mathbf{F}'$ (rules 18, 22, 30, 41, 45, 54, 106, 110, 122, 126 and 146) have $\mu > p_c$, $\lambda > 0$ and the damage spreads.

The values of the MLE for the "chaotic" CA of cases (i) and (iii) agree well with the random matrix approximation, as shown in fig. 1. This is also trivially true for the automata of case (ii).

For the CA with $0 < \mu < p_c$, λ depends on the initial condition; this is revealed by choosing a special initial condition $N(0)$ having only one nonzero component in the map (3). For those automata whose evolution leads to a nonhomogeneous periodic space pattern (class 2 CA), the MLE is the logarithm of the largest eigenvalue of the product of the Jacobian matrices over the periodic state. The measured value of λ is always nonnegative. This suggests that the asymptotic state is unstable ($\lambda > 0$) or marginally stable ($\lambda = 0$). One can think that the "freezing" of the evolution occurs because there are no "close" configurations which can be used as an intermediate state towards a more stable state. Therefore, we "heated" the evolution by exchanging the states of a small number s of pairs of randomly chosen sites at each time step. In fig. 2, we show the values of μ and λ for all the minimal CA for which $\lambda \geqslant 0$ starting with $\alpha_0 = 0.5$ in the presence of a small amount of noise.

After the introduction of noise the CA can be divided roughly in three groups. In the first group with $\lambda = -\infty$, we find all class 1 CA and some class 2 CA (rules 1, 3, 5, 7, 11, 13, 14, 19, 23, 43, 50, 72, 77, 104, 142, 178, 200 and 232). Rules 50, 77 and 178 show very long transients of the order of 15000 time steps. The CA in this group have a small μ in the absence of noise ($\mu < 0.373$). Rule 232, a majority rule,

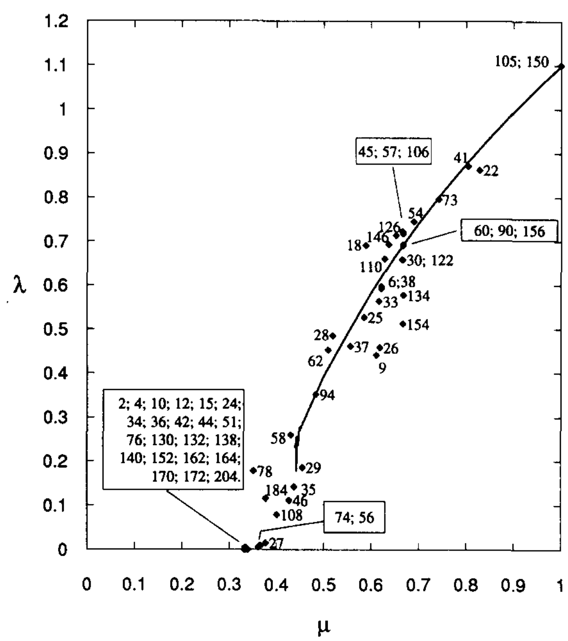

Fig. 2. The curve is the same as the one shown in fig. 1. The diamonds show the values of μ and λ for all the minimal CA with $\lambda \geqslant 0$ in the presence of a small amount of noise $s=2$ and $T=5000$, $L=512$, $\alpha_0=0.5$.

illustrates well a typical behavior. Configurations $\{0\}^L$ and $\{1\}^L$ are fixed points for this CA. A single defect in these configurations is recovered in one time step. On the other hand, an arbitrary initial configuration will relax in a few time steps to a pattern of strips. By adding a noise as described above, the borders of the strips perform a sort of random motion, thus allowing their merging. Finally, one of the two fixed points is reached, according to the initial density of the configuration.

The second group of CA has a positive MLE. It contains the class 3 CA and rules 6, 9, 25, 26, 28, 33, 37, 38, 57, 62, 73, 94, 134, 154 and 156 which are not class 3 but show local damage spreading. The values of μ and λ are slightly affected by the noise. The CA in this group have $\mu > p_c$ and λ close to the curve of the random matrix approximation.

CA in the third group have $\lambda \sim 0$, a value which is never found in the product of random matrices. The CA in this group have an intermediate value of μ ($0.281 < \mu < 0.54$ without noise and $\frac{1}{3} < \mu < p_c$ in the presence of noise). Contrary to the prediction of the random matrix approximation N does not vanish for long times. The CA in this group are rules 2, 4, 10, 12, 15, 24, 27, 29, 34, 35, 36, 42, 44, 51, 56, 58, 74, 76, 78, 108, 130, 132, 138, 140, 152, 162, 164, 170, 172, 184 and 204. Moreover, rules 4, 10, 12, 15, 34, 42, 51, 76, 138, 140, 170 and 204 have conserved additive quantities [12].

In this Letter we have shown how the MLE can be defined for CA using the Boolean derivative. A positive Lyapunov exponent is associated to local damage spreading and on the other hand reflects the exponential growth of paths on directed percolation clusters. For CA with $0 < \mu < p_c$ which do not spread damage but have a positive Lyapunov exponent the introduction of a small noise produces a collapse to $\lambda = 0$ or $\lambda = -\infty$. A random matrix model is directly suggested by the CA dynamics and displays a directed percolation phase transition. The same phase transition is observed in the CA rule space in the presence of a small amount of noise. The extension of our definition of Lyapunov exponent to other discrete systems, and possibly to probabilistic dynamics will be the subject of future investigations.

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