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Boundary layers on rotating cones, discs and axisymmetric surfaces with a concentrated heat source

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Summary. A concentrated heat source is situated at the tip of an otherwise adiabatic rotating cone. Due to centrifugal forces, velocity and thermal boundary layers spread on the surface. After a similarity transform, the governing equations reduce to a set of nonlinear, ordinary differential equations which are then integrated numerically. The related cases of rotating discs and other axisymmetric surfaces are considered.

1 Introduction

Forced convective heat transfer due to centrifugal forces is important in the design of rotating machinery. Reviews on this topic were written by Dorfman [1] and Kreith [2]. The constant temperature rotating disc, which yields an exact solution of the Navier-Stokes and energy equations, was solved by Millsaps and Pohlhausen [3], Sparrow and Gregg [4]. The boundary layer on rotating cones was studied by Tien [5], [6], Hartnett and Delang [7] for constant and variable surface temperatures. Geis [8], Hayday [9], Dorfman and Serazetdinov [10] extended the results to a family of curved surfaces of revolution which includes the disc and cone as special cases. The present note studies, for the first time, the convection on adiabatic rotating surfaces of revolution with a heat source at the tip. Similarity boundary layer solutions will be sought. Similarity solutions are important since they show the exact effects of parametric variations.

2 General formulation

The boundary layer equations on a body of revolution are (e.g. Rosenhead, [11])

$$uu_x + vu_y - \frac{w^2}{r}\frac{dr}{dx} = \nu u_{yy} \tag{1}$$

$$uw_x + vw_y + \frac{uw}{r}\frac{dr}{dx} = \nu w_{yy} \tag{2}$$

$$(ru)_x + (rv)_y = 0 \tag{3}$$

$$uT_x + vT_y = \frac{\nu}{P} T_{yy}. \tag{4}$$

Here x, y are intrinsic coordinates along and normal to the surface and u, v are the corresponding velocity components. T is the temperature, ν is the kinematic viscosity, P is the

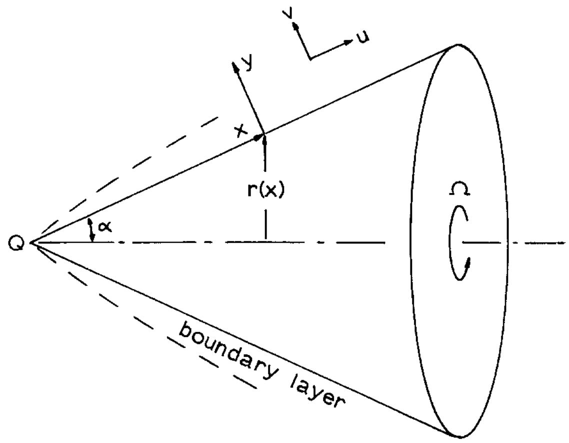

Fig. 1. The coordinate system

Prandtl number, w is the azimuthal velocity and $r(x)$ is the surface distance to the axis of revolution. Let the surface be described by

$$r = Ax^s \tag{5}$$

where A and s are positive constants. The heat source of strength Q is at the tip $r = x = 0$ (Fig. 1). Set

$$u = A\Omega x^s F'(\eta) \tag{6}$$

$$v = -\sqrt{A\Omega\nu}\, x^{\frac{s-1}{2}} \left[\left(\frac{3s+1}{2}\right) F + \left(\frac{s-1}{2}\right) \eta F'\right] \tag{7}$$

$$w = A\Omega x^s G(\eta) \tag{8}$$

$$T = T_\infty + \frac{Q}{2\pi\varrho C_p A \sqrt{A\Omega\nu}}\, x^{\frac{-3s-1}{2}} \theta(\eta) \tag{9}$$

$$\eta \equiv \sqrt{\frac{A\Omega}{\nu}}\, y x^{\frac{s-1}{2}}. \tag{10}$$

Here Ω is the angular velocity of rotation, ϱ is the density and C_p is the specific heat. Equations (1)—(4) then reduce to

$$F''' + \left(\frac{3s+1}{2}\right) FF'' + s[G^2 - (F')^2] = 0 \tag{11}$$

$$G'' + \left(\frac{3s+1}{2}\right) FG' - 2sF'G = 0 \tag{12}$$

$$\theta'' + P\left(\frac{3s+1}{2}\right)(\theta F)' = 0. \tag{13}$$

The boundary conditions are

$$F(0) = F'(0) = \theta'(0) = 0, \qquad G(0) = 1 \tag{14}$$

$$F'(\infty) = G(\infty) = \theta(\infty) = 0. \tag{15}$$

The total heat flux through any $x =$ constant surface, within the boundary layer approximation, is

$$Q = \varrho C_p \int_0^\infty 2\pi r u (T - T_\infty)\, dr \tag{16}$$

which yields the condition

$$\int^\infty F'\theta \, d\eta = 1 . \tag{17}$$

3 Rotating cones and discs

These important geometries have been studied previously, but not for the concentrated source case considered in this paper. The family of cones are obtained by setting $s = 1$, $A = \sin \alpha$ in Eq. (5). When the vertex angle 2α becomes 2π, the surface is a flat disc. The governing equations are

$$F''' + 2FF'' + G^2 - (F')^2 = 0 \tag{18}$$

$$G'' + 2FG' - 2F'G = 0 \tag{19}$$

$$\theta'' + 2P(\theta F)' = 0 . \tag{20}$$

Equations (18) and (19) are the fluid dynamic equations for Karman's rotating disc problem [12]. Accurate numerical solutions were given by Rogers and Lance [13], e.g. $F''(0) = 0.510233$, $G'(0) = -0.615922$, $F(\infty) = 0.442235$. After $F(\eta)$ is obtained, the solution to Eq. (20) is

$$\theta = c \exp\left[-2P \int_0^\eta F(\eta)\, d\eta\right] \tag{21}$$

where c is obtained from Eq. (17) by double quadrature. However, direct numerical integration may be easier. Change Eq. (17) to a differential equation

$$\frac{dK}{d\eta} = F'\theta , \qquad K(0) = 0 . \tag{22}$$

Choose $\theta(0) = 1$ and integrate Eqs. (20), (22) numerically until $K(\eta)$ converges to some value say k^*. Then the true initial value is

$$\theta(0) = 1/k^* . \tag{23}$$

Table 1 shows our results. We see that the normalized surface temperature $\theta(0)$ increases with Prandtl number. Figure 2 shows that the normalized temperature profile $\theta(\eta)$ is quite different from constant temperature or constant flux cases. The thickness of the thermal boundary layer decreases with increased P. Figure 3 shows two discs rotating with the same angular velocity, thus the radial velocities are the same. The isotherms for $P = 7$ are more parallel than those of $P = 0.7$ due to the thinner thermal boundary layer for higher Prandtl numbers. Both solutions are invalid near the origin, since our assumptions require the boundary layer thickness to be much smaller than the local radial distance r $(= x)$. Note the vertical scale η can be contracted by increasing Ω, thus increasing the range of validity for the boundary layer solutions. The cone angle enters through A in Eqs. (6)—(10).

Table 1. Values for normalized maximum temperature $\theta(0)$ for $s = 1$ and various Prandtl numbers

P	0.07	0.2	0.7	2	7	20	70	200
$\theta(0)$	2.3554	2.5224	3.0915	4.2706	7.4326	12.988	26.969	51.383

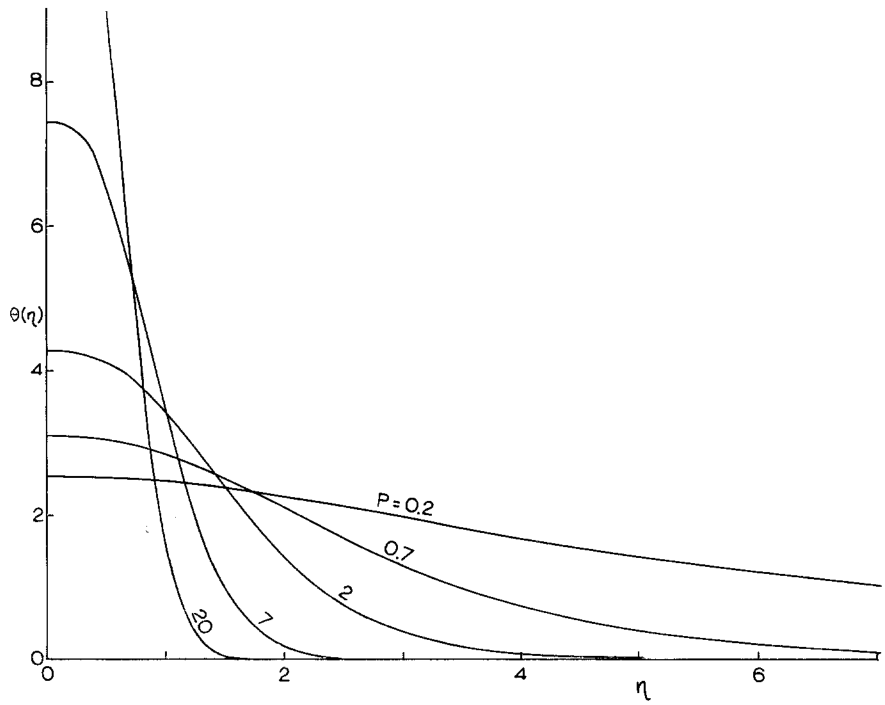

Fig. 2. Similarity temperature function $\theta(\eta)$ for various Prandtl numbers ($s = 1$)

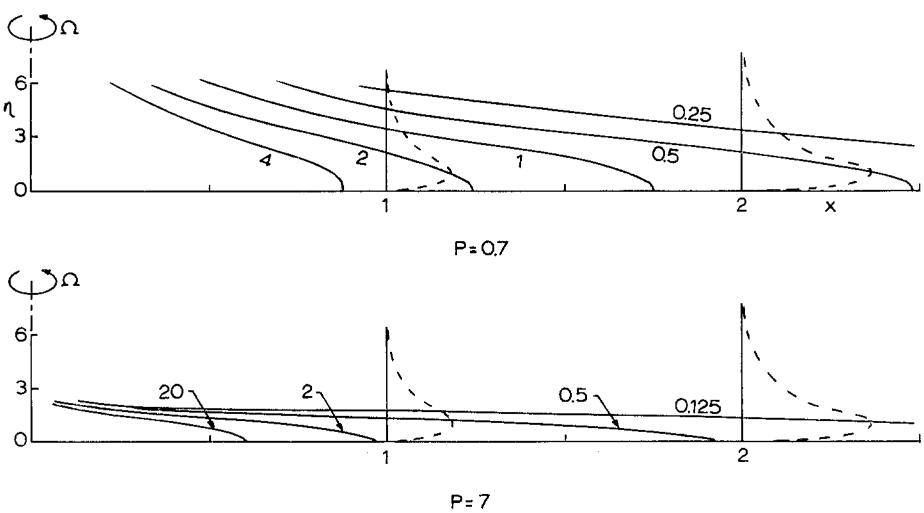

Fig. 3. Isotherms on rotating cones or discs. Values are for $2\pi\varrho C_p A \sqrt{A\Omega\nu}\,(T - T_\infty)/Q$. Dashed lines show radial velocity profiles

4 Rotating axisymmetric surfaces

For $s \neq 1$ let $\bar{x}$ be the axial distance from the tip. The shape of the surface $(\bar{x}, r)$ can be found by the parametric equations

$$r = Ax^s \tag{24}$$

$$\bar{x} = \int_0^x \sqrt{1 - \left(\frac{dr}{dx}\right)^2}\, dx = \int_0^x \sqrt{1 - A^2 s^2 x^{2s-2}}\, dx. \tag{25}$$

The form of Eqs. (6), (8) dictates that the boundary layer necessarily starts at $r = 0$. Thus Dorfman and Serazetdinov [10] are erroneous to assume a non-zero starting radius. Since $x \geqq 0$ Eq. (25) shows $s \geqq 1$ for $\bar{x}$ to be real. Also, there exists a maximum arc length of $x = (As)^{\frac{-1}{s-1}}$. Figure 4 shows the surfaces of revolution for $s > 1$ are pointed, with zero slope at tip at $x = 0$. These concave surfaces are common for impellers. Increase in the constant A would not alter the cuspidal nature. For $s \geqq 2$ even the second derivative is zero and the tip is too sharp to have practical significance.

An asymptotic analysis of Eqs. (11)—(15) for large η shows F', G decay as $\exp\left[-\frac{1}{2} \times (3s + 1)\, F(\infty)\, \eta\right]$ and θ decay as $\exp\left[-\frac{1}{2}(3s + 1)\, PF(\infty)\, \eta\right]$. The numerical integration is as follows. We guess $F''(0)$, $G'(0)$ and integrate Eqs. (11), (12) as an initial value problem

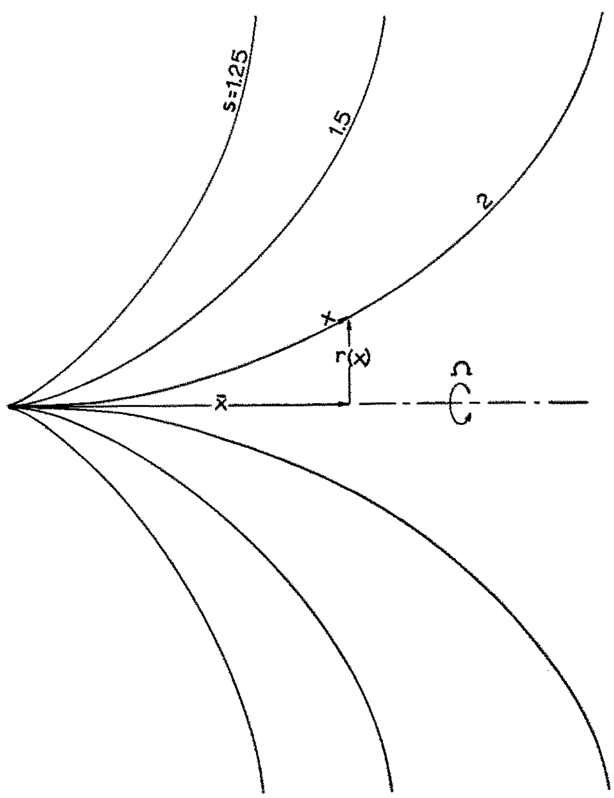

Fig. 4. Geometric cuspidal shapes of revolution for $s > 1$

Table 2. Initial and final values for the flow field and maximum temperature $\theta(0)$ for various s and P

		$s = 1.25$	$s = 1.5$	$s = 2$
	$F''(0)$	$= 0.5736$	0.6307	0.7317
	$G'(0)$	$= -0.6864$	-0.7503	-0.8641
	$F(\infty)$	$= 0.4060$	0.3777	0.3366
$P =$	$\theta(0)$	$=$		
0.2		2.7424	2.9447	3.3080
0.7		3.3553	3.6000	4.0455
2		4.6291	4.9631	5.5734
7		8.0451	8.6169	9.6642
20		14.044	15.030	16.838

by the fifth order Runge-Kutta-Fehlberg algorithm. The minimum value of $J = |F'| + |G|$ is noted. Using two dimensional shooting, a solution is found if J is minimized to less than 10^{-4}. Our results for $F''(0)$, $G'(0)$ agree with Rogers and Lance [13] ($s = 1$) and Hayday [9] ($s = 1.25, 1.5, 2$). Thus $F(\eta)$ is obtained.

The thermal problem can be integrated by a method similar to the conical case. The results are given in Table 2. We see that the maximum temperature increases with s.

5 Discussion

We find in the case of concentrated heat source at the origin, the temperature decreases as $(-1.5s - 0.5)$ power of arc length x due to conservation of total thermal energy. For cones and discs the temperature decreases as x^{-2}. In contrast, for heated surfaces studied by previous authors, the temperature either is independent or increases with x.

Similarity solutions also show the effects of various parameters. The influence of geometric parameter A, the rotation rate Ω and the viscosity ν are evident from the similarity transform Eqs. (6)—(10).

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