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FLOW PAST A SPHERE EMBEDDED IN A POROUS MEDIUM BASED ON THE BRINKMAN MODEL

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ABSTRACT

In this paper we present a closed form, exact solution for the forced flow past a sphere which is embedded in a porous medium using the Brinkman model. The theory shows that there is no flow separation for this flow configuration. However, for typical materials that occur in practice there is a velocity overshoot in the vicinity of the sphere and a mathematical explanation of this phenoma is presented.

Introduction

The general subject of the flow in a porous medium has received a great deal interest during the past three decades and the vast amount of literature that has been devoted to this subject has been recently reviewed by Nield and Bejan [1].

The objective of this paper is to report a theoretical investigation of the problem of forced convection flow past a sphere which is immersed in a porous medium based on the Brinkman model [2]. A closed form exact analytical solution of the governing equations is obtained and this leads to an expression for the separation parameter, similar to the one reported by Pop and Cheng [3] for the corresponding problem of a circular cylinder. The variation of the separation parameter and the velocity profile tangential to the surface of the sphere for different particle/sphere diameter ratios are investigated. It is found that in practical situations the flow does not separate from the sphere and a velocity overshoot occurs near to the surface of the sphere. The method of matched asymptotic expansions is employed in order to explain this velocity overshoot.

Basic Equations

Consider the steady, incompressible flow (with a constant free stream velocity U_∞) past a sphere of radius a, which is embedded in a porous medium of uniform porosity Φ. We use a spherical polar coordinate system $(\bar{r}, \theta, \epsilon)$, with the origin at the centre of the sphere and the axis $\theta = 0$ along the direction of the undisturbed flow as shown in Fig.1. Due to the symmetry of the problem we have $\partial/\partial\epsilon = 0$. It is now convenient to non-dimensionalise all variables by writing

$$r = \bar{r}/a, \quad u = \bar{u}/U_\infty, \quad v = \bar{v}/U_\infty, \quad p = \mu\bar{p}/(aKU_\infty) \tag{1}$$

where $\bar{u}$ and $\bar{v}$ are the velocity components along the r and θ directions, respectively, $\bar{p}$ is the pressure, μ is the viscosity and K is the permeability of the porous medium, which is related to the porosity Φ by the relation [1]

$$K = \frac{d^2\Phi^3}{A(1-\Phi)^2} \tag{2}$$

where A is the Ergun [4] constant and d is the mean particle diameter. Using [1], the governing equations for the problem under consideration can be written in dimensionless form as

$$\frac{\partial}{\partial r}(r^2 u sin\theta) + \frac{\partial}{\partial\theta}(r v sin\theta) = 0 \tag{3}$$

$$-\frac{\partial p}{\partial r} = u - \sigma^2\left(\nabla^2 u - \frac{2u}{r^2} - \frac{2}{r^2}\frac{\partial v}{\partial \theta} - \frac{2v cot\theta}{r^2}\right) \tag{4}$$

$$-\frac{1}{r}\frac{\partial p}{\partial \theta} = v - \sigma^2\left(\nabla^2 v - \frac{2}{r^2}\frac{\partial u}{\partial \theta} - \frac{v}{r^2 sin\theta}\right) \tag{5}$$

where

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{cot\theta}{r^2}\frac{\partial}{\partial \theta} \tag{6}$$

and σ is a dimensionless parameter defined as

$$\sigma = \frac{1}{a}\sqrt{\frac{K}{\Phi}} = \gamma\frac{\Phi}{(1-\Phi)A^{1/2}} \tag{7}$$

where $\gamma = d/a$. Typically, in a porous medium $\Phi = 0.4$ and $A = 180$ and hence

$$\sigma = 0.0497\gamma \tag{8}$$

In most physical problems of interest γ is small and hence σ is a very small parameter.

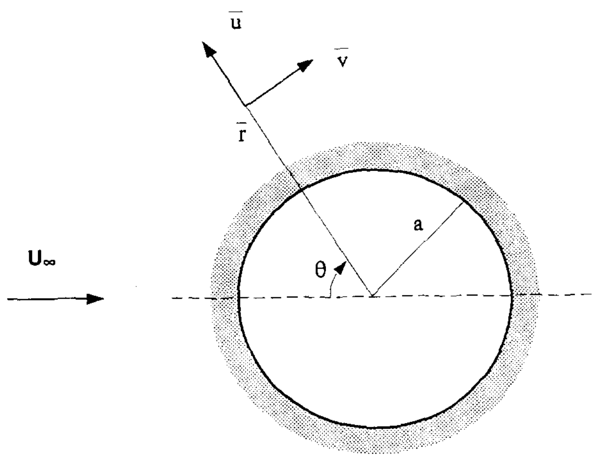

FIG. 1
Physical model and coordinate system

If we make use of the stream function ψ, defined such that

$$u = -\frac{1}{r^2 sin\theta}\frac{\partial \psi}{\partial \theta}, \quad v = \frac{1}{r sin\theta}\frac{\partial \psi}{\partial r} \tag{9}$$

and eliminate the pressure between Eqs (4) and (5), we obtain

$$\sigma^2\Delta^4\psi - \Delta^2\psi = 0 \tag{10}$$

where

$$\Delta^2 = \frac{\partial^2}{\partial r^2} + \frac{sin\theta}{r^2}\frac{\partial}{\partial\theta}\left(\frac{1}{sin\theta}\frac{\partial}{\partial\theta}\right) \tag{11}$$

Equation (10) has now to be solved subject to the boundary conditions

$$\psi = \frac{\partial\psi}{\partial r} = 0 \;\; at \;\; r = 1 \;\; \text{for all values of} \;\; \theta \tag{12}$$

$$\psi \sim \frac{1}{2}\left(r^2 - \frac{1}{r}\right) sin^2\theta \;\; \text{for all} \;\; r > 1 \;\; \text{and all values of} \;\; \theta \tag{13}$$

Method of solution

From the boundary conditions (12) and (13) it is obvious that one should look for a solution of Eq (10) in the form

$$\psi(r,\theta) = f(r)sin^2\theta \tag{14}$$

which on substitution into Eq (10) we obtain

$$f^{iv} - \frac{2}{r^2}f'' + \frac{8}{r^3}f' - \frac{8}{r^4}f - \frac{1}{\sigma^2}\left(f'' - \frac{2}{r^2}f\right) = 0 \tag{15}$$

and the boundary conditions (12) and (13) become

$$f(1) = f'(1) = 0 \tag{16}$$

$$f(r) \sim \frac{1}{2}r^2 \;\; as \;\; r \to \infty \tag{17}$$

respectively, where primes denote differentiation with respect to r. We further denote

$$f'' - \frac{2}{r^2}f = (r)^{1/2}g(r) \tag{18}$$

where g is given by the equation

$$r^2g'' + rg' - \left[\left(\frac{3}{2}\right)^2 + \left(\frac{r}{\sigma}\right)^2\right] g = 0 \tag{19}$$

The solution of this equation can be expressed in terms of the modified Bessel functions as

$$g(r) = BK_{3/2}(r/\sigma) + CI_{3/2}(r/\sigma) \tag{20}$$

where B and C are two unknown constants. On imposing the boundary condition (17) we obtain C = 0. Thus, we find that

$$f(r) = Dr^2 + \frac{E}{r} + B(r)^{1/2}K_{3/2}(r/\sigma) \tag{21}$$

where D and E are two unknown constants. In order to determine the constants, B, D and E, we impose the boundary conditions (16) and (17) to give

$$f(r) = \frac{1}{2}r^2 - \frac{1}{2}\left[1 + 3\sigma\frac{K_{3/2}(1/\sigma)}{K_{1/2}(1/\sigma)}\right]\frac{1}{r} + \frac{3\sigma}{2K_{1/2}(1/\sigma)}(r)^{1/2}K_{3/2}(r/\sigma) \tag{22}$$

Results and Discussion

To investigate whether the flow separates from the surface of the sphere, we calculate the dimensionless vorticity on the surface of the sphere, which is given by

$$\left.\frac{\partial^2\psi}{\partial r^2}\right|_{r=1} = f''(1)sin^2\theta \tag{23}$$

According to Underwood [5], the separation parameter can be defined as

$$SEP = f''(1) \tag{24}$$

which on substituting Eq (22) into Eq (24) gives

$$SEP = \frac{3}{\sigma}\frac{K_{3/2}(1/\sigma)}{K_{1/2}(1/\sigma)} \tag{25}$$

In order to simplify the expressions (22) and (25) we use the fact that

$$K_{1/2}(x) = \sqrt{\frac{\pi}{2x}}e^{-x} \tag{26}$$

$$K_{3/2}(x) = \sqrt{\frac{\pi}{2x}}\ e^{-x}(1 + \frac{1}{x}) \tag{27}$$

see Abramowitz and Stegun [6]. Thus the expression for SEP, as given in expression (25) becomes

$$SEP = \frac{3}{2}(\frac{1}{\sigma} + 1) \tag{28}$$

which is always positive for all physical values of the parameter σ. We therefore conclude that there is no flow separation occurring for the flow past a sphere which is embedded in a constant porosity medium based on the Brinkman model. Further, the smaller the value of σ, the larger is the value of SEP and hence the larger is the skin friction on the surface of the sphere.

The tangential velocity profile can be calculated from Eqs (9), (14) and (22) as

$$v = \frac{f'(r)}{r} sin\theta \tag{29}$$

where

$$\frac{f'(r)}{r} = 1 + \frac{1}{2}\left[1 + 3\sigma\frac{K_{3/2}(1/\sigma)}{K_{1/2}(1/\sigma)}\right]\frac{1}{r^3} - \frac{3}{2K_{1/2}(1/\sigma)}(r)^{1/2}K_{1/2}(r/\sigma) - \frac{3\sigma}{2K_{1/2}(1/\sigma)}(r)^{3/2}K_{3/2}(r/\sigma) \tag{30}$$

On using the expressions (26) and (27) for the modified Bessel functions, expression (30) can be simplified to

$$\frac{f'(r)}{r} = 1 + \frac{1}{2}\ [1 + 3\sigma(1+\sigma)]\frac{1}{r^3} - \frac{3}{2}\sigma\ e^{\frac{1}{\sigma}(1-r)}\ \left[\frac{\sigma}{r^3} + \frac{1}{r\sigma} + \frac{1}{r^2}\right] \tag{31}$$

The function $f'(r)/r$ is presented in Fig.2 as a function of r for $\sigma = 0.01, 0.1$, and 1. In all cases the tangential velocity increases from zero at the surface of the sphere ($r = 1$) and asymptotes to unity far away from the sphere. It is observed that for $\sigma = 0.01$ and 0.1 there is a substantial velocity overshoot whereas when $\sigma = 1$, this overshoot is small. As has been observed earlier, in practice σ is usually very small and hence in such circumstances there is always a velocity overshoot.

The velocity overshoot behaviour, as shown in Fig. 2, will now be examined further. Since the dimensionless particle diameter is much less than unity then Eq. (8) shows that, in general, σ is small. Thus Eq (10) can be approximated by

$$\Delta^2\psi \simeq 0 \tag{32}$$

in a large region of the solution domain. Hence, the solution of the outer flow problem is that of potential flow past a sphere, i.e.

$$\psi = \frac{1}{2}\left(r^2 - \frac{1}{r}\right) sin^2\theta \tag{33}$$

The tangential velocity is then given by

$$v = \frac{1}{r sin\theta}\frac{\partial\psi}{\partial r} \sim (1 + \frac{1}{2r^3}) sin\theta \tag{34}$$

and the function $(1 + \frac{1}{2r^3})$ is also shown in Fig. 2. From expression (34) it is seen that $v \sim (3/2) sin\theta$ near the surface of the sphere, where $r = 1$. On a detailed investigation of expression (31), see Fig. 2, it is seen that as $r \to 1$ then $f'(r)/r \to 3/2$ as $\sigma \to 0$.

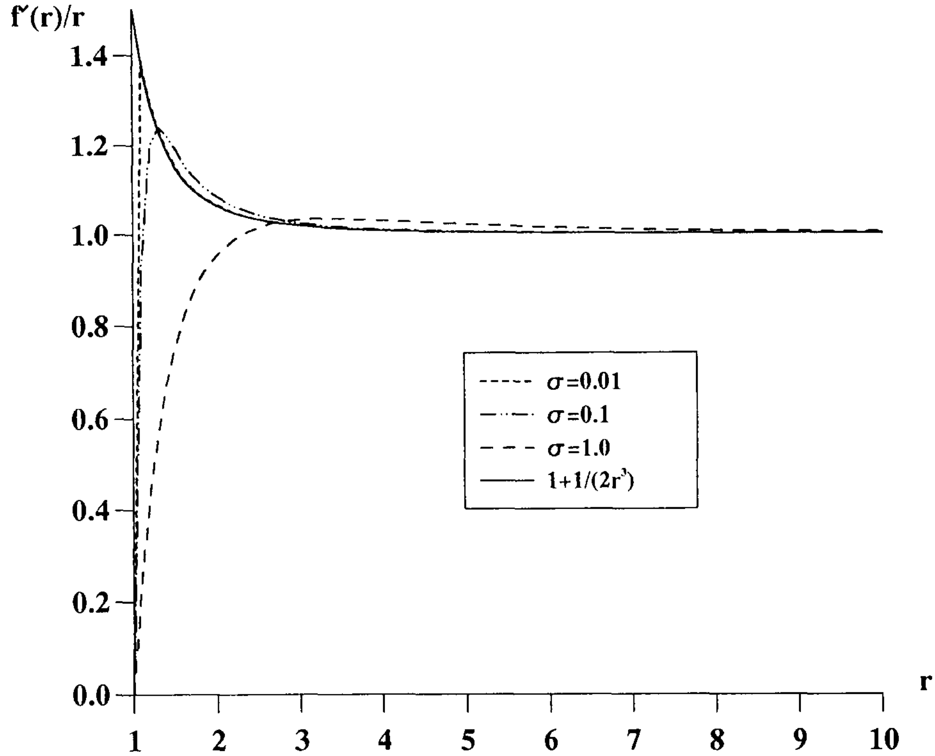

FIG. 2
Variation of $f'(r)/r$ as a function of r

We now obtain an approximate solution of Eq (10) by the method of matched asymptotic expansions for $\sigma << 1$. In the inner solution (viscous sublayer), we have

$$\begin{aligned} f(R) &= \frac{1}{2}(1+\sigma R)^2 - \frac{1}{2}[1+3\sigma(1+\sigma)]/(1+\sigma R) \\ &+ \frac{3}{2}\sigma e^{-R}(1+\sigma/(1+\sigma R)) \end{aligned} \tag{35}$$

where $R = (r-1)/\sigma$, whilst in the outer flow region, we have

$$f(r) = \frac{1}{2}r^2 - \frac{1}{2}[1+3\sigma(1+\sigma)]/r \quad + \frac{3}{2}\sigma e^{\frac{1}{\sigma}(1-r)}(1+\sigma/(1+\sigma/r)) \tag{36}$$

Substituting Eq. (35) into (24) yields

$$SEP = \frac{1}{\sigma^2}\frac{\partial^2 f}{\partial R^2}\bigg|_{R=0} = \frac{3}{2}(\frac{1}{\sigma}+1) \tag{37}$$

which agrees exactly with Eq (28).

In order to illustrate the nature of the fluid flow, Fig 3 shows the streamlines for various values of σ. The values of $\Psi = 0.02, 0.4, 0.6, 0.8, 1.0$ and 1.2 for $\sigma = 0.01, 0.1$ and 1 are presented. It is observed that

(i) the smaller the value of σ the closer together are the streamlines and this illustrates that as σ tends to zero that there is a thinner boundary-layer being developed on the surface of the sphere. This confirms the need for the asymptotic analysis when σ is very small.

(ii) the flow is symmetrical both before and after the sphere. This is clear from the assumed form of the stream function given in Eq. (14).

(iii) no separation occurs near the rear stagnation point. This is confirmed in expression (28) where SEP is always greater than zero.

Nomenclature

a	radius of the sphere
A	Ergun constant
B,C,D,E	integration constants
d	particle diameter
f,g	functions of r
$I_{3/2}$	modified Bessel functions
$K_{1/2}, K_{3/2}$	modified Bessel functions
K	permeability of the porous medium
$\bar{p}$	pressure
$\bar{r}$	radial coordinate
R	transformed radial coordinate
SEP	separation parameter
$\bar{u}, \bar{v}$	velocity components along $\bar{r}$ and θ directions, respectively
Greek symbols	
γ	dimensionless particle diameter
σ	small dimensionless parameter
μ	viscosity
Φ	porosity
θ	polar angle
ϵ	azimuthal angle
ψ	stream function
subscript	
∞	condition far from the sphere

superscript

dimensional variables

differentiation with respect to r

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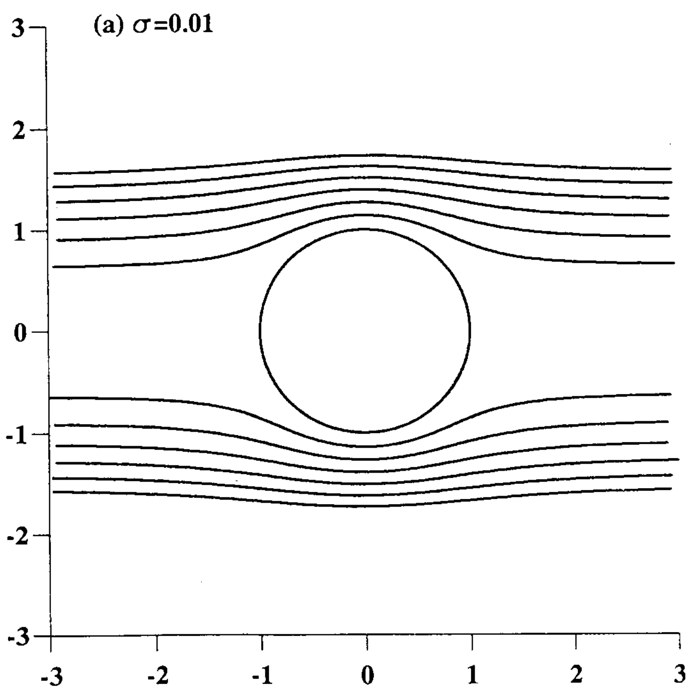

FIG. 3
Streamlines: (a) $\sigma = 0.01$; (b) $\sigma = 0.1$; (c) $\sigma = 1.0$

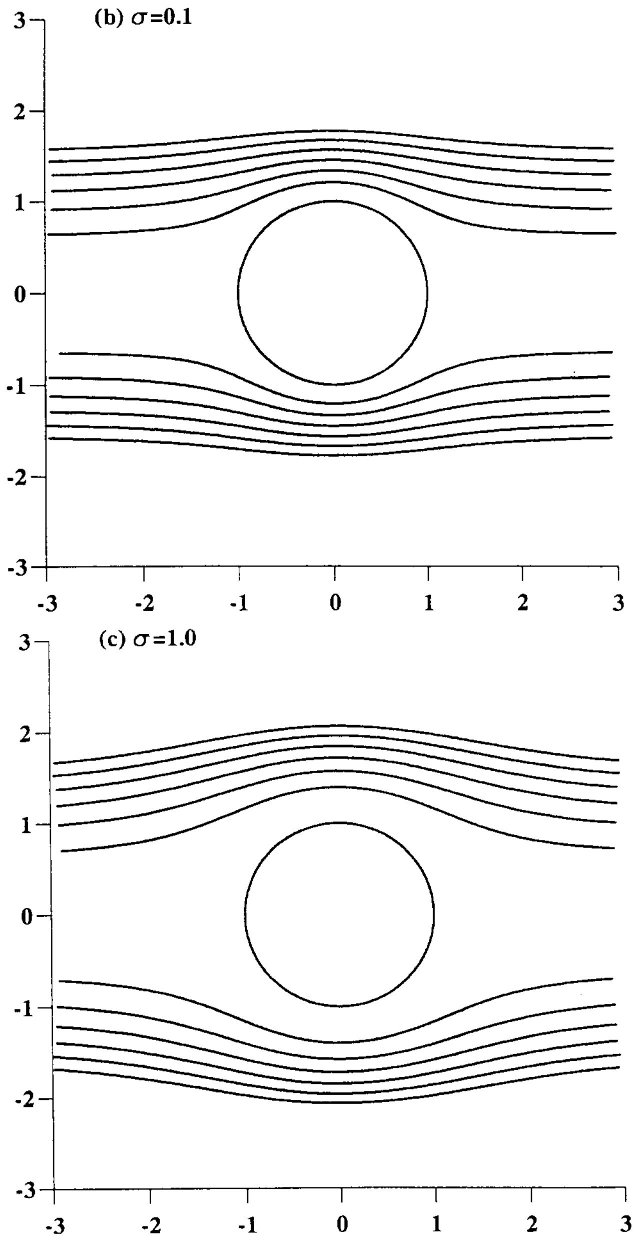

FIG. 3
Streamlines: (a) $\sigma = 0.01$; (b) $\sigma = 0.1$; (c) $\sigma = 1.0$
(continued)

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