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Risk-aversion, prudence and temperance: A unified approach[☆]

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Abstract

Risk-aversion can be defined either by the negative sign of the second derivative of the utility function or by the rejection of any mean-preserving increase in risk. The more recent notions of prudence and temperance have so far been defined exclusively by the sign of the third and the fourth derivative of the utility function. In this paper we show that, as risk-aversion, prudence and temperance can also be interpreted as systematic attitudes towards transformation of a density function.

Keywords: Prudence; Temperance; Translation of an increase in risk

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1. Introduction

Because all risk-averters dislike mean-preserving increases in risk there exists a one-to-one correspondence between a behavioral assumption – concavity of the utility function ($u'' < 0$) – and a statistical transformation (a mean-preserving spread).

Recently, Kimball (1990, 1992) has proposed the definition of prudence and temperance besides that of risk-aversion. These notions have been defined, so far, by the signs of the third and fourth derivatives of the utility function ($u''' > 0$ and $u'''' < 0$ respectively). However, contrarily to risk-aversion, no relationship has been established to date between the

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behavioral assumption of prudence and temperance, on the one hand, and transformation of a density function, on the other hand.

The purpose of this paper is to fill this gap in the literature. We show that prudence and temperance can – as risk aversion – be related to the transformation of a density function. The basic idea for prudence is that a risk-averter who is forced to undergo a mean-preserving increase in risk (which he dislikes) will prefer to see it attached to the best outcomes of a lottery rather than to the worst ones. We formalize this in Section 2 through the notion of a shift of an increase in risk (SIR). The correspondence between the effect of a SIR and the sign of successive derivatives of the utility function is established in Proposition 1 (Section 3).

2. Shifts of an increase in risk

The basic element of our analysis is a Rothschild–Stiglitz increase in risk described by a change of the distribution of a random variable $\tilde{z}$ from cumulative distribution function F to cumulative distribution function (CDF) G_0. Namely, the function $H = G_0 - F$ used to define a mean-preserving spread (MPS) is assumed to satisfy the following conditions:

$$T(x) = \int_{-\infty}^{x} H(z)\,\mathrm{d}z \geqslant 0\,, \quad \forall x \in \mathbb{R}\,, \tag{1}$$

$$T(+\infty) = 0\,, \tag{2}$$

and $H(-\infty) = H(+\infty) = 0$. Eq. 2 means that the expectation of $\tilde{z}$ is unaffected by the change in risk. As shown by Rothschild and Stiglitz (1970), an increase in risk can be obtained by a sequence of mean-preserving spreads. A shift of an increase in risk (SIR) is obtained by translating this sequence of MPSs. The resulting distribution G_t is obtained from F by adding the same sequence of MPSs characterized by function H, but after applying a translation of these mean-preserving spreads by a distance t to the right. It yields

$$G_t(z) = F(z) + H(z - t)\,, \tag{3}$$

with $t > 0$. We say that G_t is an upwards SIR with respect to G_0.[1] Said crudely, an upwards shift of an increase in risk induces more risk in wealthier states and less risk elsewhere. Notice also that a SIR does not change the variance of $\tilde{z}$. This is proved by the following sequence of equalities:

[1] Notice that, given H and F, not all shifts $t \in \mathbb{R}$ are allowed since G_t must be a CDF, i.e. G_t' must be non-negative. In the discrete case, it limits the set of relevant translations of an increase in risk to the distance between the atoms of the random variable. In the remainder of the paper we will consider the case of continuous random variables. Our results may easily be extended to the discrete case.

$$\begin{aligned}\mathrm{Var}(G_t) &= \mathrm{Var}(G_0) + \int_{-\infty}^{+\infty} z^2\,\mathrm{d}(H(z-t) - H(z)) \\ &= \mathrm{Var}(G_0) - 2\int_{-\infty}^{+\infty} z\{H(z-t) - H(z)\}\,\mathrm{d}z \\ &= \mathrm{Var}(G_0) + 2\int_{-\infty}^{+\infty} \{T(z-t) - T(z)\}\,\mathrm{d}z = \mathrm{Var}(G_0)\,. \end{aligned} \tag{4}$$

Since a SIR preserves both the mean and the variance of $\tilde{z}$, it belongs to the class of mean-variance-preserving transformations introduced by Menezes et al. (1980).

3. Effects of a positive shift of an increase in risk

Let u denote the utility function with $u' \geq 0$. The expected utility of final wealth $\tilde{z}$ distributed following CDF G_t is written as

$$U(t, H) = \int u(z)\,\mathrm{d}G_t(z) = \int u(z)\,\mathrm{d}F(z) + \int u(z)\,\mathrm{d}H(z-t)\,. \tag{5}$$

The expected utility $U(t, H)$ depends upon the initial distribution F only through the constant $E_F u = \int u(z)\,\mathrm{d}F(z)$. Namely, the effect of a SIR on the expected utility is independent of F. If risk-aversion is assumed, $U(t, H)$ is bounded above by $E_F u$, for all t and H satisfying conditions (1) and (2). Integrating by parts twice in Eq. (5) yields

$$U(t, H) = E_F u + \int u''(z) T(z-t)\,\mathrm{d}z = E_F u + \int u''(z+t) T(z)\,\mathrm{d}z\,. \tag{6}$$

Let $U^{(n)}(t, H)$ and $u^{(n)}(z)$ denote the nth derivative of these functions. Eq. (6) implies that, for $n = 1, 2, \ldots,$

$$U^{(n)}(t, H) = \int u^{(n+2)}(z+t) T(z)\,\mathrm{d}z\,. \tag{7}$$

Our main result is a simple consequence of this equation.[2]

Proposition 1. Let $n = 1, 2, \ldots, U^{(n)}(t, H) \geq 0$ for any $t \in \mathbb{R}$ and H satisfying (1) and (2) if and only if $u^{(n+2)}(z) \geq 0$ for all z. Similarly, $U^{(n)}(t, H) \leq 0$ for any $t \in \mathbb{R}$ and H satisfying (1) and (2) if and only if $u^{(n+2)} \leq 0$ for all z.

[2] The proof of Proposition 1 indicates that the mean-preserving condition (2) could be suppressed without altering the result. It means that this analysis can immediately be extended to shifts of second-degree stochastic dominance changes in risk.

Proof. Sufficiency of $u^{(n+2)}(\cdot) \geq 0$ is a direct consequence of Eqs. (1) and (7). Necessity of $u^{(n+2)}(\cdot) \geq 0$ is proved by contradiction. Suppose that there exists an interval $[z_1, z_2]$ where $u^{(n+2)}(z)$ is negative. Then consider a function H such that T is positive in interval $[z_1, z_2]$ and zero elsewhere. It follows by using Eq. (7) that $U^{(n)}(0, H)$ is negative, a contradiction. □

Corollary 1. Upwards shifts of any increase in risk are beneficial to expected utility if and only if the individual is prudent, i.e. $u'''(z)$ is uniformly positive.

Corollary 1 provides a new definition of prudence based on a welfare analysis rather than on a comparative statics problem as in Kimball (1990). Notice that the relevant concept for signing the effect of SIRs is prudence, not decreasing absolute risk-aversion. Since all DARA utility functions exhibit positive prudence, all decreasingly risk-averse individuals are made better off by upwards SIRs.

We are also interested in determining whether U is a concave function of t. The intuition suggests that marginal gains in expected utility due to successive upwards SIRs are decreasing. By definition, we say that the individual whose preferences satisfy this property is temperate. An equivalent definition is that a temperate person always prefers a given shift of an increase in risk to a risky one with the same expectation. Temperance is aversion to risky SIRs. If we assume risk-aversion and prudence, we know that U is increasing in t and bounded above by $E_F u$. It implies that under risk-aversion and prudence, U may not be convex everywhere.

Corollary 2. An individual is temperate, i.e. marginal gains in expected utility for successive upwards shifts of any increase in risk are decreasing, if and only if $u''''(z)$ is uniformly negative.

While Proposition 1 and its two corollaries enable us to interpret the positive (negative) sign of $u'''(u'''')$ in terms of the effect of a SIR on an individual's welfare, it should be noted that there exist other interpretations of prudence ($u''' > 0$) and temperance ($u'''' < 0$). For instance, Kimball (1990) shows that prudence is necessary and sufficient for individuals to save more in the presence of uncertainty. Kimball (1993), Eeckhoudt et al. (1994) and Gollier and Pratt (1994) also show that temperance is necessary for rational people to reduce their demand for a risky asset when their independent human capital becomes riskier in some specific sense.

To summarize, we draw in Fig. 1 U as a function of t for a given increase in risk H and for a given individual u who is risk-averse, prudent and temperate. Risk-aversion implies that $U(t, H)$ is less than $E_F u$. Prudence implies that U is increasing in t and temperance makes U concave.

Let us now measure the effect of a shift of an increase in risk on expected utility. Define the loss in expected utility due to an increase in risk $H(z - t)$ as

$$V(t, H) = E_{FU} - U(t, H) = -\int u(z + t)\, \mathrm{d}H(z)\,. \tag{8}$$

Under risk-aversion, it is positive. Under prudence, it is a decreasing function of t. The sensitivity of V to a SIR can be measured by its elasticity η_{V_t}, as follows:

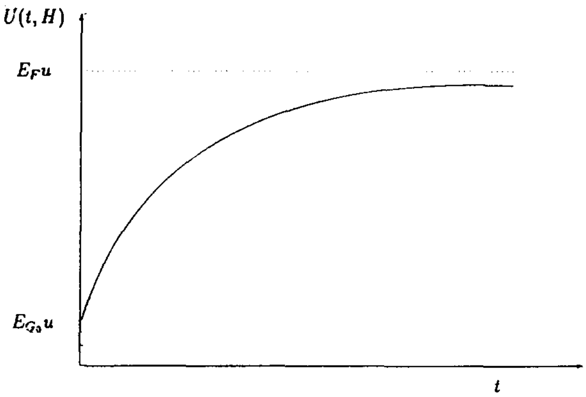

Fig. 1. Shape of $U(t, H)$ for a risk-averse, prudent and temperate person.

$$\eta_{V_t} \equiv -\frac{\partial V}{\partial t}\frac{t}{V} = -\frac{t \int u'(z+t)\,\mathrm{d}H(z)}{\int u(z+t)\,\mathrm{d}H(z)} . \tag{9}$$

η_{V_t} is the percentage reduction in the expected utility loss for a one-percent increase in the shift of the increase in risk. If the increase in risk H is small enough, the elasticity can be approximated by the following formula:

$$\eta_{V_t} \cong -t\frac{u'(t) \int \mathrm{d}H(z) + u''(t) \int z\,\mathrm{d}H(z) + \frac{1}{2}u'''(t) \int z^2\,\mathrm{d}H(z)}{u(t) \int \mathrm{d}H(z) + u'(t) \int z\,\mathrm{d}H(z) + \frac{1}{2}u''(t) \int z^2\,\mathrm{d}H(z)} \cong -\frac{tu'''(t)}{u''(t)} . \tag{10}$$

The right-hand side of this equation is relative prudence, as defined by Kimball (1990) and used by Ormiston and Schlee (1992). Relative prudence is nothing else than the elasticity of the expected utility loss of a small increase in risk when it is shifted.

4. The effect of a SIR on saving

An interesting extension of our result is obtained by considering the saving problem of a risk-averse agent under uncertainty that is written as

$$\max_s u(w-s) + Ev(rs + \tilde{x}) , \tag{11}$$

where s is saving, r is one plus the interest rate, and $\tilde{x}$ is the future risky income. The first-order condition yields $u'(w-s) = rEv'(rs + \tilde{x})$. We consider the effect of a shift of an increase in risk of $\tilde{x}$ on optimal saving. The intuition is that an upwards SIR reduces the precautionary saving. This is the case if an upwards SIR reduces Ev'. As the positivity of u''' is equivalent to the positive effect of a SIR on expected utility, the negativity of u'''' is equivalent to the negative effect of a SIR on the expected marginal utility.

Corollary 3. A risk-averse agent reduces his precautionary saving in the face of an upwards SIR if and only if he is temperate ($u'''' < 0$).

5. Conclusion

By translating to the right a mean-preserving increase in risk, one does not change either the expected value or the variance of the initial density. However, the welfare of a decision-maker is affected by such a change. If the translation to the right increases his expected utility, u''' is positive, implying a prudent behaviour. If, besides, the decision-maker prefers a given translation to any random one of equal mean, he is temperate ($u'''' < 0$).

Thanks to these results, prudence and temperance become – as risk-aversion – properties that characterize both a behavioural assumption and a transformation of a density function.

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