*Agricultural Water Management*, 6 (1983) 161—175
Elsevier Science Publishers B.V., Amsterdam — Printed in The Netherlands


# IMPLICATIONS OF SOME ANALYTICAL SOLUTIONS FOR DRAINAGE OF SOIL WATER

P.A.C. Raats

## INTRODUCTION

To describe movement of water in unsaturated soils, just over 50 years ago, Richards (1931) proposed the simplest possible balance of mass and balance of momentum, the latter expressed in Darcy's law. Restricting attention to one-dimensional vertical flows, the balance of mass for the water may be written as:

$$\frac{\partial \theta}{\partial t} = - \frac{\partial}{\partial z} \theta v, \tag{1}$$

where t is the time, z is a vertical coordinate with its origin at the soil surface and taken positive downward, θ is the volumetric water content, and v is the velocity of the water. The volumetric flux θv is given by Darcy's law:

$$\theta v = - k[\theta] \frac{\partial h}{\partial z} + k[\theta], \tag{2}$$

$$= - D[\theta] \frac{\partial \theta}{\partial z} + k[\theta], \tag{3}$$

where h is the tensiometer pressure head, k is the hydraulic conductivity, and D is the diffusivity defined by

$$D = k \frac{dh}{d\theta}. \tag{4}$$

Symbols enclosed in square brackets denote functional dependence. Unlike the dependence of k upon θ, the dependence of h upon θ tends to be subject to hysteresis. As a consequence, equation (3) is, strictly, only valid for monotonic changes in water content from some initial condition with uniform θ and h.

Introducing (3) into (1) gives

$$\frac{\partial \theta}{\partial t} = \frac{\partial}{\partial z} D[\theta] \frac{\partial \theta}{\partial z} - \varkappa[\theta] \frac{\partial \theta}{\partial z}, \tag{5}$$

0378-3774/83/$03.00 

where

$$\varkappa[\theta] = dk/d\theta. \tag{6}$$

If the gradient of the tensiometer pressure head << 1, then the first term on the right hand sides of (2), (3), and (5) can be neglected. Thus for such purely gravitational flows

$$\theta v = k[\theta], \quad \text{or } v = k[\theta]/\theta, \tag{7}$$

$$\frac{\partial\theta}{\partial t} = -\varkappa[\theta]\frac{\partial\theta}{\partial z}. \tag{8}$$

Equation (8) is a kinematic wave equation with wave speed $\varkappa[\theta]$ (cf. Whitham, 1974). According to (8), surfaces of constant $\theta$ will propagate downward at speeds $\varkappa[\theta]$. The ratio $R[\theta]$ of the wave speed $\varkappa[\theta]$ to the speed $v_g$ of a parcel of water due to the gravitational force is given by:

$$R[\theta] = \frac{\varkappa}{v_g} = \frac{dk/d\theta}{k/\theta} = \frac{k^{-1}\,dk}{\theta^{-1}\,d\theta} = \frac{d\ \ln k}{d\ \ln\theta}. \tag{9}$$

Thus R is equal to the ratio of the relative rates of change of k and of $\theta$. This ratio will be shown to play an important role in gravitational drainage.

The main difficulty in applying Richards' theory is the spatial variability of the parameters. In recent years the distribution of the parameters has been determined in numerous field experiments. Often the distributions are found to be lognormal. Lognormal distributions satisfy the following reproductive rule: if the variable $\chi$ is lognormally distributed with mean $\mu$ and variance $\sigma^2$, then $e^a\chi^b$ is lognormally distributed with mean $a + b\mu$ and variance $(b\sigma)^2$ (Aitchison and Brown, 1957). The theory of scaling and similarity of unsaturated soils implies that various parameters and physical properties $\nu$ are proportional to integer powers n of a microscopic length scale $\lambda$ (see Miller, 1980 for a review of this theory). Thus, if the length scale is lognormally distributed, then it follows that various parameters and physical properties will also be lognormally distributed. A summary of the results is shown in Table 1. A detailed derivation and evaluation is presented elsewhere (Raats, 1983).

The main purpose of this paper is to describe the theoretical background of some experimental techniques commonly used to collect the field data showing the distribution of the physical properties. Much of the analysis will be valid for arbitrary dependencies of h and k upon $\theta$. In some parts one of the following two classes of soils will be considered:

A. Soils with a linear retention curve and an exponential dependence of the hydraulic conductivity upon the water content;

TABLE 1

Scaling rules and implied means and variances for a set of similar media with lognormally distributed length scales.

| $\nu$ | n | Mean of $\nu$ | Variance of $\nu$ |
|---|---|---|---|
| t | -3 | $-3\ \mu$ | $9\ \sigma^2$ |
| h | -1 | $-\ \mu$ | $\sigma^2$ |
| H | -1 | $-\ \mu$ | $\sigma^2$ |
| x, y, z | -1 | $-\ \mu$ | $\sigma^2$ |
| $\theta$ | 0 | - | - |
| $\nabla$ | 1 | $\mu$ | $\sigma^2$ |
| $d\theta/dh$ | 1 | $\mu$ | $\sigma^2$ |
| D | 1 | $\mu$ | $\sigma^2$ |
| $k^{-1}\,dk/dh$ (= $\alpha$ for A) | 1 | $\mu$ | $\sigma^2$ |
| k | 2 | $2\ \mu$ | $4\ \sigma^2$ |
| $\theta v$ | 2 | $2\ \mu$ | $4\ \sigma^2$ |
| v | 2 | $2\ \mu$ | $4\ \sigma^2$ |
| $\varkappa = dk/d\theta$ | 2 | $2\ \mu$ | $4\ \sigma^2$ |
| dk/dh | 3 | $3\ \mu$ | $9\ \sigma^2$ |

B. Soils with power function dependencies of the pressure head and the hydraulic conductivity upon the water content.

These relationships are listed in Table 2, together with the implied dependencies of the diffusivity upon the water content and of the hydraulic conductivity and diffusivity upon the pressure head.

Note that the implied dependencies for class A are all exponential functions and for class B are all power functions. The exponential dependence of k upon h implies that (2) may be replaced by

$$\theta v = -\alpha^{-1}\frac{\partial k}{\partial z} + k, \tag{10}$$

i.e. the flux is linear in k. Introducing (10) into (1) and using the exponential relationship for $k[\theta]$ listed in Table 1 gives (Raats, 1976):

$$(\beta k)^{-1}\frac{\partial k}{\partial t} = \alpha^{-1}\frac{\partial^2 k}{\partial z^2} - \frac{\partial k}{\partial z}. \tag{11}$$

TABLE 2

Two classes of soils.

| | | Mildly nonlinear soils (A) | Power function soils (B) |
|---|---|---|---|
| *primary relationships* | $h[\theta]$ | $h_r + \gamma(\theta - \theta_r)$ | $h_s(\theta/\theta_s)^{\ell}$ |
| | $k[\theta]$ | $k_r \exp \beta(\theta - \theta_r)$ | $k_s(\theta/\theta_s)^m$ |
| *derived relationships* | $D[\theta]$ | $D_r \exp \beta(\theta - \theta_r)$ where $D_r = \gamma k_r$ | $D_s(\theta/\theta_s)^n$ where $D_s = (\ell k_s h_s/\theta_s)$, $n = \ell + m - 1$ |
| | $k[h]$ | $k_r \exp \alpha(h - h_r)$ where $\alpha = \beta/\gamma$ | $k_s(h/h_s)^p$ where $p = m/\ell$ |
| | $D[h]$ | $D_r \exp \alpha(h - h_r)$ | $D_s(h/h_s)^q$ where $q = n/\ell = (\ell + m - 1)/\ell$ |

The mild nonlinearity of (11) guarantees a qualitatively correct description. If the exponential relationship between k and h were combined with the linear relationship between k and θ given by

$$k = k_r + \omega\,(\theta - \theta_r), \tag{12}$$

then the factor $(\beta k)^{-1}$ on the left hand side of (11) would be replaced by $\omega^{-1}$. The resulting linear flow equation can be used to describe certain integral features of water movement, but does not adequately describe the distribution of the water content (Ababou *et al.*, 1979).

Class A was used in studies of drainage of soil profiles by Raats (1976) and Parlange (1982) and in an analysis of constant-flux infiltration from a hemispherical cavity by Clothier and Scotter (1982). Class B was used in many studies including an analysis of redistribution by Gardner *et al.* (1970). Both classes, sometimes in slightly different and/or incomplete forms, have been used as a basis for evaluating field observations (Nielsen *et al.*, 1973; Warrick *et al.*, 1977 a, b; Simmons *et al.*, 1979; Libardi *et al.*, 1980)

## 2. INSTANTANEOUS PROFILE METHODS

A soil is characterized by the dependencies of h and k upon θ. Simultaneous measurements of distributions of θ and h yield directly $h[\theta]$. If the distribution of θv could also be measured, then $k[\theta]$ could also be calculated. A method for measuring the distribution of $\theta v$ does not exist at present and is not likely to be forthcoming. But one can get around this shortcoming of technology by integrating the balance of mass:

$$(\theta v)\big|_{z_2[t]} - (\theta v)\big|_{z_1[t]} = -\int_{z_1[t]}^{z_2[t]} \frac{\partial\theta}{\partial t}\, dz, \tag{13}$$

or, using Leibniz's rule,

$$(\theta v)\big|_{z_2[t]} - (\theta v)\big|_{z_1[t]} = \theta[z_2, t]\, \frac{dz_2}{dt} - \theta[z_1, t]\, \frac{dz_1}{dt} - \frac{d}{dt}\int_{z_1[t]}^{z_2[t]} \theta dz. \tag{14}$$

Equation (14) shows that one of the fluxes on the left hand side can be calculated, provided the other flux, the time course of the distribution of the water content, and the velocities $dz_1/dt$ and $dz_2/dt$ are known. Two important special cases of (14) are:

*1*. $z_1$ is the soil surface with flux equal to zero and $z_2$ is independent of time

$$(\theta v)\big|_{z_2} = -\frac{d}{dt}\int_{z_1}^{z_2} \theta dz. \tag{15}$$

*2*. $z_1$ is independent of time and $z_2[t]$ is the location of a zero-flux plane

$$(\theta v)\big|_{z_1} = \frac{d}{dt}\int_{z_1}^{z_2[t]} \theta dz - \theta[z_2, t]\, \frac{dz_2}{dt}. \tag{16}$$

Cooper (1979, 1980) gave a time-integrated version of (16). Although, following Richards *et al.* (1956), the zero-flux plane method for calculating fluxes has often been used, I have not found in the literature a derivation of Equation (16) as such.

In this paper I restrict my attention to the first special case. Introducing (2) into (15) and solving for k gives:

$$k = \frac{\dfrac{d}{dt}\displaystyle\int_{z_1}^{z_2} \theta dz}{\dfrac{\partial h}{\partial z} - 1}. \tag{17}$$

To calculate k from (17), the time courses of the distributions of $\theta$ and h must be known. Collecting and processing such data is expensive. This explains the keen interest of soil physicists in any features in data sets that may relax the data requirements and/or simplify the data analysis. The main such features in some data sets are:

*1*. The gradient of the tensiometer pressure head is << 1 i.e., the flow is nearly purely gravitational. If this so-called unit gradient approximation is acceptable, then only the time course of the volumetric water content needs to be measured (Black *et al.*, 1968; Davidson *et al.*, 1969).

*2.* The average water content *above depth z* is a linear function of the water content at *depth z* (Simmons *et al.*, 1979; Libardi *et al.*, 1980).

*3.* The water content distributions at successive times are parallel (Black *et al.*, 1968; Davidson *et al.*, 1969).

In Section 3 the implications of feature *1* will be described for soils with arbitrary dependencies of h and k upon θ. In Section 4 the influence of capillarity will be evaluated for soils of class A. It will be shown that features *2* and *3* can be rationalized to some extent.

## 3. GRAVITATIONAL DRAINAGE

Drainage of uniform profiles with a deep water table requires solution of (8) subject to the initial condition

$$\theta[z,0] = \theta_i, \qquad k[z,0] = k_i, \qquad \text{for } z > 0, \tag{18}$$

where $\theta_i$ and $k_i$ are the uniform initial water content and hydraulic conductivity in the soil profile. Sisson *et al.* (1980) observed that if one adds to (18) a fictive condition

$$\theta[z,0] = \theta_m, \qquad k[z,0] = k_m = 0, \qquad \text{for } z < 0, \tag{19}$$

where $\theta_m$ is a fictive, uniform water content in the region $z < 0$, chosen such that the hydraulic conductivity $k_m$ corresponding to $\theta_m$ is zero, then conditions (18) and (19) describe an initial shock and solution of (8) subject to these conditions describes the decay of this shock. The solution of this type of problem is due to Lax (1972, 1973). Values of water content in the range $\theta_m < \theta < \theta_i$ will propagate downward from the soil surface at speeds $\varkappa[\theta]$. Noting that

$$\varkappa[\theta] = \frac{\partial z}{\partial t}\bigg|_\theta , \tag{20}$$

it follows that the depth z at which the water content will be θ at time t is given by

$$z[\theta,t] = \varkappa[\theta]t. \tag{21}$$

For depths larger than $z > \varkappa[\theta_i]t$ the water content will be $\theta_i$. If k is given as a function of θ, then the profiles $\theta[z,t]$ can be easily determined graphically (Raats, 1982). In Fig. 1 the depth-time courses of particular water contents are shown in the first quadrant: according to (21) these are straight lines with slopes $\varkappa[\theta]$. Intersections of the straight lines with vertical lines of constant t give pairs of θ and z at constant t. From this the successive profiles of θ shown in the third quadrant can be plotted. Intersections of the straight lines with horizontal lines of constant z give pairs of $\theta$ and t at constant z. From this the successive time course of $\theta$ shown in the first quadrant can be plotted.

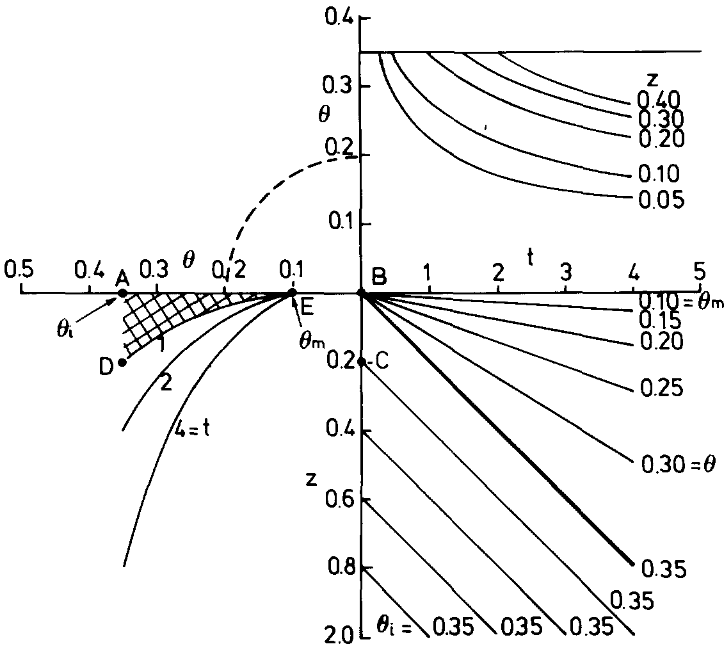


Fig. 1. Depth-time course of water content for gravitational drainage

Sisson *et al.* (1980) derived explicit expressions for $\theta[z,t]$ for the classes of soils listed in Table 2. Introducing the $k[\theta]$ relationship for Class A into (21) gives an explicit expression for the successive water content profiles:

$$\theta = \theta_i + \beta^{-1} \ln \frac{z}{\beta k_i t}, \quad 0 < \theta < \theta_i. \tag{22}$$

Note that for class A the condition (19) is not satisfied in the physically relevant range $0 < \theta < \theta_i$. As was pointed out already by Sisson *et al.*, purely gravitational drainage will for soils of class A imply $\theta = 0$ for $z < \varkappa[\theta = 0, t]t$. Thus the complete solution is

$$\begin{aligned} \theta &= 0, & z &< \varkappa[\theta = 0]t, \\ \theta &= \theta_i + \beta^{-1} \ln \frac{\alpha z}{t/\tau_i}, & \varkappa[\theta = 0]t &< z < \varkappa[\theta = \theta_i]t, \\ \theta &= \theta_i, & z &> \varkappa[\theta = \theta_i]t, \end{aligned} \tag{23}$$

where $\alpha$ is defined in Table 2 and $\tau_i$ is defined by

$$\tau_i = (\alpha\beta k_i)^{-1} . \tag{24}$$

If the expression for $\theta$ in (23) is considered over the entire range $-\infty < \theta < \infty$, then curves of $\beta\theta$ versus $\alpha z$ at successive $t/\tau_i$ are seen to be parallel to each other (see Fig. 2).

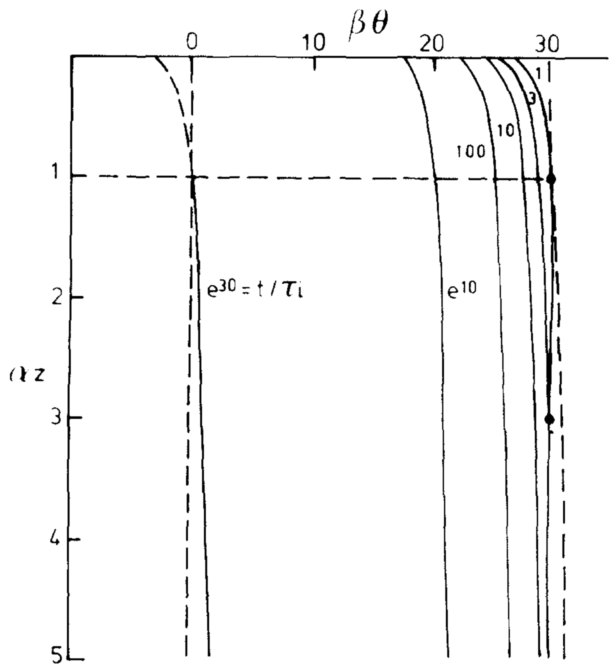


Fig. 2. Water content profiles during gravitational drainage of soils of class A.

Earlier it was pointed out that this is one of the features in some data sets. In section 4 it will be shown that, even if the effect of capillarity is not neglected, for class A, the curves of $\theta$ versus z at successive times will still be parallel, but then with a zone $\theta < 0$ appearing only after a very long time.

Returning to the general case, the amount of water stored at time t above depth z is given by

$$w = \int_0^z \theta dz . \tag{25}$$

For purely gravitational flow a surprisingly simple expression for w, valid for any $k[\theta]$ will now be derived. Integration of (19) by parts gives

$$w = \theta z - \int_{\theta_m}^{\theta} z \, d\theta. \tag{26}$$

Introducing (21) into (26) and evaluating the integral at constant t, recalling that $k_m = 0$, gives

$$w = \theta z - kt. \tag{27}$$

Although Sisson *et al.* did not derive this general expression for w, for specific exponential and power function dependencies of the hydraulic conductivity upon the water content, they did derive expressions for w that are consistent with (27). Equations (26) and (27) allow a simple graphical interpretation. In the third quadrant of Fig. 1, at t = 1, z = 0,2, and $\theta$ = 0,35 the term $\theta z$ corresponds to the rectangle ABCD, while the integral in (26) and kt in (27) correspond to the shaded area. Solving (27) for k gives:

$$k[\theta] = \frac{\theta z - w}{t}. \tag{28}$$

This equation gives a very simple recipe for calculating the $k[\theta]$ curve from a single observed profile of the water content at time t

The average water content above depth z is given by

$$\bar{\theta} \equiv w/z. \tag{29}$$

Dividing both sides of (27) by z and using (29), (21), and (9) gives

$$\bar{\theta} = (1 - R^{-1}[\theta])\theta. \tag{30}$$

From purely gravitational drainage, equation (30) relates the average water content $\bar{\theta}$ *above* depth z to the water content *at* depth z. This relationship is surprisingly simple: it does not involve the depth z and time t explicitly. For classes A and B in Table 2, equation (30) reduces to, respectively

$$\bar{\theta} = \theta - \beta^{-1}, \quad \text{for class A,} \tag{31}$$

$$\bar{\theta} = (1 - m^{-1})\theta, \quad \text{for class B.} \tag{32}$$

It appears that these two are the only classes yielding a linear relationship between $\bar{\theta}$ and $\theta$. As mentioned earlier, such a linear relationship is sometimes observed (Simmons *et al.*, 1979; Libardi *et al.*, 1980).

## 4. THE INFLUENCE OF CAPILLARITY

The relatively simple forms of the flux $\theta v$ and the flow equation for soils of class A, given by (15) and (16) can be used to evaluate the influence of capillarity upon the drainage process. The method of separation of variables can be used to show that (Raats, 1976)

$$\theta = \theta_{io} + \beta^{-1} \ln \frac{1 + \alpha z}{1 + t/\tau_{io}} \tag{33}$$

is an exact solution of equation (11). The parameter $\tau_{io}$ in (33) is given by

$$\tau_{io} = (\alpha\beta k_{io})^{-1}, \tag{34}$$

with $k_{io}$ being the hydraulic conductivity corresponding to the initial water content $\theta_{io}$ at the soil surface. Setting $t = 0$ in (33) shows that (33) describes water content profiles evolving from the initial distribution

$$\theta_i = \theta_{io} + \beta^{-1} \ln(1 + \alpha z). \tag{35}$$

[In equation (25) of my earlier paper (Raats, 1976), the sum $(1 + \alpha z)$ is mistakenly multiplied by $k_{io}$]. A plot of (35) is shown in Fig. 3.

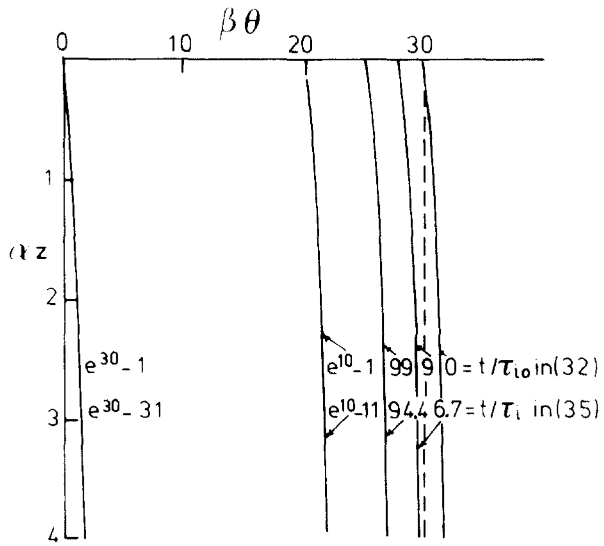


Fig. 3. Water content profiles during drainage of soils of class A, including the influence of capillarity.

The profiles described by (22) and (33) are closely related. The plots of (22) are transformed in plots of (33) by shifting the origin of the z-axis a distance $\alpha^{-1}$ downward and reducing t by a period $\tau_{io}$. Just as for (22), the curves of $\theta$ versus z at successive t described by (33) are parallel to each other.

The only remaining problem is that (33) with $\tau_{io}$ given by (34) evolves from the very specific, nonuniform initial condition (35) rather than the uniform initial condition (18). Some recent comments of Parlange (1982) on the paper by Sisson *et al.* (1980) resolve this dilemma. Parlange also considered soils of class A in Table 2. He derived an expression for $dk/d\theta$ from which, in the notation of this paper, follows:

$$\theta = \theta_i + \beta^{-1} \ln \frac{1 + \alpha z}{(\tau + t)/\tau_i}, \tag{36}$$

with $\tau$ given by

$$\tau = \frac{k_o}{k_i - k_o} t + \frac{\theta_i - \theta_o}{k_i - k_o} \alpha^{-1} . \tag{37}$$

To obtain his expression for $dk/d\theta$, Parlange integrated (10) to obtain an expression for k, and used equations (7) and (21) with t replaced by an arbitrary function F(t) to get an estimate of $\theta v$. He evaluated F(t) on the basis of a mass balance for the entire profile. The resulting solution satisfies the initial condition (18), implies the exact cumulative drainage $k_i t$ and approximately accounts for the influence of capillarity. The shape of the water content profile is not exact, but a zone with $\theta = 0$ will emerge only after some time.

In (37) the subscript o denotes time dependent values of $\theta$ and k at the soil surface. Using the exponential relationship between k and $\theta$ and setting $z = 0$, equations (36) and (37) can be written as:

$$\tau + t = (k_i/k_o)\tau_i, \tag{38}$$

$$\tau = \frac{k_o}{k_i - k_o} \{t + ((k_i/k_o) \ln (k_i/k_o))\tau_i\}, \tag{39}$$

Solving (38) and (39) for t and $\tau$, respectively, gives:

$$t/\tau_i = \{(k_i/k_o) - 1 - \ln (k_i/k_o)\}, \tag{40}$$

$$\tau/\tau_i = \{1 + \ln (k_i/k_o)\}. \tag{41}$$

Using (41) to eliminate $k_i/k_o$ from (40) gives

$$t/\tau_i = \{\exp(\tau/\tau_i - 1) - \tau/\tau_i\}. \tag{42}$$

Introducing the exponential dependence of k and θ in (40) and (41) gives:

$$t/\tau_i = \exp \beta(\theta_i - \theta_o) - 1 - \beta(\theta_i - \theta_o), \tag{43}$$

$$\tau/\tau_i = 1 + \beta(\theta_i - \theta_o). \tag{44}$$

Equations (42), (43), and (44) are plotted in Fig. 4.

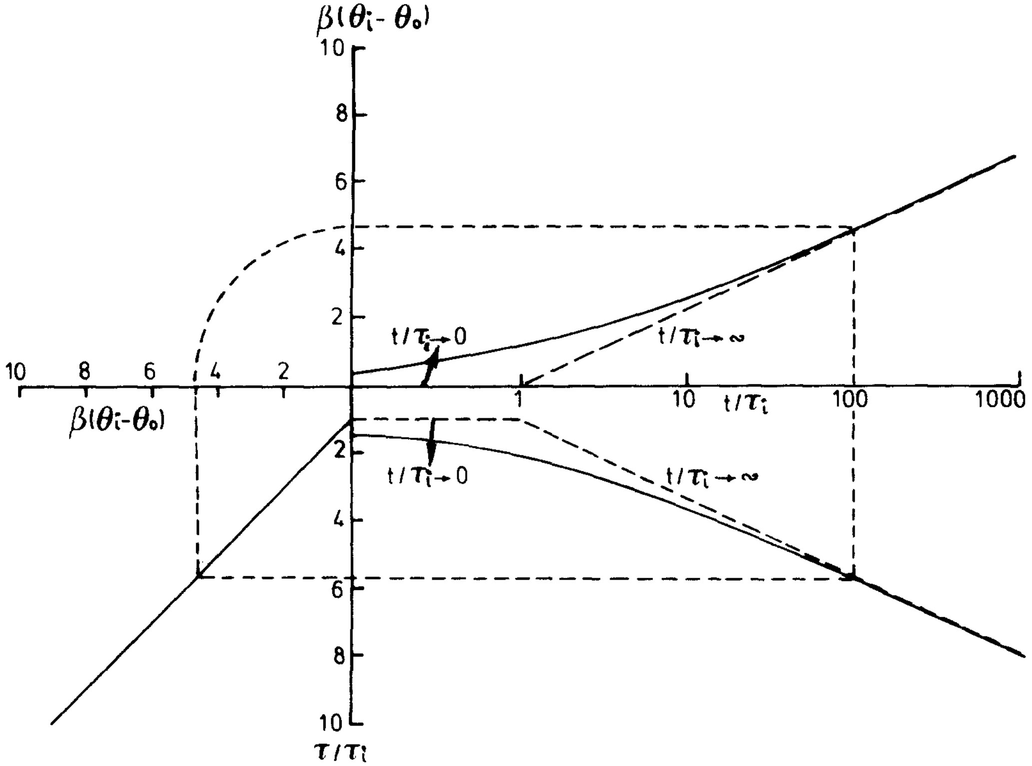


Fig. 4. Relationships among $t/\tau_i$, $\tau/\tau_i$, and $\beta(\theta - \theta_i)$.

In the limit of small $t/\tau_i$ the parameter $\tau/\tau_i$ approaches unity and, hence, (37) reduces to (33). In the limit of large $t/\tau_i$, the parameter $\tau/\tau_i << t/\tau_i$ and, hence, (37) reduces to

$$\theta = \theta_i + \beta^{-1} \ln \frac{1 + \alpha z}{t/\tau_i} \,. \tag{45}$$

For intermediate times the deviation from these limits is large. Equations (33) and (36) describe the same set of water content profiles, but the times at which the various profiles occur differ (see Fig. 3).

Introducing (32) into (26) gives

$$w = \theta z - (\tau + t)(k - k_o) + \alpha^{-1}(\theta - \theta_o). \tag{46}$$

Solving (46) for k gives

$$k = k_o - \frac{\theta z - w + \alpha^{-1}(\theta - \theta_o)}{\tau + t}. \tag{47}$$

Introducing (46) into (29) gives

$$\bar{\theta} = \theta - \frac{(\tau + t)(k - k_o) - \alpha^{-1}(\theta - \theta_o)}{z}. \tag{48}$$

If $\tau$, $\alpha^{-1}$, and $k_o$ are zero then (46), (47), and (48) reduce to (27), (28), and (30). As was pointed out before, for soils of class A in Table 2 the influence of gravity upon the water content profiles is merely a shift of the profiles in space and in time. Unfortunately the resulting expressions for w, k, and $\theta$ are far more complicated. To use (47) for calculating k from observed water content profiles requires prior knowledge of $\alpha$, $\beta$, and $k_o$. Of course, one could use (47) as the basis of an interative procedure. Using (38) in (47) gives

$$\exp \beta(\theta - \theta_o) + \beta(\theta - \theta_o) + \alpha\beta(\theta z - w) = 1. \tag{49}$$

Equation (49) can be used to iteratively estimate $\alpha$ and $\beta$ from the water content profile above an arbitrary depth z. It would be worthwhile to analyze some of the existing data sets on the basis of (28) and (49), particularly any sets in which the successive water content profiles are parallel for some period.

## 5. CONCLUDING REMARKS

Instantaneous profile methods are often used for *in situ* determination of physical properties. Among this group of methods the one based on the process of drainage of a uniform profile from a uniform initial condition is commonly used to study spatial variability of field soils. To relax the data requirements and/or simplify the data analysis, special physical and empirical assumptions are often used. In this paper some of the commonly used assumptions have been justified.

## 6. REFERENCES

1 Ababou, R., M. Vauclin, and G. Vachaud, 1979. Etude numérique de l'infiltration de l'eau suite à une irrigation localisée. In: Séminaires sur l'irrigation localisée. Institut d'Agronomie de l'Université de Bologne, Bologne, Italy, pp. 9-19.
2 Aitchison, J. and J.A.C. Brown, 1957. The lognormal distribution, with special reference to its uses in economics. University Press, Cambridge.
3 Black, T.A., G.W. Thurtell, and C.B. Tanner, 1968. Hydraulic load-cell lysimeter, construction, calibration, and tests. Soil Sci. Soc. Am. Proc., 32: 623-629.
4 Clothier, B.E. and D.R. Scotter, 1982. Constant-flux infiltration from a hemispherical cavity. Soil Sci. Soc. Am. J., 46: 696-700.
5 Cooper, J.D., 1979. Water use of a tea estate from soil moisture measurements. E. Afr. Agr. For. J., 45: 102-121.
6 Cooper, J.D., 1980. Measurement of moisture fluxes in unsaturated soil in Thetford Forest. Institute of Hydrology, Wallingford, U.K. Report No. 66, 97 p.
7 Davidson, J.M., L.R. Stone, D.R. Nielsen, and M.E. La Rue, 1969. Field measurement and use of soil water properties. Water Resour. Res., 5: 1312-1321.
8 Gardner, W.R., D. Hillel, and Y. Benyamini, 1970. Post irrigation movement of soil water: I. Redistribution. Water Resour. Res., 6: 851-861.
9 Lax, P.D., 1972. The formation and decay of shock waves. Am. Math. Monthly, 79: 227-241.
10 Lax, P.D., 1973. Hyperbolic systems of conservation laws and the mathematical theory of shock waves. Society for Industrial and Applied Mathematics. Philadelphia, Penn., USA, 48 + iii pp.
11 Libardi, P.L., K. Reichardt, D.R. Nielsen, and J.W. Biggar, 1980. Simple field methods for estimating soil hydraulic conductivity. Soil Soc. Am. J., 44: 3-7.
12 Miller, E.E., 1980. Similitude and scaling of soil-water phenomena In: Hillel, D. (ed.) Application of soil physics, pp. 300-318.
13 Nielsen, D.R., J.W. Biggar, and K.T. Erh, 1973. Spatial variability of field measured soil water properties. Hilgardia, 42: 215-259.
14 Parlange. J.Y. 1982. Simple method for predicting drainage from field plots. Soil Sci. Soc. Am. J., 46: 887-888.
15 Raats, P.A.C., 1976. Analytical solutions of a simplified flow equation. Trans. ASAE, 19: 683-689.
16 Raats, P.A.C., 1982. Convective transport of ideal tracers in unsaturated soils. Paper presented at Symposium on Unsaturated Flow and Transport Modeling, held at Battelle Seattle Research Center on March 22-24, 1982. (In press.)
17 Raats, P.A.C., 1983. Heterogeneous soils with lognormally distributed length scales. Soil Sci. Soc. Am. J.(prepared)
18 Richards, L.A., 1931. Capillary conduction of liquids through porous mediums. Physics, 1: 318-333.

19 Richards, L.A., W.R. Gardner, and G. Ogata, 1956. Physical processes determining the water loss from soil. Soil Sci. Soc. Am. Proc., 20: 310-314.
20 Simmons, C.S., D.R. Nielsen, and J.W. Biggar, 1979. Scaling of field- measured soil water properties, I. Methodology, II. Hydraulic conductivity and flux. Hilgardia, 47: 73-173.
21 Sisson, J.B., A.H. Ferguson, M.Th. van Genuchten, 1980. Simple method for predicting drainage from field plots. Soil Sci. Soc. Am. J., 44: 1147-1152.
22 Warrick, A.W., G.J. Mullen, and D.R. Nielsen, 1977a. Prediction of the soil-water flux based upon field-measured soil-water properties. Soil Sci. Soc. Am. J., 41: 4-19.
23 Warrick, A.W., G.J. Mullen, and D.R. Nielsen, 1977b. Scaling field-measured soil hydraulic properties using similar-media concept. Water Resour. Res., 13: 355-362.
24. Whitham, G.B., 1974. Linear and nonlinear waves. Wiley, New York, 636 pp.