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Inertia and Interception in the Deposition of Particles from Boundary Layers

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The treatment of Parnas and Friedlander for the problem of particle capture by interception in laminar boundary layers is generalized to include the effect of inertia. For a circular cylinder in cross-flow it is seen that the collection efficiency is initially reduced by inertia at small Stokes number, Stk, and depends mainly on the product of Stk and the square root of the Reynolds number. An approximate method of extending this work to other two-dimensional or axisymmetric geometries is sketched.

INTRODUCTION

In a recent article in this journal, Parnas and Friedlander (1984) have studied the problem of particle diffusion and interception to spheres and cylinders using boundary layer theory. In their simple and elegant treatment of the interception effect, these authors observe that the particles initially approach the wall as the fluid accelerates away from the stagnation point, but they eventually start moving away from the obstacle due to the "displacement velocity" associated with the boundary layer growth (Schlichting, 1968).

The purpose of this paper is to incorporate the effect of inertia into the Parnas–Friedlander analysis. Our original motivation to explore such an effect arose from a previous study (Fernandez de la Mora and Friedlander, 1982) where data of Chamberlain (1966) on particle deposition on roughness elements were analyzed using boundary layer theory. The conclusions from that work were somewhat puzzling because it seemed that the predictions from interception theory were still valid well into the region where inertia should have been important. More recently, Schack et al. (1984) have studied a much wider body of data, strengthening considerably our previous conclusions. The reasons why inertia seems to play no role, when ordinarily it would be expected to act vigorously, are rather perplexing, and will be considered in this paper.

THE MATHEMATICAL AND PHYSICAL PICTURES

There is no basic difficulty in studying the deposition of particles by inertia and interception. The problem is fully deterministic and can be handled by straightforward integration of Newton's equations for the particles (Fuchs, 1964; Friedlander, 1977). Such a treatment, however, would require extensive numerical computations, and would permit the extraction of limited amounts of general information. Besides, the role of inertia at Stokes numbers above the critical is so much greater than that of interception that we are interested only in the region of subcritical Stokes numbers. A general procedure to conveniently treat such a problem has been described in Fernandez de la Mora and Rosner (1982). The particle velocity field (including inertia) is obtained as a function of the fluid local properties by expansion in powers of the Stokes numbers. Denoting by $\mathbf{u}_p$ and $\mathbf{U}$ the particle and fluid velocity fields and τ the particle stopping time, then, to

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first order in the Stokes number, Stk,

$$\mathbf{u}_p = \mathbf{U} - \tau(U \cdot \nabla)\mathbf{U} + O(\mathrm{Stk}^2) \tag{1}$$

for a steady state flow. Equation (1) has been used repeatedly in the literature (Marble, 1970; Michael, 1968; Michael and Norey, 1970, etc.). Stechina et al (1969) refer to a paper by Voloshuck (which I have not seen) where it is pointed out that the expansion (1) is valid for subcritical Stokes numbers. That claim is somewhat optimistic, though the expansion has been used for supercritical Stokes numbers by Cleaver and Yates (1975) and Yeh and Liu (1974), among others. It is difficult to give precise conditions for (1) to hold, since it is an asymptotic rather than a convergent expansion. For linear flows such as the stagnation point flow, a solid body rotation or a Couette flow, Eq. (1) contains the two leading terms of a series that converges for all subcritical Stokes numbers. However, even then it is only the outer solution of a singular perturbation expansion which is not valid close to the points where particles are injected or reflected. Therefore, one ought to use Eq. (1) with extreme care. Fortunately, for the problem of particle motion in viscous fluids it can be stated generally (Fernandez de la Mora and Rosner, 1982) *that Eq. (1) is valid in a thin strip close* to the obstacle's surface for all subcritical Stokes numbers. The reason for that powerful fact is that the local Stokes number (based on the product of τ with the local velocity gradient) tends to zero near a surface. As a result, the particle velocity can be given analytically near the wall, the particle streamlines can be inferred, and the local analysis of Parnas and Friedlander (1984) can be extended without much extra labor. A first step in that direction for boundary layer problems is given in Fernandez de la Mora (1980, Sect. 6.4) for the case of a cylinder. But neither interception nor the important effect of the velocity displacement were included there. The work, however, contains the inertial ingredient that has to be added on to the Parnas–Friedlander approach: if the obstacle is curved, centrifugal effects on the particles act together with the fluid displacement to bring the point of minimum separation closer to the stagnation point.

ANALYSIS FOR A CYLINDER IN TWO DIMENSIONS

The fluid velocity field in the proximity of the wall can be written in cylindrical coordinates r, θ as

$$U_r = -(\omega\nu)^{1/2}\eta^2 G'(\theta)/2 \tag{2}$$

$$U_\theta = \omega R \eta G(\theta), \tag{3}$$

with

$$\eta = (r - R)(\omega/\nu)^{1/2} \tag{4}$$

being the nondimensional length in the direction normal to the obstacle's surface. R is the cylinder radius, ν the fluid kinematic viscosity, θ the angular position along the surface measured from the stagnation point, $G(\theta)$ is a known function for the cylinder (Schlichting, 1968, p. 151) given by

$$G(\theta) = G_1\theta + G_2\theta^3 + G_3\theta^5 + \cdots, \tag{5}$$

with

$$G_1 = 1.2326$$

$$G_2 = -0.48293$$

$$G_3 = 0.05160$$

and ω is the rate of deceleration in the inviscid stagnation point region, given in the case of the cylinder by

$$\omega = 2U_\infty/R, \tag{6}$$

where U_∞ is the free stream velocity far from the cylinder. Evaluating now the correction $(\mathbf{U} \cdot \nabla)\mathbf{U}$ in cylindrical coordinates, and neglecting terms of order larger than η or η^2 for $U_{p\theta}$ and U_{pr}, respectively, we obtain

$$U_{pr} = U_r + \tau U_\theta^2/R$$

$$= -\frac{1}{2}(\omega\nu)^{1/2}(G' + \xi G^2) \tag{7}$$

$$U_{p\theta} = U_\theta, \tag{8}$$

with

$$\xi \equiv 2\omega\tau\left(\omega R^2/\nu\right)^{1/2}. \tag{9}$$

Therefore, only the centrifugal correction $\tau U_\theta^2/R$ is significant close to the wall, and the corresponding velocity drift is proportional to the unusual group ξ. This is indeed unexpected, because inertial effects are generally considered to be of the order of the Stokes number ($\omega\tau$ here). But ξ is a Stokes number magnified by a half power of the Reynolds number:

$$\mathrm{Re} \equiv 2U_\infty R/\nu = \omega R^2/\nu. \tag{10}$$

Although not widely known, this result is not new in the literature, having been first discussed in 1968 by Michael, and in more detail in the above-mentioned 1980 Ph.D. thesis. The behavior of particles in boundary layers thus departs drastically from that of fluids. The main difference is that there is no pressure term in the particle momentum conservation equation normal to the wall to dominate the picture as in the standard boundary layer theory (see Fernandez de la Mora, 1982, Sect. 3). The term next in importance is the centrifugal one, which becomes leading for the particle phase, but is still negligible for the fluid. All other terms in the $\tau(\mathbf{U}\cdot\nabla)\mathbf{U}$ particle velocity drift are of the order of $\omega\tau$, and therefore much smaller than ξ in the $\mathrm{Re} \gg 1$ limit, where boundary layer theory is applicable. Other discussions on the centrifugal effect can be found in Rosner and de la Mora (1982) and also in Brun and Dorsh (1955) for a situation with a supercritical Stokes number.

The streamlines corresponding to the velocity field (7, 8) can be obtained straightforwardly, giving

$$\eta^2 G(\theta)\exp\left[-\xi\int_0^\theta G\,d\theta\right] = \psi_p = \text{const}, \tag{11}$$

where in the limit $\xi = 0$ our ψ_p is the same streamfunction of Parnas and Friedlander (except for a constant). The point of closest approach to the cylinder is again independent of ψ_p (the particular trajectory picked), and given by the condition:

$$G'(\theta_*) - \xi G^2(\theta_*) = 0. \tag{12}$$

Solving this transcendental equation to obtain θ_* as a function of ξ is facilitated by noticing that the series giving $G(\theta)$ converges very rapidly for values of θ below 1 radian. Two terms give a reasonable description, while keeping three terms leads to the same results of Parnas and Friedlander within less than one-half of one percent. The description becomes even better for $\xi \neq 0$, since inertia tends to reduce the value of θ_*, improving the rapidity of convergence of (5). Keeping terms up to θ_*^4 in (12) yields the following quadratic equation:

$$G_1 + \theta_*^2\left(3G_2 - \xi G_1^2\right) + \theta_*^4\left(5G_3 + 2G_1G_2\xi\right) = 0, \tag{13}$$

whose only meaningful root gives the function $\theta_*(\xi)$ tabulated below. The table (Table 1) contains also the ratio $\psi_p(\xi)/\psi(0)$ (evaluated keeping only three terms in the expansion for $G(\theta)$) or a correction factor by which the Parnas–Friedlander capture efficiency should be multiplied when $\xi \neq 0$. The outcome is quite surprising, and seemingly unphysical. Inertia reduces the rate of particle collection. A similar effect was found by Fernandez de la Mora and Rosner (1982) for the capture of particles on spheres by inertia and diffusion at low Reynolds numbers. Odd as this conclusion seems, we have not been able to find any flaw in the process leading to it. On the other hand, at the point of closest approach the particle path has the same curvature as the cylinder, so that it must have been significantly centrifuged for an important portion of its previous history. On the very early part of the trajectory it is clear that the inertial drift pushes the particles towards the obstacle rather than away

TABLE 1.

ξ	0	0.05	0.10	0.15	0.20
$\theta_*^2(\xi)$	1.045	0.918	0.830	0.763	0.709
$\psi_p(\xi)/\psi_p(0)$	1	0.9720	0.9438	0.9169	0.8919

from it, so that some compensation for the ulterior centrifugal resistance does exist. This effect is of the order of $\omega\tau$, and thus much smaller than the centrifugal one proportional to ξ. This can be shown quantitatively by following the particle streamline from the small η region previously studied to the region far upstream. The operation can be performed analytically to lowest order in the Stokes number. The corresponding analysis is, however, cumbersome and physically less interesting than the preceding portion of the paper.

Since the fluid velocity field is given accurately by Eqs. (2) and (3) up to values of η as large as 0.3, and our description (Eqs. (7) and (8)) for the particle velocity field is also excellent within that interval of η and up to the critical Stokes number $\omega\tau = 1/4$ (see Figure 4 of Fernandez de la Mora and Rosner, 1981), particle trajectories are given quite precisely by Eq. (11) up to $\eta \sim 0.3$–0.4. Furthermore, the sizes of particles for which this theory is valid are such that $\eta \ll 1$ at the point of interception, so that (11) will be valid from the capture point ($\theta = \theta_*$) where $\eta \ll 1$ to the upstream point where $\eta \simeq 0.3$–0.4, where θ is now rather small. In this region the η dependence of the particle velocity field is more complex than before, but the θ dependence becomes trivial, and the particle streamline can again be found analytically. Indeed, for small values of θ one is effectively in the stagnation point where the particle velocity field admits a similarity solution (Fernandez de la Mora and Rosner, 1981):

$$u_{pr} = -(\omega\nu)^{1/2} F_1(\eta) \tag{14}$$

$$u_{p\theta} = \omega R \theta F_2(\eta), \tag{15}$$

where F_1 and F_2 are given by solving the first order differential equations:

$$\omega\tau F_1 F_1' - F_1 + f = 0 \tag{16}$$

$$\omega\tau(F_2' F_1 - F_2^2) + f' - F_2 = 0, \tag{17}$$

and $f(\eta)$ is the function giving the fluid velocity field at the stagnation point (Schlichting, 1968, p. 88, uses the notation ϕ for our f). The asymptotic forms for $\eta \gg 1$ are for $\omega\tau < 1/4$,

$$F_1 \to a(\eta - \eta_0) \qquad \eta \gg 1 \tag{18}$$

$$F_2 \to b \qquad \eta \gg 1, \tag{19}$$

with

$$\eta_0 = 0.6479 \tag{20}$$

$$\omega\tau a^2 - a + 1 = 0 \tag{21}$$

$$\omega\tau b^2 + b - 1 = 0. \tag{22}$$

Then the particle streamline is given through the differential equation:

$$\frac{d\theta}{\theta} + \frac{F_2(\eta)\, d\eta}{F_1(\eta)} = 0, \tag{23}$$

which involves only integrating the ratio F_2/F_1, which is itself a universal function of η for every value of $\omega\tau$. The asymptotic forms of the trajectory are

$$\theta(\eta - \eta_0)^{b/a} = \text{const} = \phi_\infty \qquad \text{as} \quad \eta \gg 1 \tag{24}$$

$$\theta\eta^2 = \text{const} = \phi_0 \qquad \text{as} \quad \eta \to 0. \tag{25}$$

Clearly ϕ_0 is related to our previous ψ_p, while ϕ_∞ is related to the mass flow between the trajectory considered and the stagnation line. If n is the particle density away from the viscous region, but still in the neighborhood of the stagnation point region, the particle mass conservation equation is

$$-a(\eta - \eta_0)\frac{\partial n}{\partial \eta} + b\theta \frac{\partial n}{\partial \theta} + n(b - a) = 0, \tag{26}$$

with characteristics (trajectories) given by (24), along which n is

$$n = n_\infty (\eta - \eta_0)^{b/a - 1}, \tag{27}$$

where n_∞ can depend on ϕ_∞ (the particular trajectory considered), but is a constant for this problem, where all relevant streamlines are so close to the stagnation line as to follow basically the same degree of compression during the process of approach. The mass flow between the stagnation line and a streamline characterized by ϕ_∞ is then

$$\dot{m}(\phi_\infty) = \int_0^\theta n_\infty(\eta - \eta_0)^{b/a} a(\omega\nu)^{1/2} R\, d\theta, \tag{28}$$

implying that

$$\phi_\infty = \frac{\dot{m}(\phi_\infty)}{n_\infty aR(\omega\nu)^{1/2}}. \tag{29}$$

Denoting by n_0 the particle concentration far upstream, it turns out that $n_\infty > n_0$ due to the compressibility of the particle phase. The ratio n_∞/n_0 is a function of Re and $\omega\tau$ given by Fernandez de la Mora and Rosner (1981). Now, since ψ_p is fixed by the interception condition at $\theta = \theta_*$, we can determine ϕ_0 through (11), while the mass of captured matter is given through ϕ_∞ by (29). It remains to connect ϕ_∞ to ϕ_0 by integrating (23). The result is to first order in the Stokes number:

$$\ell n\left(\frac{\phi_0 G_1}{2\phi_\infty}\right) = \omega\tau[1 + C], \tag{30}$$

where C is given by

$$C \equiv 2\left[\ell n(\eta_1 - \eta_0) - \int_0^{\eta_1} \frac{f'^2}{f} d\eta\right];\ \eta_1 > 3. \tag{31}$$

This conclusion results from the fact that to first order in $\omega\tau$(from (16) and (17)),

$$\frac{F_2}{F_1} = \frac{f'}{f} + \omega\tau\left(f'' - \frac{2f'^2}{f}\right) + \cdots. \tag{32}$$

The corresponding capture efficiency for a spherical particle of radius a_p, defined as

$$\eta_c \equiv \frac{\int_0^{a_p} n_p u_p\, dy}{n_0 R U_\infty} \tag{33}$$

becomes

$$\eta_c = \left[\frac{n_\infty}{n_0} a e^{-C\omega\tau}\right] \frac{a_p^2}{R^2} \times \mathrm{Re}^{1/2}\Bigg\{\left(G_1\theta_* + G_2\theta_*^3 + G_3\theta_*^3\right) \times \exp\left(-\xi\left[\frac{G_1\theta_*^2}{2} + \frac{G_2\theta_*^4}{4} + \frac{G_3\theta_*^6}{6}\right]\right)\Bigg\}. \tag{34}$$

In the Parnas–Friedlander approach the first term in square brackets (the inertial compression) becomes unity, while the last one in curly brackets (the centrifugal depletion) takes the value of $G(\theta_*)_{\xi=0} = 0.798$ (in our approximation of retaining only three terms in the expansion (5), $G(\theta_*)_{\xi=0}$ is 0.8017). As previously claimed on the basis of qualitative arguments, the centrifugal depletion initially dominates over the inertial compression. The effects, however, depend weakly on ξ, which itself changes with $\mathrm{Re}^{1/2}$. Also, the compensating contribution of the two inertial terms leads to a minimum somewhere in the η_c versus $\omega\tau$ curve, flattening further the response of the capture efficiency to the Stokes number. No dramatic inertial effects should thus be observable, even in the proximity of the critical Stokes number, a result which rationalizes in part the observations of Fernandez de la Mora and Friedlander (1982).

As a final remark, we point out that the treatment of Parnas and Friedlander, and our own Eq. (11) are general for two-dimensional laminar boundary layers around any form of obstacle, provided the function $G(\theta)$ is determined appropriately and that ξ is based on the obstacle's local radius of curvature (a function of the location θ). The generalization for three-dimensional axially symmetric geometries is also trivial. For the determination of $G(\theta)$, one can follow standard approximate methods of boundary layer theory such as the Karman Pohlhausen integral approximation. As a result, one can attack analytically a wide variety of problems of particle deposition in natural or artificial filters. In particular, it would be instructive to study the problem of two-dimensional elipses or even blades and compare the results with the data of Chamberlain on particle deposition to blade-shaped roughness elements. Perhaps the anomalously low deposition rates inferred from the oversimplified model of Ferrandez de la Mora and Friedlander (1982) could be improved by the more precise interception theory sketched above.

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