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Theoretical aspects of morphological filters by reconstruction

José Crespo[a,*], Jean Serra[b], Ronald W. Schafer[c]

[a] *Departamento de Lenguajes y Sistemas Informáticos e Ingeniería de Software, Facultad de Informática, Universidad Politécnica de Madrid, 28660 Boadilla del Monte (Madrid), Spain*

[b] *Centre de Morphologie Mathématique, École des Mines de Paris, 35 rue Saint Honoré, 77305 Fontainebleau, France*

[c] *School of Electrical Engineering, Georgia Institute of Technology, Atlanta, GA 30332, USA*

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Abstract

This paper investigates morphological connected filters and, in particular, the so-called filters by reconstruction. A brief background is offered on the theory of morphological filtering. Then, the concept of connectivity is introduced within the morphological framework, which makes it possible to establish connected filters as those that do not introduce discontinuities or, in other words, that extend the input image flat zones. An important subset of connected filters is the class of filters by reconstruction, which allows to build connected filters that treat both the peaks and valleys of an input image while possessing a robustness property called the strong-property. The focus of our research is on the combination, by means of the sup- and inf-operations, of alternating filters by reconstruction when their component filters belong to a granulometry and an antigranulometry (by reconstruction). These operators will be investigated by means of the study of their grain and pore properties. Some commutation properties are introduced that facilitate the manipulation of filters by reconstruction. An important theoretical result of this paper is the establishment of a new family of strong morphological filters. Although most theoretical expressions refer to set operators, results are automatically extendable for non-binary (gray-level) functions.

Zusammenfassung

In diesem Artikel werden morphologische zusammenhängende Filter, insbesondere sogenannte 'Filter durch Rekonstruktion', untersucht. Es wird eine kurze Diskussion der Theorie der morphologischen Filterung geboten. Anschließend wird in dieser Theorie der Begriff der 'connectivity' eingeführt, wodurch es möglich wird, zusammenhängende Filter zu definieren. Diese zeichnen sich dadurch aus, daß sie keine Unstetigkeiten bewirken, d.h. Gebiete konstanter Intensität im Eingangsbild werden aufgeweitet. Ein wichtiger Spezialfall zusammenhängender Filter sind Filter durch Rekonstruktion. Diese erlauben den Entwurf von zusammenhängenden Filtern, welche zur Bearbeitung sowohl der Spitzen als auch der Täler eines Bildes geeignet sind und dabei ein als 'starke Eigenschaft' bezeichnetes robustes Verhalten besitzen. Unsere Forschung konzentriert sich auf die Kombination alternierender Filter durch Rekonstruktion mithilfe der sup- und inf-Operationen, wenn die Komponentenfilter (durch Rekonstruktion) einer Granulometrie und Antigranulometrie angehören. Diese Operatoren werden durch Untersuchung ihrer Korn- und Poren-Eigenschaften studiert. Es werden Kommutations-Eigenschaften eingeführt, die die Behandlung von Filtern durch Rekonstruktion vereinfachen. Ein

* Corresponding author. Tel.: (34) (1) 336 7455; fax: (34) (1)336 7412. E-mail: jcrespo@asterix.fi.upm.es. José Crespo was formerly at Universidad Alfonso X El Sabio, 28691 Villanueva de la Cañada (Madrid), Spain.

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wichtiges theoretisches Ergebnis dieser Arbeit ist die Definition einer neuen Familie von starken morphologischen Filtern. Obwohl sich die meisten theoretischen Ausdrücke auf Mengenoperatoren beziehen, sind die Ergebnise automatisch auf nichtbinäre (Graustufen-) Funktionen erweiterbar.

Résumé

Ce papier présente une étude portant sur les filtres connectés morphologiques et en particulier ceux dits par reconstruction. Un bref aperçu est d'abord donné sur la théorie du filtrage morphologique. Ensuite, le concept de connectivité est situé dans le cadre morphologique, rendant possible la conception de filtres connectés qui n'introduisent pas de discontinuités ou, en d'autres termes, qui étendent les régions uniformes de l'image d'entrée. Un sousensemble important des filtres connectés est la classe des filtres par reconstruction, qui permettent de construire des filtres connectés traitant à la fois les pics et les vallées d'une image d'entrée tout en possédant une propriété de robustesse dite 'Propriété Forte'. Le but de notre recherche est basé sur la combinaison, par l'utilisation des opérateurs sup et min, de filtres alternés par reconstruction quand les filtres les composant appartiennent a une granulométrie ou une anti-granulométrie (par reconstruction). Ces opérateurs seront étudiés par leurs propriétés de pore et de grain. Des propriétés de commutation sont introduites qui facilitent la manipulation de filtres par reconstruction. Un résultat théorique important est l'etablissement d'une nouvelle famillie de filtres morphologiques forts. Bien que la plupart des expressions théoriques se réfèrent à des opérateurs d'ensemble, les résultats sont automatiquement extensibles à des fonctions non binaires (niveaux de gris).

Keywords: Mathematical morphology; Image analysis; Connectivity; Filter by reconstruction; Pyramid

Introduction

Mathematical morphology is a non-linear branch of the signal processing field that was born in the 1960s. Mathematical morphology concerns the application of set theory concepts to image analysis. References in this field are [17, 18–22, 35, 37]. The starting point of the treatment of connectivity, which allows us to talk about connected components of a set (or binary image) such as grains and pores, is the work by Matheron and Serra in the middle 1980s [26, 28]. However, in [26, 28] the notion of a connected operator is different from the grain-removing or pore-filling action that has finally prevailed (in fact, grain-removing and pore-filling operations appeared in [28] but they were not called connected). After this work, the treatment of connectivity has taken a somewhat different path, possibly motivated by the influence of reconstruction algorithms that have been used to implement connected operators. The work by Serra and Salembier [41], updated in [42], was the next step, when they established the notion of a connected operator as a grain-removing and pore-filling operation.

Grain-removing and pore-filling binary operations have been used since the beginning of the image analysis field. The complete elimination of small connected groups of pixels is an operation that must have been used by 'all' researchers on some occasion. Mathematical morphology offers a framework in which to study those operations when they satisfy the increasingness requirement.

Connected operators [4, 7, 41] do not introduce discontinuities. When they are applied to binary images, for example, either connected components of the foreground (*grains*) are removed or those of the background (*pores*) are filled. They are called morphological when they are *increasing*. Morphological connected filters, which are those morphological connected operators that are *idempotent*, will be extensively treated in this paper. (Terms that have a precise meaning in mathematical morphology are written in italics in this section; their definitions can be found in the next sections).

For gray-level functions such as non-binary images, morphological connected filtering uses what can be called a planar (or spatial) approach. For planar filters, the absolute differences in intensity values of the image pixels are not a factor. What matters are the structures that the input image possesses at each intensity level. Thus, each intensity level is processed independently (although this is not how the filters are implemented in practice for efficiency reasons), and this fact makes morphological connected filtering robust and insensitive to certain illumination changes.

Filters by reconstruction [4, 41, 42] are a class of connected filters that are composed of *openings* and *closings by reconstruction*, denoted in the following by $\tilde{\gamma}$ and $\tilde{\varphi}$. When applied to a binary image (equivalently represented by a set), the openings and closings by reconstruction treat each grain or pore independently from the rest of the grains and pores.

An important theoretical property of filters by reconstruction is the classical theorem (Theorem 1) that establishes the *strong* property of the *alternating filter* by reconstruction $\tilde{\varphi}\tilde{\gamma}$ (and of $\tilde{\gamma}\tilde{\varphi}$). Thus, an alternating filter by reconstruction is a tool (1) that is connected, (2) that possesses the strong property (which provides a desirable robustness against noise), and (3) that can treat both grains and pores (peaks and valleys for gray-level images). In the following it will be shown that $\tilde{\varphi}\tilde{\gamma}$ (and $\tilde{\gamma}\tilde{\varphi}$) are not the only connected operators that satisfy these desirable conditions.

The focus of the research presented in this paper is on *sup*- and *inf*-combinations of alternating filters by reconstruction. This way (of parallel nature) to combine alternating filters by reconstruction is different to the purely sequential manner that is employed by the 'classical' family of *alternating sequential filters* (by reconstruction). (Notice that both the sequential composition and the sup- and inf-composition of increasing operators is also increasing – the base of mathematical morphology.) Our goal is to introduce a new multi-scale operator composed of alternating filters by reconstruction that is idempotent, i.e., that is a morphological filter. The new filter, which possesses the strong property, constitutes a novel family that provides an alternative to the alternating sequential filter family as a multi-scale image analysis tool.

The work presented in this paper comes in great part from the thesis work by Crespo [4], although some results (among them the important Theorem 4 that establishes the idempotence and strong property of the new family of operators) appeared stated by Crespo et al. in [7]. In [6], some more results were outlined by Crespo and Serra as well as the practical application for segmentation of the novel family. In this paper, as in [4], we have attempted to study filters by reconstruction in a novel and systematic way by means of their grain and pore properties. Our study will focus on the effects of combinations of $\tilde{\gamma}$ and $\tilde{\varphi}$ on the grains and pores of an input set. Two groups of families will be distinguished:

(1) *Granulometry* $\{\tilde{\gamma}_i\}$, *antigranulometry* $\{\tilde{\varphi}_i\}$ and alternating filters $\{\tilde{\varphi}_i\tilde{\gamma}_i\}$ (composed of openings and closings by reconstruction).
(2) The families formed by compositions under the sup- and inf-operators of alternating filters by reconstruction $\{\tilde{\varphi}_i, \tilde{\gamma}_i\}$.

Regarding group (1), although the effects of (general) non-connected families of openings $\{\gamma_i\}$, closings $\{\varphi_i\}$ and alternating filters $\{\varphi_i\gamma_i\}$ are well known, this is not the case when the component filters (openings and closings) are filters by reconstruction. In fact, the grain and pore properties of the first group of basic filters will enable us to investigate the more complicated combinations of filters constituted by the families in group (2).

Idempotence is the defining property of morphological filters. When openings and closings by reconstruction are combined in order to build new operators, the idempotence of the resulting operator is not guaranteed. We will see later both a case where idempotence exists (the new family of strong filters) and a case of non-idempotence in studying group (2). Another important result in this paper will be the determination of some instances when filters commute with the sup- and inf-operations (when the latter are applied to alternating filters by reconstruction). These properties greatly facilitate the manipulation and simplification of mathematical expressions.

The outline of the paper is as follows. Section 1 gives a general overview of morphological filters, and Section 2 focuses on the treatment of connectivity within the framework of mathematical morphology. Filters by reconstruction are discussed in Section 3. The following two sections will investigate the grain and pore properties of filters by reconstruction. Section 4 is devoted to the grain–pore properties of the granulometry $\{\tilde{\gamma}_i\}$, of the antigranulometry $\{\tilde{\varphi}_i\}$ and of the family of alternating filters $\{\tilde{\varphi}_i\tilde{\gamma}_i\}$. The following section, Section 5, studies the more complex properties of combinations under the sup- and the inf-operators of members of $\{\tilde{\varphi}_i\tilde{\gamma}_i\}$.

Specifically, the originalities of our work are the following. In Section 3, Proposition 1 is new. The way to study in Section 4 the well-known families $\{\tilde{\gamma}_i\}$ $\{\tilde{\varphi}_i\}$ and $\{\tilde{\varphi}_i\tilde{\gamma}_i\}$ by means of their grain and pore properties has not been done previously, and, in particular, some results in Section 4.3 are novel and fundamental to our work (expression (13) and, especially, expression (15)). Finally, the whole Section 5 is new.

This paper introduces some theoretical results. Only the most important proofs are included. The interested reader can find the other proofs in [4].

1. Morphological filtering

The morphological filtering framework was established by Matheron and Serra in the 1980s. This section will provide a brief overview of the subject. References in this area are [13, 14, 23–25, 27, 35–37].

Morphological operators operate on an algebraic structure called a *complete lattice* [3, 37], which is the minimal structure required.

Definition 1. A set T is a complete lattice if: (a) there exists a partial ordering $\leqslant$ over T, and (b) for any (finite or infinite) family $\{A_i\}$ of elements in T, there exists: a smallest majorant $\bigvee_i A_i$ called the 'sup' (for supremum), and a greatest minorant $\bigwedge_i A_i$ called the 'inf' (for infimum).

In practically all theoretical expressions in this paper, we will be working on the lattice $\mathscr{P}(E)$, where E is a given set of points called a *space* and $\mathscr{P}(E)$ denotes the set of all subsets of E (i.e., $\mathscr{P}(E) = \{A: A \subseteq E\}$). In other words, inputs and outputs will be supposed to be sets or, equivalently, binary functions. The order relation $\leqslant$ employed in the lattice $\mathscr{P}(E)$ is the set inclusion $\subseteq$, whereas the sup $\vee$ and inf $\wedge$ operations are the set union $\cup$ and set intersection $\cap$, respectively. Binary images are examples of binary functions $f: E \to T$ where the space of points E is a grid of points (a subset of $\mathbb{Z}^2$, where $\mathbb{Z}$ denotes the set of integers) and T is a set of two ordered elements (for example, $T = \{0, 1\}$). Even though we will work on the lattice $\mathscr{P}(E)$, results are extendable for gray-level functions by means of the so-called flat operators [12, 17, 18, 35, 40]. For the reader who is interested only in gray-level images, let us notice that the input set can be a thresholded version of gray-level image, and that the output set will be the thresholded (at identical threshold) output gray-level image. This is in fact exactly what happens using the (flat) operators treated in this work.

1.1. Building pieces: erosions ε, dilations δ, openings γ and closings φ

Mathematical morphology deals with *increasing* mappings. A mapping (or transformation) ψ is increasing if it preserves the ordering, i.e., if two inputs are ordered then their outputs are likewise ordered. For an increasing set operator $\psi: \mathscr{P}(E) \to \mathscr{P}(E)$, $A \leqslant B \Rightarrow \psi(A) \leqslant \psi(B)$, where $A, B \in \mathscr{P}(E)$. The sup, the inf and the sequential composition of increasing operators is increasing.

Two elementary morphological operations are *erosions* and *dilations*, denoted, respectively, by ε and δ.

Definition 2. Let E be any space. The mappings $\psi: \mathscr{P}(E) \to \mathscr{P}(E)$ that commute with the inf (or, respectively, the sup) are called *erosions* ε (respectively, *dilations* δ). That is, for all $A_i \in \mathscr{P}(E)$, $\varepsilon(\bigwedge_i A_i) = \bigwedge_i \varepsilon(A_i)$ (respectively, $\delta(\bigvee_i A_i) = \bigvee_i \delta(A_i)$).

Before defining what a morphological filter is, let us establish the idempotence concept. A transformation ψ is *idempotent* when if ψ is applied twice it leaves the first output unchanged. Mathematically, this can be expressed as $\psi\psi(A) = \psi(\psi(A)) = \psi(A)$, $\forall A \in \mathscr{P}(E)$, or equivalently, working directly with operators, as $\psi\psi = \psi$.

Definition 3. A mapping ψ is a *morphological filter* if and only if it is increasing and idempotent.

An operator ψ is *anti-extensive* (or, respectively, *extensive*) if $\psi \leqslant I$ (respectively, $\psi \geqslant I$), where I represents the identity operator (for all $A \in \mathscr{P}(E)$, $I(A) = A$).

Definition 4. An *opening* γ (or, respectively, a *closing* φ) is an anti-extensive (respectively, extensive) morphological filter.

The names 'algebraic openings' and 'algebraic closings' are also used in the literature to refer to these most general types of openings and closings. (Algebraic closings are also called Moore-family closings.)

In a space E provided with translation such as the Euclidean space $\mathbb{R}^2$ ($\mathbb{R}$ symbolizes the set of real numbers) or $\mathbb{Z}^2$, well-known types of openings and closings are the *'standard' (or structural) morphological openings* and *closings*, symbolized by γ_B and φ_B, which consist of a Minkowski set addition followed by the dual Minkowski set substraction [11, 32] (or vice versa). The subscript B denotes the so-called structuring element that is used by these filters. The erosions and dilations that employ Minkowski set operations are often denoted by ε_B and δ_B. We are not going to study the Minkowski addition and substraction nor to give their definitions, which can be found in [11, 20–22]. Notice that slightly different definitions are commonly used, fact that must be taken into account for computing γ_B and φ_B.

Any opening has a dual closing (and vice versa), and in general each morphological operation has a dual operator. Two operators ψ_1 and ψ_2 are the dual of each other if

$$\psi_1 = I^c \psi_2 I^c, \tag{1}$$

where I^c is the complementation operator. For all $A \in \mathscr{P}(E)$, $I^c(A) = A^c$, where A^c is the complement of the set A ($A \vee A^c = E$, $A \wedge A^c = \emptyset$). Clearly, if ψ_1 and ψ_2 satisfy (1), then $\psi_2 = I^c \psi_1 I^c$.

The alternating compositions of an opening and a closing $\varphi\gamma$ and $\gamma\varphi$ are idempotent, i.e., they are filters, called *alternating filters*.

Let us define next the important concept of granulometry and antigranulometry, formalized by Matheron [21, 22].

Definition 5. A family of openings $\{\gamma_i\}$ (or, respectively, of closings $\{\varphi_i\}$), where $i \in S = \{1, \ldots, n\}$, is a *granulometry* (respectively, an *antigranulometry*) if for all $i, j \in S$: $i \leqslant j \Rightarrow \gamma_i \geqslant \gamma_j$ (respectively, $\varphi_i \leqslant \varphi_j$).

1.2. $\wedge$-Filters, $\vee$-filters and strong filters

Matheron [24, 25, 27] (all concepts presented in this section have been introduced by Matheron) has investigated the following expressions for increasing mappings ψ: $\psi(I \wedge \psi)$ and $\psi(I \vee \psi)$. Using the fact that for increasing operators ψ and ψ_i, $\psi(\bigwedge_i \psi_i) \leqslant \bigwedge_i (\psi\psi_i)$ and $\psi(\bigvee_i \psi) \geqslant \bigvee_i (\psi\psi_i)$, it can be shown that

$$\psi(I \wedge \psi) \leqslant \psi \wedge \psi\psi \leqslant \psi, \tag{2}$$

$$\psi(I \vee \psi) \geqslant \psi \vee \psi\psi \geqslant \psi. \tag{3}$$

However, there exist some mappings for which the first or the second (or both) of the previous sets of inequalities (2) and (3) is an equality. These are the $\wedge$*-overfilters* and the $\vee$*-underfilters*, which are defined in the following. First, however, let us define the concepts of *overfilter* and *underfilter*.

Definition 6. An increasing mapping ψ is an $\wedge$-*overfilter* (or, respectively, an *underfilter*) if and only if $\psi\psi \geqslant \psi$ (respectively, $\psi\psi \leqslant \psi$).

Definition 7. An increasing mapping ψ is an $\wedge$-*overfilter* (or, respectively, an $\vee$*-underfilter*) if and only if $\psi = \psi(I \wedge \psi)$ (respectively, $\psi = \psi(I \vee \psi)$).

Definition 8. An increasing mapping ψ is an $\wedge$-*filter* (or, respectively, an $\vee$*-filter*) if and only if ψ is an $\wedge$-overfilter and an underfilter (respectively, an $\vee$-underfilter and an overfilter).

Next, the important concept of *strong filter* is defined. As stated in Corollary 1, strong filters are robust in the sense that input variations (such as, for example, noise) within certain boundaries cause no variation in the output.

Definition 9. A filter ψ is *strong* if and only if ψ is both an $\wedge$-filter and an $\vee$-filter, i.e., $\psi = \psi(I \wedge \psi) = \psi(\mathrm{I} \vee \psi)$.

Corollary 1. *Let ψ be a filter from $\mathscr{P}(E)$ to $\mathscr{P}(E)$. If ψ is strong, then for all $A, B \in \mathscr{P}(E)$: $A \wedge \psi(A) \leqslant B \leqslant A \vee \psi(A) \Rightarrow \psi(A) = \psi(B)$.*

Corollary 1 can be interpreted as follows. If A is an uncorrupted input signal, then a strong filter

ψ computes the same output for corrupted signals whose values are within the boundaries $A \wedge \psi(A)$ and $A \vee \psi(A)$. Fig. 1 illustrates this for functions.

Table 1 summarizes the properties of the most common morphological filters γ, φ, $\varphi\gamma$ and $\gamma\varphi$. We can observe that $\varphi\gamma$ (or, respectively, $\gamma\varphi$) is an $\wedge$-filter (respectively, $\vee$-filter). In fact, the implication in the other sense is also true; i.e., all mappings that are $\wedge$-filters (or, respectively, $\vee$-filters) can be expressed in the form of $\varphi\gamma$ (respectively, $\gamma\varphi$). When γ and φ satisfy more restricted conditions than just Definition 4, other properties can be inferred as will be seen later in the paper.

1.3. Pyramids

The following definition of pyramid [41] applies both to morphological and non-morphological pyramids such as the Gaussian pyramid.

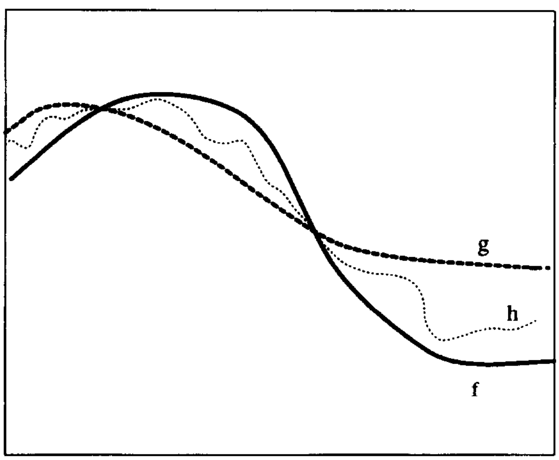


Fig. 1. Strong filter example. If f is an input function and $g = \psi(f)$, where ψ is a strong filter, for any function h between f and g, it is true that $\psi(h) = \psi(f) = g$.

Table 1
Properties of openings γ, of closings φ, and of alternating filters $\varphi\gamma$ and $\gamma\varphi$

	Idempotent	$\wedge$-over-filter	$\vee$-under-filter	Strong filter
γ	Yes	Yes	Yes	Yes
φ	Yes	Yes	Yes	Yes
$\varphi\gamma$	Yes	Yes	No	No
$\gamma\varphi$	Yes	No	Yes	No

Definition 10. A family of operators $\{\psi_i\}$, where $i \in S = \{1, \ldots, n\}$, forms a *pyramid* if and only if for all $j, k \in S$, $j \geqslant k$, $\exists l$ such that $\psi_j = \psi_l \psi_k$.

In words, $\{\psi_i\}$ is a pyramid if any level j of the hierarchy can be reached by applying a member of $\{\psi_i\}$ to a finer (smaller index) level k. Let us note, however, that sometimes the term of pyramid is applied only to a family of operators that constitute a semigroup.

Within the class of morphological filters, cases that satisfy the previous definition are (a) granulometries $\{\gamma_i\}$, (b) antigranulometries $\{\varphi_i\}$, and (c) *alternating sequential filters* (ASF). An ASF [38] is an ordered sequential composition of $\varphi_i\gamma_i$ (or of its dual) such as $\mathrm{ASF}_i = \varphi_i\gamma_i \cdots \varphi_j\gamma_j \cdots \varphi_1\gamma_1$, where $i \geqslant j \geqslant 1$, and where γ_i and φ_i belong, respectively, to a granulometry and an antigranulometry.

2. Connectivity in mathematical morphology

Connectivity is introduced in mathematical morphology by the operation that extracts the *connected components* of a set. As will be seen in this section, those operators that do not break the connected components of either the foreground or the background of an image are called *connected operators*.

2.1. The point opening γ_x

Let us assume that the space E is provided with a definition of connectivity. For all pairs of points x, y in E, it is possible to establish whether they are connected or not. For example, when the space of points E is $\mathbb{R}^2$ or $\mathbb{Z}^2$ (associated with the *usual* connectivity), a pair of points x, y in a set A is said to be connected if there exists a path linking x and y that is also included in A.

Connectivity is established more generally in [37] by means of the *connected class* concept. A connected class $\mathscr{C}$ in $\mathscr{P}(E)$ is a subset of $\mathscr{P}(E)$ such that (a) $\emptyset \in \mathscr{C}$ and for all $x \in E$, $\{x\} \in \mathscr{C}$; and (b) for each family $\mathscr{C}_i$ in $\mathscr{C}$, $\bigwedge_i C_i \neq \emptyset$ implies $\bigvee_i C_i \in \mathscr{C}$. No definition of neighborhood relationships (i.e., no particular topology) has been assumed for E in the definition of the connected class $\mathscr{C}$.

The subclass $\mathscr{C}_x$ that has all members of $\mathscr{C}$ that contain x (i.e., $\mathscr{C}_x = \{C\colon\ x \in C \in \mathscr{C}\}$) defines an opening called *point opening* [37]. The point opening of a point x, denoted by γ_x, has as invariant class (i.e., the class formed by those sets that are left unchanged by γ_x) $\mathscr{C}_x \cup \{\emptyset\}$. For all $x \in E$, $A \in \mathscr{P}(E)$

$$\gamma_x(A) = \bigvee \{C\colon\ C \in \mathscr{C}_x, C \leqslant A\}. \tag{4}$$

The operation γ_x is therefore idempotent (i.e., $\gamma_x(\gamma_x(A)) = \gamma_x(\mathrm{A})$ or, equivalently, $\gamma_x\gamma_x = \gamma_x$) and anti-extensive (i.e., $\gamma_x(A) \leqslant A$ or, equivalently, $\gamma_x \leqslant I$). Properties satisfied by $\gamma_x(A)$ are: (a) $\forall x \in E, \gamma_x(\{x\}) = \{x\}$; (b) $\forall A \in \mathscr{P}(E)$, $\forall x, y \in E$, $\gamma_x(A)$ and $\gamma_y(A)$ are equal or disjoint; and (c) $\forall A \in \mathscr{P}(E)$, $x \notin A$ implies $\gamma_x(A) = \emptyset$.

When we associate, for example, the operation γ_x with the usual connectivity in $\mathbb{Z}^2$, the opening $\gamma_x(A)$, $A \in \mathscr{P}(\mathbb{Z}^2)$, can be defined as the union of all paths that contain x and that are included in A. Fig. 2 shows an example of the results of $\gamma_x(A)$, where the set A comprises the black regions and x belongs to a connected component of A.

When a space E is equipped with the opening γ_x, connectivity issues in E can be expressed using γ_x. We can establish, for example, whether or not a set $A \in \mathscr{P}(E)$ is connected (a set A is connected if and only if $A = \gamma_x(A)$, $x \in A$), and whether or not a pair of points x, y belong to the same connected component in A (x, y belong to the same connected component in A if and only if $x \in \gamma_y(A)$ or, equivalently, if and only if $\gamma_x(A) = \gamma_y(A) \neq \emptyset$).

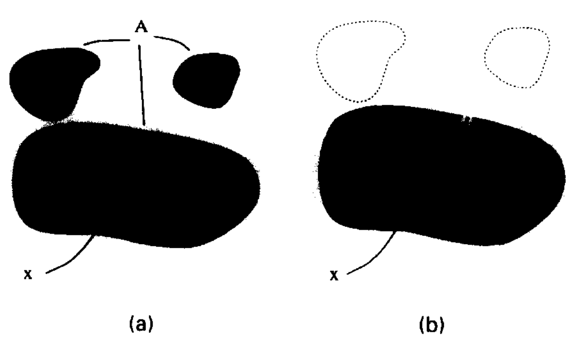


Fig. 2. Connected component extraction. The opening $\gamma_x(A)$ extracts the connected component of A to which x belongs. (a) Input set A (in black), (b) $\gamma_x(A)$.

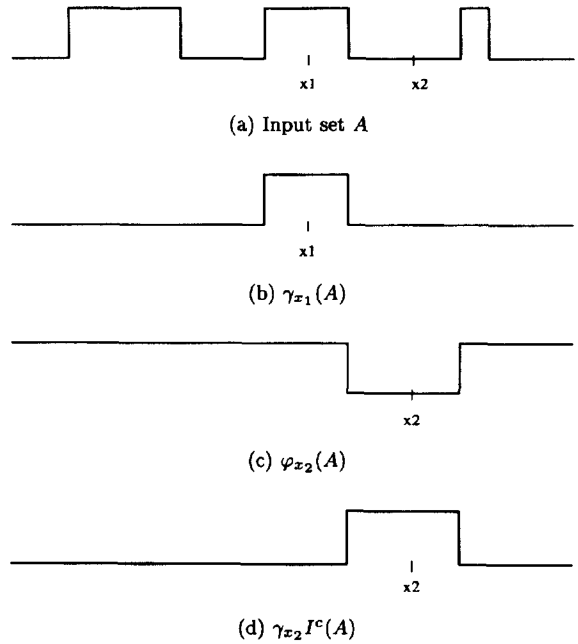


Fig. 3. γ_x, φ_x, $\gamma_x I^c$: one-dimensional example.

The dual operation of γ_x is the closing φ_x that is equal to $E \backslash \gamma_x I^c(A)$, for all $A \in \mathscr{P}(E)$, where $\backslash$ denotes set difference. Fig. 3 shows a one-dimensional example of the dual operations γ_x and φ_x, along with the pore extraction operation.

The operation that extracts the pore to which a point x of the space E belongs is not the dual operation of the grain extraction operation. Fig. 3(d) shows a pore extraction operation. For a point x of E, two equivalent ways to extract the pore to which x belongs are $\gamma_x I^c$ or $I^c \varphi_x$. In the following, the first way $\gamma_x I^c$ has been (arbitrarily) chosen.

2.2. Connected filters

Connected filters belong to a class of operators that *consider* the connectivity of an input set A, $A \in \mathscr{P}(E)$. Connected operators ensure that if two points x, y in E are connected in A or in A^c (foreground and background are regarded symmetrically), then the pair x, y will be connected operators to process grains and pores in an all-or-nothing way. If a grain is to be removed (i.e., the grain is modified), then all its component points will be removed. Likewise for pores: either they are filled or they are left unchanged. On the other hand, non-connected operators process sets without any restriction on changes of connectivity from the input set to the output set. In particular, a morphological non-connected operator must satisfy only the increasingness requirement.

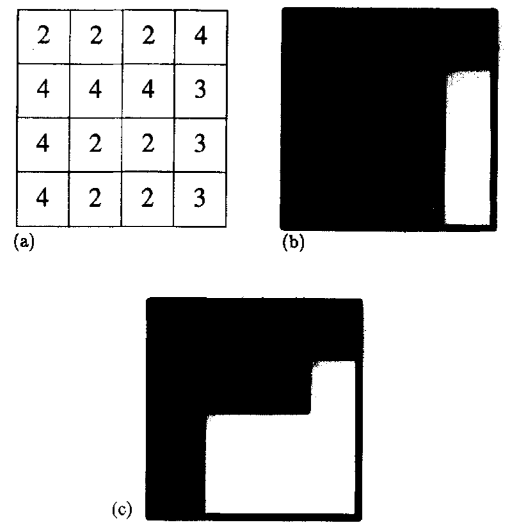
The following definition of a connected operator is due to Serra and Matheron. Let us define first the concept of flat zone [41], which is defined more generally for functions rather than for sets.

Definition 11. Let E be a space equipped with γ_x and T a complete lattice. The flat zones of a function $f: E \to T$ are defined as the (largest) connected components of points $x \in E$ with the same function value.

Definition 12. An operator ψ is *connected* if and only if it extends the flat zones of the input function.

Definition 12, which applies both to binary and gray-level morphology, does not establish how each intensity level of an input function is processed. However, in the following all connected operators are supposed to be flat, in the sense that they process each intensity level of an input function independently from the rest. In addition, even though only increasing connected operators will be studied in the following, observe that Definition 12 also applies to non-increasing operators. For the binary case, an equivalent definition of connected operator is that in [41], which applies only to binary morphology: an operator $\psi: \mathscr{P}(E) \to \mathscr{P}(E)$ is said to be connected if and only if both set subtractions $A \setminus \psi(A)$ and $\psi(A) \setminus A$ are formed exclusively by connected components of A or of its complement A^c.

Fig. 4 (where the space connectivity is eight-connectivity) illustrates the previous definitions. All pixels in an image belong to a flat zone, and isolated pixels constitute their own flat zone. Another gray-level example is shown in Fig. 5. In this figure, connected filters have been cascaded (to form a connected alternating sequential filter (ASF) pyramid), and a certain flat zone has been marked at several stages to show the flat zone inclusion property.



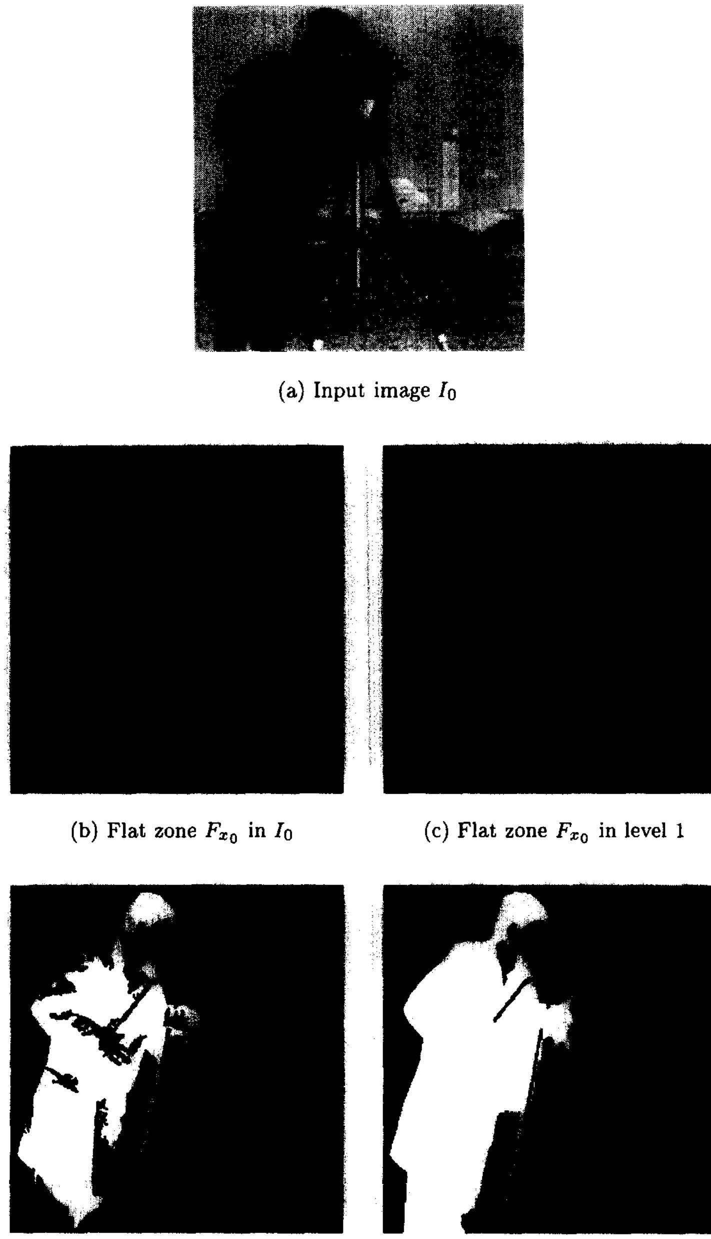
2	2	2	4
4	4	4	3
4	2	2	3
4	2	2	3

Fig. 4. Flat zones example. For an input gray level image (a), part (b) shows its four flat zones, i.e., those connected regions with a same function value. Notice that there are *two* flat zones with intensity value 2 (and not one) because pixels with value 2 form two separated regions. Part (c) shows the flat zones of a connected operation performed on (a). Observe the flat zone inclusion relation between (b) and (c). Note: eight-connectivity has been assumed for the space of points. (a) Input image I_0. (b) Flat zones of I_0 (four). (c) Flat zones of $\psi(I_0)$ (where ψ is a connected operator).

In the following, only sets will appear in theoretical expressions. The flat zone of a point x is the connected component of the set or of its complement to which x belongs. This can be expressed using the operator F_x employed in [4, 7]. The operator F_x is defined as $F_x = \gamma_x \vee \gamma_x I^c$, i.e., F_x is the grain or pore, whichever is not empty, to which x belongs. (Notice that $\gamma_x(A) \neq \emptyset \Rightarrow \gamma_x I^c(A) = \gamma_x(A^c) = \emptyset$ and vice versa.) For set operators $\psi: \mathscr{P}(E) \to \mathscr{P}(E)$, Definition 12 can be expressed as follows: an operator is connected if and only if $F_x \psi \geqslant F_x$.

Clearly, the class of connected operators is closed under the operations sup, inf and the sequential composition of connected operators [41]. Figs. 6 and 7 show that discontinuities can be introduced by non-connected operators and that they modify the shape of the preserved connected components.

The 'standard' morphological openings γ_B and closings φ_B are connected in one-dimensional



(a) Input image I_0

(b) Flat zone F_{x_0} in I_0

(c) Flat zone F_{x_0} in level 1

(d) Flat zone F_{x_0} in level 2

(e) Flat zone F_{x_0} in level 3

Fig. 5. Flat zone extension: gray-level example. In this figure, a certain flat zone F_{x_0} has been marked at the outputs of an ASF connected pyramid (x_0 is a pixel that belongs to the head of the cameraman). The higher the pyramid level, the more severe has been the filtering applied. Notice the flat zone inclusion relationship.

spaces when the structuring element B is connected. (In this case they are also filters by reconstruction, which will be defined later.) However, in two-dimensions γ_B and φ_B are not connected.

3. Filters by reconstruction

This section discusses an important group of connected filters, to so-called *filters by reconstruction.*

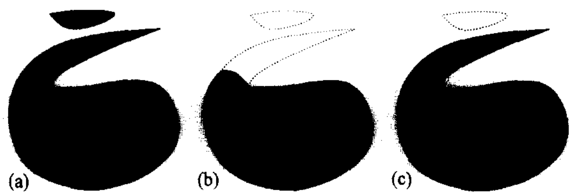


Fig. 6. Differences between a non-connected and a connected opening. In this example, one of the two grains of (a) has been *broken* in (b). Notice that image (b) shows a discontinuity that does not exist in (a). (a) Input set (in black), (b) non-connected opening, (c) connected opening.

Filters by reconstruction are defined by means of the concepts of *trivial opening* γ_0 and *trivial closing* φ_0, which appeared in [28].

Definition 13. Let E be any space. An opening $\gamma_0 : \mathscr{P}(E) \to \mathscr{P}(E)$ is a *trivial opening* if for all input sets $A \in \mathscr{P}(E)$

$$\gamma_0(A) = \begin{cases} A & \text{if } A \text{ satisfies an increasing criterion,} \\ \emptyset & \text{if } A \text{ does not satisfy the increasing criterion.} \end{cases}$$

Example 1. Examples of increasing criteria and the trivial openings defined by them are the following.
(a) Standard morphological opening γ_B: γ_0 would leave the input set A invariant if $\gamma_B(A) \neq \emptyset$; on the other hand, if $\gamma_B(A) = \emptyset$ then $\gamma_0(A) = \emptyset$.
(b) Area: $\gamma_0(A) = A$ if the area of A is larger than a certain number; otherwise, $\gamma_0(A) = \emptyset$.

We can see that the criterion used for γ_0 can be anisotropic and shape dependent (such as in case (a) of Example 1) but that γ_0 does not modify shapes since the input set is either preserved or removed.

Definition 14. Let E be any space. A closing $\varphi_0 : \mathscr{P}(E) \to \mathscr{P}(E)$ is a *trivial closing* if for all $A \in \mathscr{P}(E)$

$$\varphi_0(A) = \begin{cases} E & \text{if } A \text{ satisfies an increasing criterion,} \\ A & \text{if } A \text{ does not satisfy the increasing criterion.} \end{cases}$$

Definition 15. Let E be a space equipped with γ_x. An opening $\tilde{\gamma} : \mathscr{P}(E) \to \mathscr{P}(E)$ is an *opening by reconstruction* if and only if it is connected and

$$\tilde{\gamma} = \bigvee_{x \in E} \gamma_0 \gamma_x,$$

where γ_0 is a trivial opening.

Thus, the output of an opening by reconstruction $\tilde{\gamma}$ performed on an input set A is the set formed by all connected components of A that satisfy the increasing criterion of the trivial opening γ_0 that is associated with $\tilde{\gamma}$.

Definition 16. Let E be a space equipped with γ_x. A closing $\tilde{\varphi} : \mathscr{P}(E) \to \mathscr{P}(E)$ is a *closing by reconstruction* if and only if it is connected and

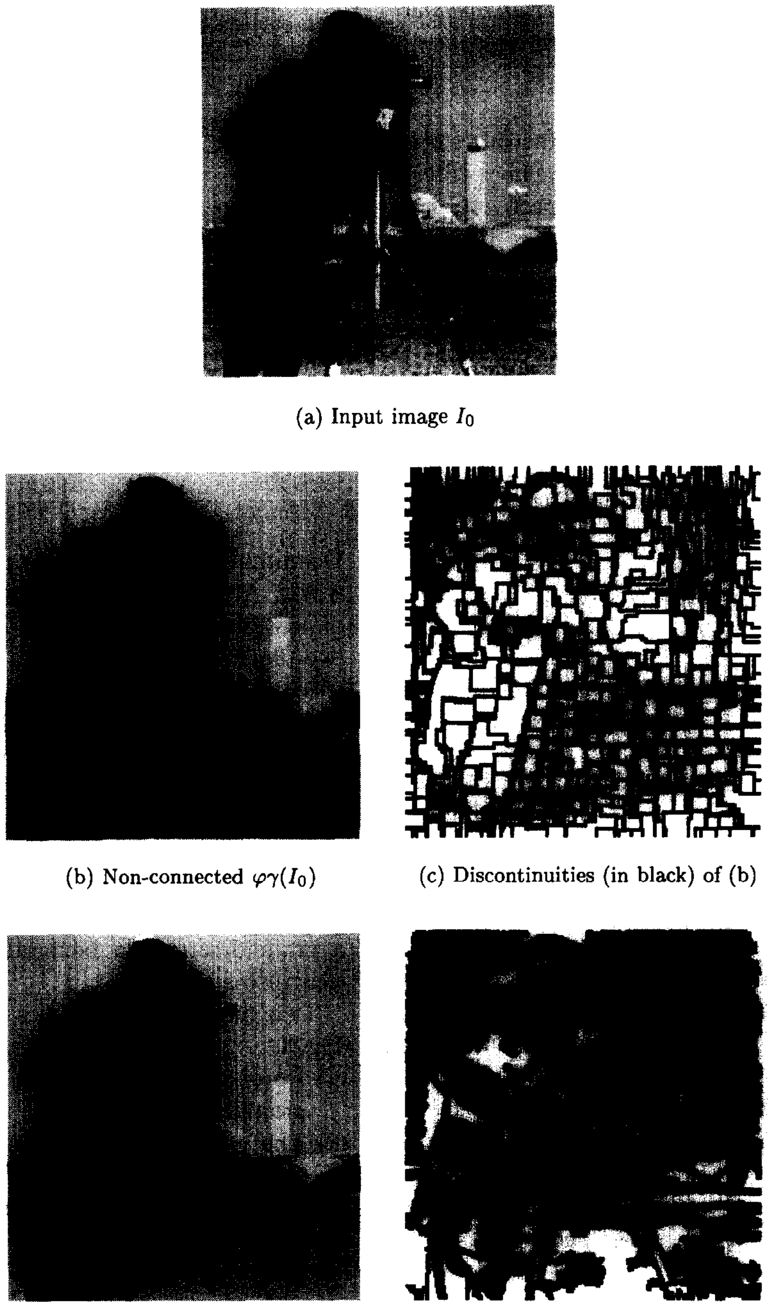
$$\tilde{\varphi} = \bigwedge_{x \in E} \varphi_0 \varphi_x,$$

where φ_0 is a trivial closing.

The concepts of opening and closing by reconstruction appeared in [28, 41] (although not under those names in [28]). From the definitions of $\tilde{\gamma}$ and $\tilde{\varphi}$, it is clear that: (1) the fact that $\tilde{\gamma}$ and $\tilde{\varphi}$ are connected implies that $\tilde{\gamma}$, which is anti-extensive, exclusively removes grains and that $\tilde{\varphi}$, which is extensive, exclusively fills pores; and (2) $\tilde{\gamma}$ treats each grain and $\tilde{\varphi}$ treats each pore *independently* from the rest of grains and pores, respectively, of the input set. In this paper, we will call *filters by reconstruction those combinations of openings $\tilde{\gamma}$ and closings $\tilde{\varphi}$ by reconstruction that are idempotent.*

The grain extraction operation γ_x and its dual φ_x (see Section 2.1) are an opening by reconstruction and a closing by reconstruction, respectively. When the grain extraction operation γ_x is sequentially composed (or cascaded) with any opening by reconstruction $\tilde{\gamma}$, it is true that $\tilde{\gamma}\gamma_x = \gamma_x\tilde{\gamma}$.

The following theorem [41] establishes that alternating filters are strong filters under certain conditions. This is a variant of a (more general) theorem by Matheron and Serra, which appeared in [28] that proved this property of the alternating filter $\gamma\varphi$ under different requirements for γ and φ.



(a) Input image I_0

(b) Non-connected $\varphi\gamma(I_0)$

(c) Discontinuities (in black) of (b)

(d) Connected $\varphi\gamma(I_0)$

(e) Discontinuities (in black) of (d)

Fig. 7. Comparison between non-connected and connected filters: connected filters do not introduce discontinuities and, as a consequence, maintain the shape of the preserved features.

Theorem 1. *Let $\tilde{\gamma}$ and $\tilde{\varphi}$ be respectively an opening and a closing by reconstruction. The connected alternating filters $\tilde{\varphi}\tilde{\gamma}$ and $\tilde{\gamma}\tilde{\varphi}$ are strong.*

As commented in Section 1.2, Matheron established that an $\wedge$-filter (or, respectively, an $\vee$-filter) can be expressed in the form of $\varphi\gamma$ (respectively, $\gamma\varphi$). Since $\tilde{\varphi}\tilde{\gamma}$ and $\tilde{\gamma}\tilde{\varphi}$ are strong (i.e., they are both an $\wedge$-filter and an $\vee$-filter), we can expect to find an equivalent expression for them in the dual form of, respectively, $\gamma\varphi$ and $\gamma\varphi$. The following new proposition gives such a dual form for the alternating filter $\tilde{\gamma}\tilde{\varphi}$ by reconstruction.

Proposition 1. *Let $\tilde{\gamma}$ and $\tilde{\varphi}$ be, respectively, an opening and a closing by reconstruction. Then, $\tilde{\gamma}\tilde{\varphi} = \tilde{\varphi}\gamma'$ where $\gamma' = I \wedge \tilde{\gamma}\tilde{\varphi}$.*

Proof of Proposition 1. (i) $\tilde{\varphi}(I \wedge \tilde{\gamma}\tilde{\varphi}) \leqslant \tilde{\varphi}\tilde{\gamma}\tilde{\varphi} = \tilde{\gamma}\tilde{\varphi}$; and (ii) $\tilde{\varphi}(I \wedge \tilde{\gamma}\tilde{\varphi}) \geqslant \tilde{\gamma}\tilde{\varphi}(I \wedge \tilde{\gamma}\tilde{\varphi}) = \tilde{\gamma}\tilde{\varphi}$. □

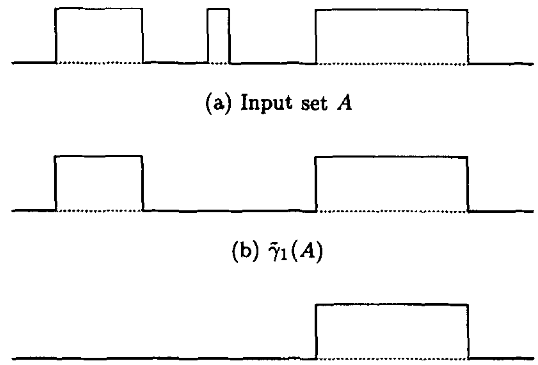
In Proposition 1, the expression $I \wedge \tilde{\gamma}\tilde{\varphi}$ is an opening because $\tilde{\gamma}\tilde{\varphi}$ is strong and, therefore, an $\wedge$-filter [27]. The proof is easy: (a) $I \wedge \tilde{\gamma}\tilde{\varphi} \leqslant I$; and (b) $(I \wedge \tilde{\gamma}\tilde{\varphi})(I \wedge \tilde{\gamma}\tilde{\varphi}) = I \wedge \tilde{\gamma}\tilde{\varphi} \wedge \tilde{\gamma}\tilde{\varphi}(I \wedge \tilde{\gamma}\tilde{\varphi}) = I \wedge \tilde{\gamma}\tilde{\varphi}$. Clearly, Proposition 1 applies dually to $\tilde{\varphi}\tilde{\gamma}$: $\tilde{\varphi}\tilde{\gamma} = \tilde{\gamma}(I \vee \tilde{\varphi}\tilde{\gamma})$.

Reconstruction algorithms [1, 15, 33, 35] are a simple way to compute filters by reconstruction, and extremely efficient algorithms based on waiting queues are available [2, 29, 30, 43–45]. Reconstruction algorithms employ what are called *geodesic operators* [13, 16]. We notice that filters by reconstruction are sometimes defined in the literature [10, 33] by means of a reconstruction algorithm (i.e., using an algorithmic definition).

4. Study of the families $\{\tilde{\gamma}_i\}$, $\{\tilde{\varphi}_i\}$ and $\{\tilde{\varphi}_i\tilde{\gamma}_i\}$: grain–pore properties

4.1. The granulometry $\{\tilde{\gamma}_i\}$

Openings by reconstruction were defined in Definition 15. In this section we are going to study the effect of these filters on grains and pores of the input set. In particular, the behavior of these types of openings when they form granulometries (see



(a) Input set A

(b) $\tilde{\gamma}_1(A)$

(c) $\tilde{\gamma}_2(A)$

Fig. 8. Granulometry by reconstruction $\{\tilde{\gamma}_i\}$: one-dimensional example.

Definition 5) is most important in the following work. Although example figures display one-dimensional examples, results are valid also for two-dimensional spaces (and in general for any space equipped with a point opening γ_x).

A granulometry $\{\tilde{\gamma}_i\}$ is by definition an ordered family. Fig. 8 shows an input set A in a one-dimensional space and two outputs $\tilde{\gamma}_1(A)$ and $\tilde{\gamma}_2(A)$. Notice that the subindex of the filters forming a granulometry just means the relative ordering (in this case $\tilde{\gamma}_1(A) \geqslant \tilde{\gamma}_2(A)$) but no assumption should be made regarding any structuring element. (In fact, a granulometry can be built without using any structuring element.)

Two major characteristics can be observed in Fig. 8:

(1) Pores grow in size as the index in the granulometry increases.

(2) The grains in $\tilde{\gamma}_i(A)$, for all i, are identical to those in the input set A.

The first effect can be expressed mathematically as, for all $x \in E$

$$\gamma_x I^c \tilde{\gamma}_n \geqslant \gamma_x I^c \tilde{\gamma}_j, \quad j \leqslant n, \tag{5}$$

and the second effect as

$$\gamma_x \tilde{\gamma}(A) \neq \emptyset \Rightarrow \gamma_x \tilde{\gamma}(A) = \gamma_x(A), \quad \forall A \in \mathscr{P}(E). \tag{6}$$

The grains in $\tilde{\gamma}_i(A)$ are in A, for all levels i. In a family $\{\tilde{\gamma}_i\}$, grains are *passed* to coarser (greater subindex) levels or eliminated. Notice that for non-connected openings γ, expression (5) would still be

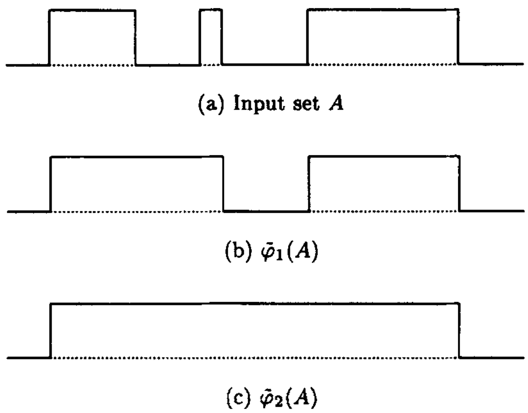


Fig. 9. Antigranulometry by reconstruction $\{\tilde{\varphi}_i\}$: one-dimensional example.

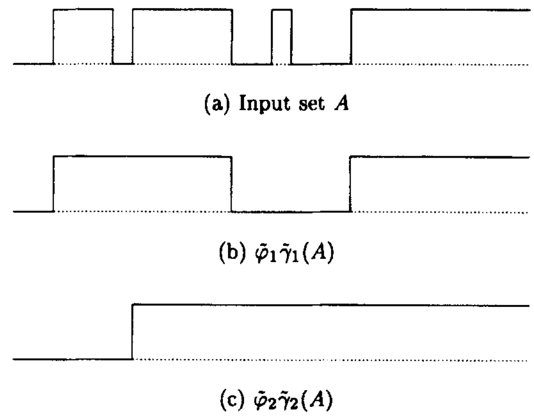


Fig. 10. Family of alternating filters by reconstruction $\{\tilde{\varphi}_i\tilde{\gamma}_i\}$: one-dimensional example.

true but not expression (6) (for non-connected openings, instead of expression (6) we have only that $\gamma_x\gamma \leqslant \gamma_x$).

Clearly, flat zones of the output $\tilde{\gamma}_i(A)$ grow as the index of the granulometry i increases.

$$F_x\tilde{\gamma}_n \geqslant F_x\tilde{\gamma}_j, \quad j \leqslant n. \tag{7}$$

4.2. *The antigranulometry* $\{\tilde{\varphi}_i\}$

Properties of granulometries by reconstruction apply *dually* to antigranulometries by reconstruction. Only the formulas are stated. An example is shown in Fig. 9.

$$\gamma_x\tilde{\varphi}_n \geqslant \gamma_x\tilde{\varphi}_j, \quad j \leqslant n, \tag{8}$$

$$\gamma_x\tilde{\varphi}(A) = \emptyset \Rightarrow \gamma_x I^c\tilde{\varphi}(A) = \gamma_x I^c(A), \quad A \in \mathscr{P}(E), \tag{9}$$

$$F_x\tilde{\varphi}_n \geqslant F_x\tilde{\varphi}_j, \quad j \leqslant n. \tag{10}$$

Compare with the corresponding expressions for the granulometry. Notice that the grain and pore properties of $\tilde{\varphi}$ can be called 'dual' of those of $\tilde{\gamma}$, but expressions (8)–(10) are not strictly the dual of, respectively, expressions (5)–(7).

4.3. *Family of alternating filters* $\{\tilde{\varphi}_i\tilde{\gamma}_i\}$

Alternating filters by reconstruction are sequential compositions of an opening and a closing by reconstruction; i.e., $\tilde{\varphi}\tilde{\gamma}$ and $\tilde{\gamma}\tilde{\varphi}$. By combining both an opening and a closing, the resulting operation both removes grains and fills pores. In this section the effects of the alternating filter $\tilde{\varphi}\tilde{\gamma}$, and in particular the effects of members of the family $\{\tilde{\varphi}_i\tilde{\gamma}_i\}$ on the grains and pores of an input set are studied. In the family $\{\tilde{\varphi}_i\tilde{\gamma}_i\}$, the component families $\{\tilde{\gamma}_i\}$ and $\{\tilde{\varphi}_i\}$ form, respectively, a granulometry and an antigranulometry. An example of alternating filters by reconstruction can be seen in Fig. 10.

In this section we notice that expression (13) and, especially, expression (15) are new and fundamental to our work. Expression (11) and expression (14) have been established in [41]. All other expressions follow easily from the general properties of openings and closings [37].

The main characteristics of alternating filters are summarized below. Properties that apply to $\tilde{\varphi}\tilde{\gamma}$ and the relationships between different levels of the family $\{\tilde{\varphi}_i\tilde{\gamma}_i\}$ are discussed.

- There is no order relation between $\tilde{\varphi}\tilde{\gamma}$ and I: $I \nleqslant \tilde{\varphi}\tilde{\gamma} \nleqslant I$. That is, the filter $\tilde{\varphi}\tilde{\gamma}$ is neither an opening nor a closing.
- There is an ordering between $\tilde{\varphi}\tilde{\gamma}$ and $\tilde{\gamma}\tilde{\varphi}$ [41]:

$$\tilde{\varphi}\tilde{\gamma} \leqslant \tilde{\gamma}\tilde{\varphi}. \tag{11}$$

Expression (11) is not generally true when the filters $\tilde{\gamma}$ and $\tilde{\varphi}$ are not filters by reconstruction.

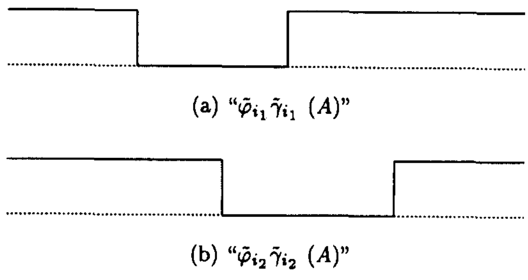


Fig. 11. Impossible case. This case is not possible: notice that pores are not disjoint and that they are not nested.

From [28], it is known that expression (11) is equivalent to the equality

$$\tilde{\gamma}\tilde{\varphi}\tilde{\gamma} = \tilde{\varphi}\tilde{\gamma}. \tag{12}$$

- The pores $\tilde{\varphi}\tilde{\gamma}(A)$ come from $\tilde{\gamma}(A)$, $A \in \mathscr{P}(E)$

$$\gamma_x\tilde{\varphi}\tilde{\gamma}(A) = \emptyset \Rightarrow \gamma_x I^c\tilde{\varphi}\tilde{\gamma}(A) = \gamma_x I^c\tilde{\gamma}(A). \tag{13}$$

  The same cannot be said for grains. The grains of $\tilde{\varphi}\tilde{\gamma}(A)$ are not identical to the grains of $\tilde{\varphi}(A)$. Only the expression $\gamma_x\tilde{\varphi}\tilde{\gamma}(A) \leqslant \gamma_x\tilde{\varphi}(A)$ is valid, as is also true for non-connected γ and φ.
- Alternating filters by reconstruction possess a leftwards absorption property [41]:

$$\tilde{\varphi}_j\tilde{\gamma}_j\tilde{\varphi}_p\tilde{\gamma}_p = \tilde{\varphi}_p\tilde{\gamma}_p, \quad j \leqslant p. \tag{14}$$

  This can be deduced from Eq. (12).
- Even though both families $\{\tilde{\gamma}_i\}$ and $\{\tilde{\varphi}_i\}$ are ordered, there is no order relation between different levels of the family $\{\tilde{\varphi}_i\tilde{\gamma}_i\}$:

$$\tilde{\varphi}_i\tilde{\gamma}_i \nleqslant \tilde{\varphi}_j\tilde{\gamma}_j \nleqslant \tilde{\varphi}_i\tilde{\gamma}_i, \quad i \neq j.$$

- All pores of $\{\tilde{\varphi}_i\tilde{\gamma}_i\}$ are not nested but *some* are. Let $A \in \mathscr{P}(E)$. For any pair of levels the following expression is true:

$$\gamma_x I^c\tilde{\varphi}_i\tilde{\gamma}_i(A) \cap \gamma_x I^c\tilde{\varphi}_j\tilde{\gamma}_j(A) \neq \emptyset$$
$$\Rightarrow \gamma_x I^c\tilde{\varphi}_i\tilde{\gamma}_i(A) \leqslant \gamma_x I^c\tilde{\varphi}_j\tilde{\gamma}_j(A), \quad i \leqslant j. \tag{15}$$

  The reason is that pores of $\tilde{\varphi}_i\tilde{\gamma}_i(A)$ are a subset of those of $\tilde{\gamma}_i(A)$, which are nested for different levels (see expression (5)). Fig. 11 shows an impossible case that does not satisfy expressions (15).
  Only a subset of pores are nested: there exist pores in finer levels that do not exist (they are filled) in coarser levels. Remember that there is no ordering in the family $\{\tilde{\varphi}_i\tilde{\gamma}_i\}$.
- Unfortunately, little can be said about grains at different levels of $\{\tilde{\varphi}_i\tilde{\gamma}_i\}$. There exists no order relation between the grains.
- There is no flat zone inclusion relation between different levels of $\{\tilde{\varphi}_i\tilde{\gamma}_i\}$ (as expected from the lack of relation between grains at several levels).

$$F_x\tilde{\varphi}_n\tilde{\gamma}_n \ngeqslant F_x\tilde{\varphi}_j\tilde{\gamma}_j, \quad j \leqslant n. \tag{16}$$

The grain and pore properties of alternating filters $\tilde{\varphi}\tilde{\gamma}$ have been discussed. As is well known, it goes without saying that the filter $\tilde{\varphi}\tilde{\gamma}$ is idempotent, i.e., it is a filter ($\tilde{\varphi}\tilde{\gamma}\tilde{\varphi}\tilde{\gamma} = \tilde{\varphi}\tilde{\gamma}$ since (a) $\tilde{\varphi}\tilde{\gamma}\tilde{\varphi}\tilde{\gamma} \geqslant \tilde{\varphi}\tilde{\gamma}\tilde{\gamma}\tilde{\gamma} = \tilde{\varphi}\tilde{\gamma}$, and (b) $\tilde{\varphi}\tilde{\gamma}\tilde{\varphi}\tilde{\gamma} \leqslant \tilde{\varphi}\tilde{\varphi}\tilde{\varphi}\tilde{\gamma} = \tilde{\varphi}\tilde{\gamma}$).

5. Compositions of alternating filters $\{\tilde{\varphi}_i\tilde{\gamma}_i\}$ under the inf and the sup: $\bigwedge_{i=1}^n \tilde{\varphi}_i\tilde{\gamma}_i$ and $\bigvee_{i=1}^n \tilde{\varphi}_i\tilde{\gamma}_i$

Compositions of members of the family $\{\tilde{\varphi}_i\tilde{\gamma}_i\}$, where $\{\tilde{\gamma}_i\}$ and $\{\tilde{\varphi}_i\}$ form a granulometry and anti-granulometry, respectively, are the focus of this section. Our motivation is to establish when the composition of several levels of alternating filters by reconstruction is a *filter*, i.e., when the resulting operator shows idempotence. In [41, 42], other multilevel morphological structures are presented. However, except the classical cases constitued by alternating sequential filters and *self-dual centers* [39] by reconstruction (the latter possessing limited simplifying capabilities – the higher the level, the output is closer to the original image –), the other multilevel pyramids (*residues* [34] and *constrast* [31] operators), are not morphological filters. Our goal is to discover a new family of operators that shows idempotence (i.e., it is a family of filters) and that constitutes an alternative to the alternating sequential filter family as a multi-scale image analysis tool.

We will study the compositions of members of the family $\{\tilde{\varphi}_i\tilde{\gamma}_i\}$ under the inf operator (i.e., $\bigwedge_{i=1}^n \tilde{\varphi}_i\tilde{\gamma}_i$) and under the sup operator (i.e., $\bigvee_{i=1}^n \tilde{\varphi}_i\tilde{\gamma}_i$). Therefore, we employ a composition manner of parallel nature that is radically different to the sequential composition way used to build alternating sequential filters. Notice that increasingness (the foundation of morphological filtering) is preserved when operators are

(1) composed sequentially or (2) composed by means of the sup- and inf-operations. Thus, in this paper we explore the second way to obtain new morphological operators. We will discover one case (the new family of filters) where idempotence and the strong property exist.

Sections 5.1 and 5.2 provide the base results that are necessary for subsequent results presented later in Sections 5.3–5.6. Therefore, if the reader is most interested in the final results, he or she can skip the first two sections and start reading Section 5.3.

5.1. Grain and pore properties of $\{\bigwedge_{i=1}^{n}\tilde{\varphi}_i\tilde{\gamma}_i\}$

The grain and pore properties of the family $\{\bigwedge_{i=1}^{n}\tilde{\varphi}_i\tilde{\gamma}_i\}$ are studied in this section. First, general properties will be presented, and then some theorems and propositions will follow. They will lead to the finding that the family $\{\bigwedge_{i=1}^{n}\tilde{\varphi}_i\tilde{\gamma}_i\}$ forms a new family of strong filters.

5.1.1. General properties

The following properties follow easily from the properties of the component families $\{\tilde{\gamma}_i\}$ and $\{\tilde{\varphi}_i\}$, which were discussed in Section 4.1.

- From the lack of ordering between $\tilde{\varphi}\tilde{\gamma}$ and I, neither does there exist any order between $\bigwedge_{i=1}^{n}\tilde{\varphi}_i\tilde{\gamma}_i$ and I:

$$I \nleqslant \bigwedge_{i=1}^{n}\tilde{\varphi}_i\tilde{\gamma}_i \nleqslant I.$$

  That is, $\bigwedge_{i=1}^{n}\tilde{\varphi}_i\tilde{\gamma}_i$ is neither an opening nor a closing.
- The pores of $\{\bigwedge_{i=1}^{n}\tilde{\varphi}_i\tilde{\gamma}_i\}$ are nested:

$$\gamma_x I^c\left(\bigwedge_{i=1}^{n}\tilde{\varphi}_i\tilde{\gamma}_i\right) \geqslant \gamma_x I^c\left(\bigwedge_{i=1}^{j}\tilde{\varphi}_i\tilde{\gamma}_i\right), \quad j \leqslant n. \tag{17}$$

  This follows from the fact that $\{\bigwedge_{i=1}^{n}\tilde{\varphi}_i\tilde{\gamma}_i\}$ is an ordered family:

$$\bigwedge_{i=1}^{n}\tilde{\varphi}_i\tilde{\gamma}_i \leqslant \bigwedge_{i=1}^{j}\tilde{\varphi}_i\tilde{\gamma}_i, \quad j \leqslant n. \tag{18}$$

- The grains of $\{\bigwedge_{i=1}^{n}\tilde{\varphi}_i\tilde{\gamma}_i\}$ are (inversely) nested. This is obvious from (18). For all $x \in E$,

$$\gamma_x\left(\bigwedge_{i=1}^{n}\tilde{\varphi}_i\tilde{\gamma}_i\right) \leqslant \gamma_x\left(\bigwedge_{i=1}^{j}\tilde{\varphi}_i\tilde{\gamma}_i\right), \quad j \leqslant n. \tag{19}$$

- There is no flat zone inclusion, i.e.,

$$F_x\left(\bigwedge_{i=1}^{n}\tilde{\varphi}_i\tilde{\gamma}_i\right) \ngeqslant F_x\left(\bigwedge_{i=1}^{j}\tilde{\varphi}_i\tilde{\gamma}_i\right), \quad j \leqslant n. \tag{20}$$

(a) Input set A

(b) $\tilde{\varphi}_1\tilde{\gamma}_1(A)$

(c) $\tilde{\varphi}_2\tilde{\gamma}_2(A)$

(d) $\bigwedge_{i\in\{1,2\}}\tilde{\varphi}_i\tilde{\gamma}_i(A)$

Fig. 12. Family of alternating filters by reconstruction $\{\tilde{\varphi}_i\tilde{\gamma}_i\}$ and $\bigwedge_{i=1}^{n}\tilde{\varphi}_i\tilde{\gamma}_i$: one-dimensional example. Notice that $\tilde{\varphi}_1\tilde{\gamma}_1 = \bigwedge_{i\in\{1\}}\tilde{\varphi}_i\tilde{\gamma}_i$ (i.e., $\tilde{\varphi}_1\tilde{\gamma}_1$ belongs to the class $\{\bigwedge_{i=1}^{n}\tilde{\varphi}_i\tilde{\gamma}_i\}$).

An example can be observed in Fig. 12. Comparing Fig. 12(b) and (d), it can be seen that the grain in the middle of $\tilde{\varphi}_1\tilde{\gamma}_1(A)$ (notice that $\tilde{\varphi}_1\tilde{\gamma}_1(A)$ is equal to $\bigwedge_{i=1}^{1} \tilde{\varphi}_i\tilde{\gamma}_i(A)$, i.e., it belongs to the class $\{\bigwedge_{i=1}^{n} \tilde{\varphi}_i\tilde{\gamma}_i\}$) is not included in any grain or pore of $\bigwedge_{i=1}^{2} \tilde{\varphi}_i\tilde{\gamma}_i(A)$.

5.1.2. Propositions

Some theoretical results follow that will be useful later to establish the family $\{\bigwedge_{i=1}^{n} \tilde{\varphi}_i\tilde{\gamma}_i\}$ (and its dual $\{\bigvee_{i=1}^{n} \tilde{\gamma}_i\tilde{\varphi}_i\}$ as a new family of strong morphological filters. On the other hand, the family of operators $\{\bigvee_{i=1}^{n} \tilde{\varphi}_i\tilde{\gamma}_i\}$ (and its dual $\{\bigwedge_{i=1}^{n} \tilde{\gamma}_i\tilde{\varphi}_i\}$) is not a family of filters as will be shown later. In the following, only the family $\{\bigwedge_{i=1}^{n} \tilde{\varphi}_i\tilde{\gamma}_i\}$ is investigated; however, all results apply dually to the family $\{\bigvee_{i=1}^{n} \tilde{\gamma}_i\tilde{\varphi}_i\}$.

As mentioned before, in general, grains in a certain level of $\{\bigvee_{i=1}^{n} \tilde{\varphi}_i\tilde{\gamma}_i(A)\}$, $A \in \mathscr{P}(E)$, are not included in grains or pores of subsequent coarser levels; the grain inclusion relation is inverted (see expression (19)). Thus, it can happen that a grain that exists at one level is *broken* in coarser levels (i.e., not included in either a grain or a pore of a coarser level). This is the reason why $F_x(\bigwedge_{i=1}^{n} \tilde{\varphi}_i\tilde{\gamma}_i) \not\geqslant F_x(\bigwedge_{i=1}^{j} \tilde{\varphi}_i\tilde{\gamma}_i), j \leqslant n$. On the other hand, expression (17) states that there is an inclusion relation of pores as the index of the family increases, i.e., as the primitive gets farther from the identity operator.

Let us define some concepts that will be used in the following discussion. The concept of adjacency between two sets formalizes the intuitive notion of contiguity. Let $A \in \mathscr{P}(E)$. Two flat zones $F_x(A)$ and $F_{x'}(A)$ are said to be *adjacent* if $F_x(A) \vee F_{x'}(A) = \gamma_x(F_x(A) \vee F_{x'}(A))$, i.e., if $F_x(A) \vee F_{x'}(A)$ is a connected set. The *adjacent flat zones* of x in an input set A, symbolized by $\mathscr{A}_x(A)$, are the pores (if $x \in A$) or the grains (if $x \notin A$) that are adjacent to $F_x(\mathscr{A}_x(A) = \bigvee_{x'} \{F_{x'}(A)\colon\ x' \in E, F_{x'}(A) \vee F_x(A) = \gamma_x(F_{x'}(A) \vee F_x(A))\})$. An example of the adjacent flat zones of a point is shown in Fig. 13.

The following propositions establish relationships between the grains and pores of $\bigwedge_{i=1}^{n} \tilde{\varphi}_i\tilde{\gamma}_i(A)$, $A \in \mathscr{P}(E)$, and those of each level $\tilde{\varphi}_i\tilde{\gamma}_i(A)$.

Proposition 2. *Let E be a space equipped with* γ_x *and* $A \in \mathscr{P}(E)$. *Let* $\{\tilde{\gamma}_i\}$ *and* $\{\tilde{\varphi}_i\}$, *where* $i \in S = \{1, \ldots, n\}$, *be, respectively, a granulometry and an antigranulometry by reconstruction from* $\mathscr{P}(E)$ *to* $\mathscr{P}(E)$. *If x belongs to a pore of* $\bigwedge_{i=1}^{n} \tilde{\varphi}_i\tilde{\gamma}_i(A)$, *then*

$$\exists i_0 \in S\colon \gamma_x I^c \left(\bigwedge_{i=1}^{n} \tilde{\varphi}_i\tilde{\gamma}_i \right)(A)$$
$$= \gamma_x I^c \tilde{\varphi}_{i_0}\tilde{\gamma}_{i_0}(A) = \gamma_x I^c \tilde{\gamma}_{i_0}(A).$$

Proposition 2 can be deduced easily from expression (13). Thus, all pores of $\bigwedge_{i=1}^{n} \tilde{\varphi}_i\tilde{\gamma}_i(A)$ exist in a member of $\{\tilde{\varphi}_i\tilde{\gamma}_i(A)\}$ (and of $\{\tilde{\gamma}_i(A)\}$).

There is a lack of ordering between $\bigwedge_{i=1}^{n} \tilde{\varphi}_i\tilde{\gamma}_i$ and I. In general, there is no ordering either between an opening γ and $\gamma(\bigwedge_{i=1}^{n} \tilde{\varphi}_i\tilde{\gamma}_i)$. However, there exists an ordering relation between some pores of the two aforesaid operators when the opening is a component of the granulometry by reconstruction used to build $\bigwedge_{i=1}^{n} \tilde{\varphi}_i\tilde{\gamma}_i$. This is stated in the following proposition.

Proposition 3. *Let E be a space equipped with* γ_x *and* $A \in \mathscr{P}(E)$. *Let* $\{\tilde{\gamma}_i\}$ *and* $\{\tilde{\varphi}_i\}$, *where* $i \in S = \{1, \ldots, n\}$, *be, respectively, a granulometry and an antigranulometry by reconstruction from* $\mathscr{P}(E)$ *to*

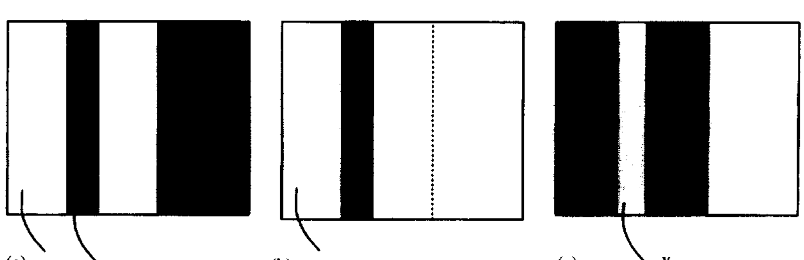


Fig. 13. Adjacent flat zones of a point. (a) Input set A (in black); (b) $\mathscr{A}_x(A)$: adjacent flat zones of x; (c) $\mathscr{A}_y(A)$: adjacent flat zones of y.

$\mathscr{P}(E)$. If x belongs to a grain of $(\bigwedge_{i=1}^{n} \tilde{\varphi}_i\tilde{\gamma}_i)(A)$, then

$$\gamma_x I^c \tilde{\gamma}_{i_0}\left(\bigwedge_{i=1}^{n} \tilde{\varphi}_i\tilde{\gamma}_i\right)(A) \leqslant \gamma_x I^c \tilde{\gamma}_{i_0}(A), \quad \forall i_0 \in S.$$

Notice that Proposition 3 does not imply that $\tilde{\gamma}_p(\bigwedge_{i=1}^{n} \tilde{\varphi}_i\tilde{\gamma}_i) \geqslant \tilde{\gamma}_p$, which is not true.

The next theorem is important since it will be used in the following section for establishing the cases for which some filters commute with the inf- and the sup-operators.

Theorem 2. *Let $\{\tilde{\gamma}_i\}$ and $\{\tilde{\varphi}_i\}$, where $i \in S = \{1, \dots, n\}$, be, respectively, a granulometry and an antigranulometry by reconstruction. Then*

$$\tilde{\varphi}_{i_0}\tilde{\gamma}_{i_0}\left(\bigwedge_{i=1}^{n} \tilde{\varphi}_i\tilde{\gamma}_i\right) \geqslant \bigwedge_{i=1}^{n} \tilde{\varphi}_i\tilde{\gamma}_i, \quad i_0 \in S.$$

Proof. Let G be any grain of $\bigwedge_{i=1}^{n} \tilde{\varphi}_i\tilde{\gamma}_i(A)$, $A \in \mathscr{P}(E)$. Let $x \in G$.

Case 1. $\gamma_x\tilde{\gamma}_{i_0}(\bigwedge_{i=1}^{n} \tilde{\varphi}_i\tilde{\gamma}_i)(A) \neq \emptyset$ (i.e., $\tilde{\gamma}_{i_0}$ does not remove G from $(\bigwedge_{i=1}^{n} \tilde{\varphi}_i\tilde{\gamma}_i)(A)$): then, $G \leqslant \gamma_x\tilde{\varphi}_{i_0}\tilde{\gamma}_{i_0}(\bigwedge_{i=1}^{n} \tilde{\varphi}_i\tilde{\gamma}_i)(A)$.

Case 2. $\gamma_x\tilde{\gamma}_{i_0}(\bigwedge_{i=1}^{n} \tilde{\varphi}_i\tilde{\gamma}_i)(A) = \emptyset$ (i.e., $\tilde{\gamma}_{i_0}$ removes G from $(\bigwedge_{i=1}^{n} \tilde{\varphi}_i\tilde{\gamma}_i)(A)$): If $\tilde{\gamma}_{i_0}$ *removes the grain G* in the operation $\tilde{\gamma}_{i_0}(\bigwedge_{i=1}^{n} \tilde{\varphi}_i\tilde{\gamma}_i)$, then $\tilde{\varphi}_{i_0}$ fills the new pore $\gamma_x I^c\tilde{\gamma}_{i_0}(\bigwedge_{i=1}^{n} \tilde{\varphi}_i\tilde{\gamma}_i)(A)$, since (from Proposition 3) $\gamma_x I^c\tilde{\gamma}_{i_0}(\bigwedge_{i=1}^{n} \tilde{\varphi}_i\tilde{\gamma}_i)(A) \leqslant \gamma_x I^c\tilde{\gamma}_{i_0}(A)$ and $x \in \tilde{\varphi}_{i_0}\tilde{\gamma}_{i_0}(A)$. Then, $\gamma_x\tilde{\varphi}_{i_0}\tilde{\gamma}_{i_0}(\bigwedge_{i=1}^{n} \tilde{\varphi}_i\tilde{\gamma}_i)(A) \geqslant \gamma_x(\bigwedge_{i=1}^{n} \tilde{\varphi}_i\tilde{\gamma}_i)(A) = G$, $\forall i_0 \in S$. □

As a corollary, it can be deduced that if $p \geqslant j$, where $p, j \in S$, then

$$\tilde{\varphi}_p\tilde{\gamma}_p\left(\bigwedge_{i=1}^{n} \tilde{\varphi}_i\tilde{\gamma}_i\right) \geqslant \tilde{\varphi}_j\tilde{\gamma}_j\left(\bigwedge_{i=1}^{n} \tilde{\varphi}_i\tilde{\gamma}_i\right).$$

The grains of $\bigwedge_{i=1}^{n} \tilde{\varphi}_i\tilde{\gamma}_i(A)$, $A \in \mathscr{P}(E)$, are not in general invariants under γ_i, $i \in S$. Proposition 4 and Corollary 2 study, for a grain of $\bigwedge_{i=1}^{n} \tilde{\varphi}_i\tilde{\gamma}_i(A)$ that is variant under a particular γ_{i_0}, $i_0 \in S$, the corresponding grains at certain levels of $\tilde{\varphi}_i\tilde{\gamma}_i(A)$ and their adjacent pores. This section concludes with Theorem 3, which states which grains are invariants under γ_i, for all $i \in S$.

Proposition 4. *Let a connected set E be a space equipped with γ_x and $A \in \mathscr{P}(E)$. Let $\{\tilde{\gamma}_i\}$ and $\{\tilde{\varphi}_i\}$, where $i \in S = \{1, \dots, n\}$, be, respectively, a granulometry and an antigranulometry by reconstruction from $\mathscr{P}(E)$ to $\mathscr{P}(E)$. Let $G \neq E$ be a grain of $\bigwedge_{i=1}^{n} \tilde{\varphi}_i\tilde{\gamma}_i(A)$ that is variant under $\tilde{\gamma}_{i_0}$, $i_0 \in S$, and let $x \in G$. Then*

$$\gamma_x\tilde{\varphi}_i\tilde{\gamma}_i(A) > G, \quad \forall i \geqslant i_0, \quad i \in S.$$

Corollary 2. *All pores of $\bigwedge_{i=1}^{n} \tilde{\varphi}_i\tilde{\gamma}_i(A)$ that are adjacent pores to G come from at least one of the finer levels in $\tilde{\varphi}_j\tilde{\gamma}_j(A)$, where $j < i_0$ and $j \in S$. That is,*

$$\forall \text{pore } P \in \mathscr{A}_x\left(\bigwedge_{i=1}^{n} \tilde{\varphi}_i\tilde{\gamma}_i(A)\right),$$

$$\exists j < i_0, j \in S\colon\ P = \gamma_{x'} I^c \tilde{\varphi}_j\tilde{\gamma}_j(A), \quad x' \in P.$$

Theorem 3. *Let E be a space equipped with γ_x and $A \in \mathscr{P}(E)$. Let $\{\tilde{\gamma}_i\}$ and $\{\tilde{\varphi}_i\}$, where $i \in S = \{1, \dots, n\}$, be, respectively, a granulometry and an antigranulometry by reconstruction from $\mathscr{P}(E)$ to $\mathscr{P}(E)$. Then, all grains of $\bigwedge_{i=1}^{n} \tilde{\varphi}_i\tilde{\gamma}_i(A)$ that are adjacent to a pore of $\tilde{\varphi}_n\tilde{\gamma}_n(A)$ are invariant under $\tilde{\gamma}_n$.*

5.2. Grain and pore properties of $\{\bigvee_{i=1}^{n} \tilde{\varphi}_i\tilde{\gamma}_i\}$

The importance of the family $\{\bigvee_{i=1}^{n} \tilde{\varphi}_i\tilde{\gamma}_i\}$ is less than that of the family $\{\bigwedge_{i=1}^{n} \tilde{\varphi}_i\tilde{\gamma}_i\}$, as will be seen later where the idempotence of the members of the latter family will be shown. Notice that $\{\bigvee_{i=1}^{n} \tilde{\varphi}_i\tilde{\gamma}_i\}$ is not the dual family of $\{\bigwedge_{i=1}^{n} \tilde{\varphi}_i\tilde{\gamma}_i\}$. A less extensive analysis than that of the previous section will be performed, and only one proposition is stated regarding the grain and pore properties of $\{\bigvee_{i=1}^{n} \tilde{\varphi}_i\tilde{\gamma}_i\}$.

As was the case for the family $\{\bigwedge_{i=1}^{n} \tilde{\varphi}_i\tilde{\gamma}_i\}$, each pore of $\bigvee_{i=1}^{n} \tilde{\varphi}_i\tilde{\gamma}_i(A)$, where $A \in \mathscr{P}(E)$, is a pore of a member of $\{\tilde{\varphi}_i\tilde{\gamma}_i(A)\}$ (and of $\{\tilde{\gamma}_i(A)\}$). That is, for any x that belongs to a pore of $\bigvee_{i=1}^{n} \tilde{\varphi}_i\tilde{\gamma}_i(A)$

$$\exists i_0 \in S\colon \gamma_x I^c\left(\bigvee_{i=1}^{n} \tilde{\varphi}_i\tilde{\gamma}_i\right)(A) = \gamma_x I^c\tilde{\varphi}_{i_0}\tilde{\gamma}_{i_0}(A) = \gamma_x I^c\tilde{\gamma}_{i_0}(A). \tag{21}$$

The following proposition takes the result of expression (21) further.

Proposition 5. *Let E be a space equipped with γ_x and $A \in \mathscr{P}(E)$. Let $\{\tilde{\gamma}_i\}$ and $\{\tilde{\varphi}_i\}$, where $i \in S = \{1, \ldots, n\}$, be, respectively, a granulometry and an antigranulometry by reconstruction from $\mathscr{P}(E)$ to $\mathscr{P}(E)$. For all $x \in E$*

$$\gamma_x\left(\bigvee_{i=1}^{n} \tilde{\varphi}_i\tilde{\gamma}_i\right)(A) = \emptyset$$

$$\Rightarrow \gamma_x I^c\left(\bigvee_{i=1}^{n} \tilde{\varphi}_i\tilde{\gamma}_i\right)(A) = \gamma_x I^c \tilde{\varphi}_1\tilde{\gamma}_1(A)$$

$$= \gamma_x I^c \tilde{\gamma}_1(A).$$

5.3. Commutation properties

In general increasing operators do not commute with the inf- or the sup-operations unless they are, respectively, erosions and dilations. As mentioned in Section 1.1, the inequalities that correspond when increasing operations are performed on the inf and sup of other operators are $\psi(\bigwedge_i \psi_i) \leqslant \bigwedge_i \psi\psi_i$, and $\psi(\bigvee_i \psi_i) \geqslant \bigvee_i \psi\psi_2$.

In this section it is shown that in certain cases the previous inequalities can be transformed into equalities when the family $\{\psi_i\}$ is a family of alternating filters by reconstruction. This can greatly simplify the manipulation of sups and infs of alternating filters by reconstruction, as will be seen in the next section. Obviously, all results apply also to the dual operations. For some of the following results, it is not strictly necessary that all filters be filters by reconstruction, as stated in more detail in [4].

The following lemma, which is true for all γ and for all φ (connected or not), will be needed later.

Lemma 1. *Let γ and φ be, respectively, an opening and a closing. Then,*
(a) $\varphi\gamma \geqslant I \Rightarrow \varphi\gamma = \varphi$.
(b) $\varphi\gamma \leqslant I \Rightarrow \varphi\gamma = \gamma$.

Proof. (a): (i) $\varphi\gamma \leqslant \varphi$, since $\gamma \leqslant$ I; and (ii) $\varphi\gamma \geqslant I \Rightarrow \varphi\varphi\gamma = \varphi\gamma \geqslant \varphi$.
(b): (i) $\varphi\gamma \geqslant \gamma$, since $\varphi \geqslant I$ and (ii) $\varphi\gamma \leqslant I \Rightarrow \varphi\gamma\gamma = \varphi\gamma \leqslant \gamma$. □

Proposition 6 states a case in which $\tilde{\varphi}$ can commute with the inf operation when the latter is performed on members of the family $\{\tilde{\varphi}_i\tilde{\gamma}_i\}$.

Proposition 6. *Let E be a space equipped with γ_x. Let $\{\tilde{\gamma}_i\}$ and $\{\tilde{\varphi}_i\}$, where $i \in S = \{1, \ldots, n\}$, be, respectively, a granulometry and an antigranulometry by reconstruction. Then*

$$\tilde{\varphi}\left(\bigwedge_{i=1}^{n} \tilde{\varphi}_i\tilde{\gamma}_i\right) = \bigwedge_{i=1}^{n} \tilde{\varphi}\tilde{\varphi}_i\tilde{\gamma}_i.$$

Proof. Let $A \in \mathscr{P}(E)$. A pore $P = \gamma_x I^c(\bigwedge_{i=1}^{n} \tilde{\varphi}_i\tilde{\gamma}_i)(A) \neq \emptyset$ belongs to a pore of a member of $\{\tilde{\varphi}_i\tilde{\gamma}_i(A)\}$ (from Proposition 2). Let us denote this member as $\tilde{\varphi}_{i_0}\tilde{\gamma}_{i_0}(A)$. The pore $P = \gamma_x I^c \tilde{\varphi}_{i_0}\tilde{\gamma}_{i_0}(A)$ (and also $P = \gamma_x I^c \tilde{\gamma}_{i_0}(A)$) is greater than or equal to $\gamma_x I^c \tilde{\varphi}_i\tilde{\gamma}_i(A)$, $\forall i \in S$. Therefore, $\tilde{\varphi}$ fills a pore $\gamma_x I^c(\bigwedge_{i=1}^{n} \tilde{\varphi}_i\tilde{\gamma}_i)(A) \neq \emptyset$ if and only if $\tilde{\varphi}$ fills the pores $\gamma_x I^c \tilde{\varphi}_i\tilde{\gamma}_i(A) \neq \emptyset$, $\forall i \in S$. (Notice that this proposition also holds when the closing on the left is a standard morphological closing φ_B when the structuring element B is a simply connected set.) □

The following proposition establishes a commutation property that arises when a filter by reconstruction that is a member of a family $\{\tilde{\varphi}_i\tilde{\gamma}_i\}$ is applied to the inf of the family.

Proposition 7. *Let E be a space equipped with γ_x. Let $\{\tilde{\gamma}_i\}$ and $\{\tilde{\varphi}_i\}$, where $i \in S = \{1, \ldots, n\}$ be, respectively, a granulometry and an antigranulometry by reconstruction. Then*

$$\tilde{\varphi}_{i_0}\tilde{\gamma}_{i_0}\left(\bigwedge_{i=1}^{n} \tilde{\varphi}_i\tilde{\gamma}_i\right) = \tilde{\varphi}_{i_0}\left(\bigwedge_{i=1}^{n} \tilde{\varphi}_i\tilde{\gamma}_i\right) = \bigwedge_{i=1}^{n} \tilde{\varphi}_{i_0}\tilde{\varphi}_i\tilde{\gamma}_i$$

$$= \bigwedge_{i=i_0}^{n} \tilde{\varphi}_i\tilde{\gamma}_i, \quad i_0 \in S.$$

Proof. From Theorem 2 and Lemma 1, $\tilde{\varphi}_{i_0}\tilde{\gamma}_{i_0}(\bigwedge_{i=1}^{n} \tilde{\varphi}_i\tilde{\gamma}_i) = \tilde{\varphi}_{i_0}(\bigwedge_{i=1}^{n} \tilde{\varphi}_i\tilde{\gamma}_i)$, $i_0 \in S$. From Proposition 6, $\tilde{\varphi}_{i_0}(\bigwedge_{i=1}^{n} \tilde{\varphi}_i\tilde{\gamma}_i) = \bigwedge_{i=1}^{n} \tilde{\varphi}_{i_0}\tilde{\varphi}_i\tilde{\gamma}_i = \bigwedge_{i=i_0}^{n} \tilde{\varphi}_i\tilde{\gamma}_i$. (Notice that $\tilde{\varphi}_{i_0}\tilde{\gamma}_i \geqslant \tilde{\varphi}_{i_0}\tilde{\gamma}_{i_0}$, $i \leqslant i_0$.) □

Table 2
Filters by reconstruction: commutation properties. Expressions in the second half of the table are dual of those in the first half

$\tilde{\varphi}(\bigwedge_{i=1}^{n} \tilde{\varphi}_i\tilde{\gamma}_i) = \bigwedge_{i=1}^{n} \tilde{\varphi}\tilde{\varphi}_i\tilde{\gamma}_i$
$\tilde{\varphi}_{i_0}\tilde{\gamma}_{i_0}(\bigwedge_{i=1}^{n} \tilde{\varphi}_i\tilde{\gamma}_i) = \tilde{\varphi}_{i_0}(\bigwedge_{i=1}^{n} \tilde{\varphi}_i\tilde{\gamma}_i) = \bigwedge_{i=1}^{n} \tilde{\varphi}_{i_0}\tilde{\varphi}_i\tilde{\gamma}_i$
$= \bigwedge_{i=i_0}^{n} \tilde{\varphi}_i\tilde{\gamma}_i,\ i_0 \in \{1, \ldots, n\}$
$\tilde{\varphi}(\bigvee_{i=1}^{n} \tilde{\varphi}_i\tilde{\gamma}_i) = \bigvee_{i=1}^{n} \tilde{\varphi}\tilde{\varphi}_i\tilde{\gamma}_i = (\bigvee_{i=1}^{n} \tilde{\varphi}_i\tilde{\gamma}_i) \vee \tilde{\varphi}\tilde{\varphi}_1\tilde{\gamma}_1.$
$\tilde{\gamma}(\bigvee_{i=1}^{n} \tilde{\gamma}_i\tilde{\varphi}_i) = \bigvee_{i=1}^{n} \tilde{\gamma}\tilde{\gamma}_i\tilde{\varphi}_i$
$\tilde{\gamma}_{i_0}\tilde{\varphi}_{i_0}(\bigvee_{i=1}^{n} \tilde{\gamma}_i\tilde{\varphi}_i) = \tilde{\gamma}_{i_0}(\bigvee_{i=1}^{n} \tilde{\gamma}_i\tilde{\varphi}_i) = \bigvee_{i=1}^{n} \tilde{\gamma}_{i_0}\tilde{\gamma}_i\tilde{\varphi}_i = \bigwedge_{i=i_0}^{n} \tilde{\gamma}_i\tilde{\varphi}_i,$
$i_0 \in \{1, \ldots, n\}$
$\tilde{\gamma}(\bigwedge_{i=1}^{n} \tilde{\gamma}_i\tilde{\varphi}_i) = \bigwedge_{i=1}^{n} \tilde{\gamma}\tilde{\gamma}_i\tilde{\varphi}_i = (\bigvee_{i=1}^{n} \tilde{\gamma}_i\tilde{\varphi}_i) \wedge \tilde{\gamma}\tilde{\gamma}_1\tilde{\varphi}_1$

As a particular case,

$$\tilde{\varphi}_{i_0}\tilde{\gamma}_{i_0}\left(I \bigwedge_{i=1}^{n} \tilde{\varphi}_i\tilde{\gamma}_i\right) = \tilde{\varphi}_{i_0}\left(I \bigwedge_{i=1}^{n} \tilde{\varphi}_i\tilde{\gamma}_i\right) = \bigwedge_{i=1}^{n} \tilde{\varphi}_{i_0}\tilde{\varphi}_i\tilde{\gamma}_i$$
$$= \bigwedge_{i=i_0}^{n} \tilde{\varphi}_i\tilde{\gamma}_i, \quad i_0 \in S.$$

The commutation between $\tilde{\varphi}$ and the sup operator when the latter is applied to members of $\{\tilde{\varphi}_i\tilde{\gamma}_i\}$ is established by Proposition 8.

Proposition 8. *Let E be a space equipped with γ_x. Let $\{\tilde{\gamma}_i\}$ and $\{\tilde{\varphi}_i\}$, where $i \in S = \{1, \ldots, n\}$, be, respectively, a granulometry and an antigranulometry by reconstruction from $\mathscr{P}(E)$ to $\mathscr{P}(E)$. Then*

$$\tilde{\varphi}\left(\bigvee_{i=1}^{n} \tilde{\varphi}_i\tilde{\gamma}_i\right) = \bigvee_{i=1}^{n} \tilde{\varphi}\tilde{\varphi}_i\tilde{\gamma}_i.$$

Proof. Let P be a pore of $\bigvee_{i=1}^{n} \tilde{\varphi}_i\tilde{\gamma}_i(A)$, $A \in \mathscr{P}(E)$, and let $x \in P$. Expression (15) implies that the corresponding pores of P in the family $\{\tilde{\varphi}_i\tilde{\gamma}_i(A)\}$ are nested. Then, $\tilde{\varphi}$ fills the pore P if it fills $\gamma_x I^c \tilde{\varphi}_i\tilde{\gamma}_i(A)$ for all $i \in S$. □

We have as well that

$$\tilde{\varphi}\left(\bigvee_{i=1}^{n} \tilde{\varphi}_i\tilde{\gamma}_i\right) = \left(\bigvee_{i=1}^{n} \tilde{\varphi}_i\tilde{\gamma}_i\right) \vee \tilde{\varphi}\tilde{\varphi}_1\tilde{\gamma}_1.$$

Table 2 summarizes the properties that have been presented. Dual expressions are also shown.

5.4. *A new family of strong filters: $\{\Psi_n = \bigwedge_{i=1}^{n} \tilde{\varphi}_i\tilde{\gamma}_i\}$*

In this section it will be shown that the operation $\bigwedge_{i=1}^{n} \tilde{\varphi}_i\tilde{\gamma}_i$ is idempotent. (As before, $\{\tilde{\gamma}_i\}$ and $\{\tilde{\varphi}_i\}$ denote a granulometry and antigranulometry by reconstruction, respectively.) Furthermore, the filter $\bigwedge_{i=1}^{n} \tilde{\varphi}_i\tilde{\gamma}_i$ is strong. The symbol Ψ_n will be used in the following to denote the filter $\bigwedge_{i=1}^{n} \tilde{\varphi}_i\tilde{\gamma}_i$, whereas Ψ_n^* symbolizes the dual filter $\bigvee_{i=1}^{n} \tilde{\gamma}_i\tilde{\varphi}_i$. Clearly, all results obtained for Ψ_n apply *dually* to Ψ_n^*.

The following theorem establishes this new family of filters. It was presented in [7] using the outline of another (more complicated) proof. In the proof of this theorem as well as in most of the expressions used in this section, the commutation properties presented in the previous section (and summarized in Table 2) are employed.

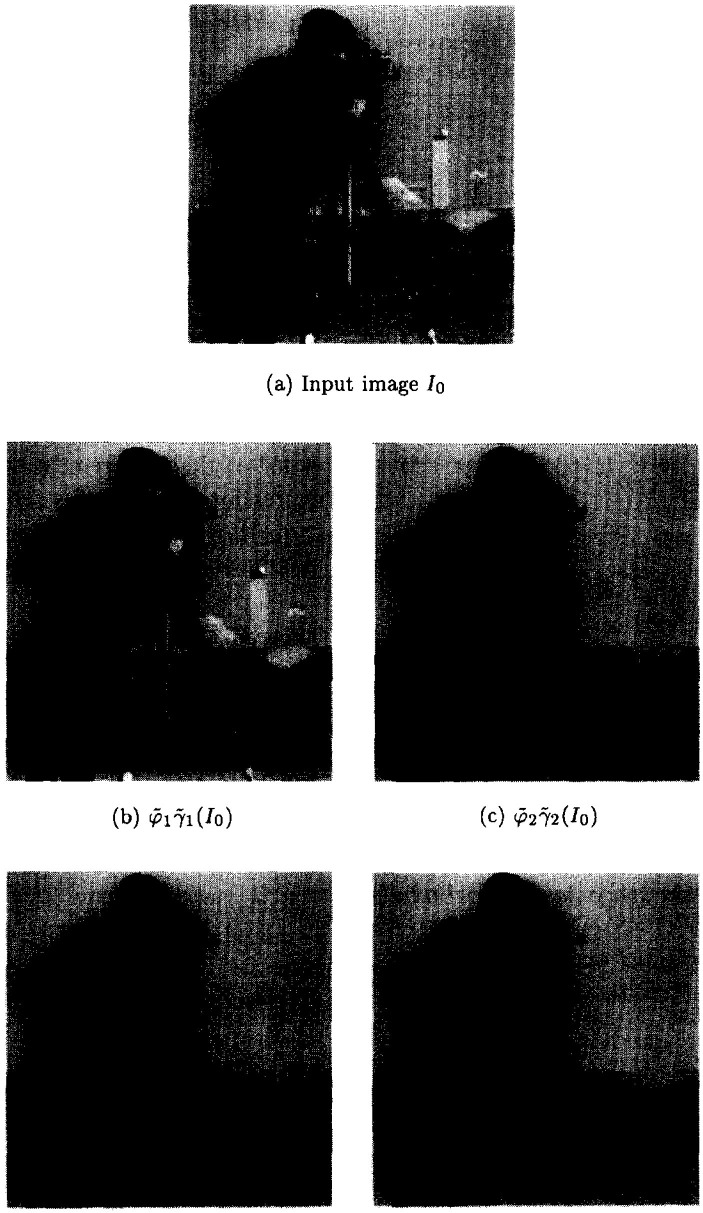
Theorem 4. *Let $\{\tilde{\gamma}_i\}$ and $\{\tilde{\varphi}_i\}$, where $i \in S = \{1, \ldots, n\}$, be, respectively, a granulometry and an antigranulometry by reconstruction. Let Ψ_n and Ψ_n^* be, respectively, $\bigwedge_{i=1}^{n} \tilde{\varphi}_i\tilde{\gamma}_i$ and $\bigvee_{i=1}^{n} \tilde{\gamma}_i\tilde{\varphi}_i$. Then Ψ_n and the dual Ψ_n^* are strong filters.*

Proof. It is known, from Theorem 1, that $\tilde{\varphi}_i\tilde{\gamma}_i$ is an $\vee$-filter (besides being an $\wedge$-filter and, therefore, a strong filter). Then, Ψ_n is an $\vee$-underfilter because it is the inf of $\vee$-filters [27]. Let us prove that Ψ_n is also an $\wedge$-overfilter, i.e., that $\Psi_n = \Psi_n\ (I \wedge \Psi_n)$. (The fact that Ψ_n is strong implies its idempotence.)
(i) $\Psi_n = \Psi_n(I \wedge \Psi_n) \leqslant \Psi_n$.
(ii) Using Proposition 7, $\Psi_n = \Psi_n(I \wedge \Psi_n) = (\bigwedge_{i=1}^{n} \tilde{\varphi}_i\tilde{\gamma}_i)(I \bigwedge_{i=1}^{n} \tilde{\varphi}_i\tilde{\gamma}_i) = \bigwedge_{j=1}^{n} (\tilde{\varphi}_j\tilde{\gamma}_j(I \bigwedge_{i=1}^{n} \tilde{\varphi}_i\tilde{\gamma}_i)) = \bigwedge_{j=1}^{n} (\bigwedge_{i=1}^{n} \tilde{\varphi}_j\tilde{\varphi}_i\tilde{\gamma}_i) \geqslant \bigwedge_{i=1}^{n} \tilde{\varphi}_i\tilde{\gamma}_i = \Psi_n$. □

Corollary 3. *The family of filters $\{\Psi_n = \bigwedge_{i=1}^{n} \tilde{\varphi}_i\tilde{\gamma}_i\}$ forms a multilevel structure whose composition laws are:*
(a) $\Psi_n\Psi_p = \Psi_p,\ p \geqslant n.$
(b) $\Psi_p(I \wedge \Psi_n) = \Psi_p,\ p \geqslant n.$

Part (b) of Corollary 3 is obvious from the fact that Ψ_n is a strong filter.

The class $\{\Psi_n\}$ is clearly ordered ($\Psi_p \leqslant \Psi_j, j \leqslant p$) and is therefore closed under the inf- and sup-operations.



(a) Input image I_0

(b) $\tilde{\varphi}_1\tilde{\gamma}_1(I_0)$

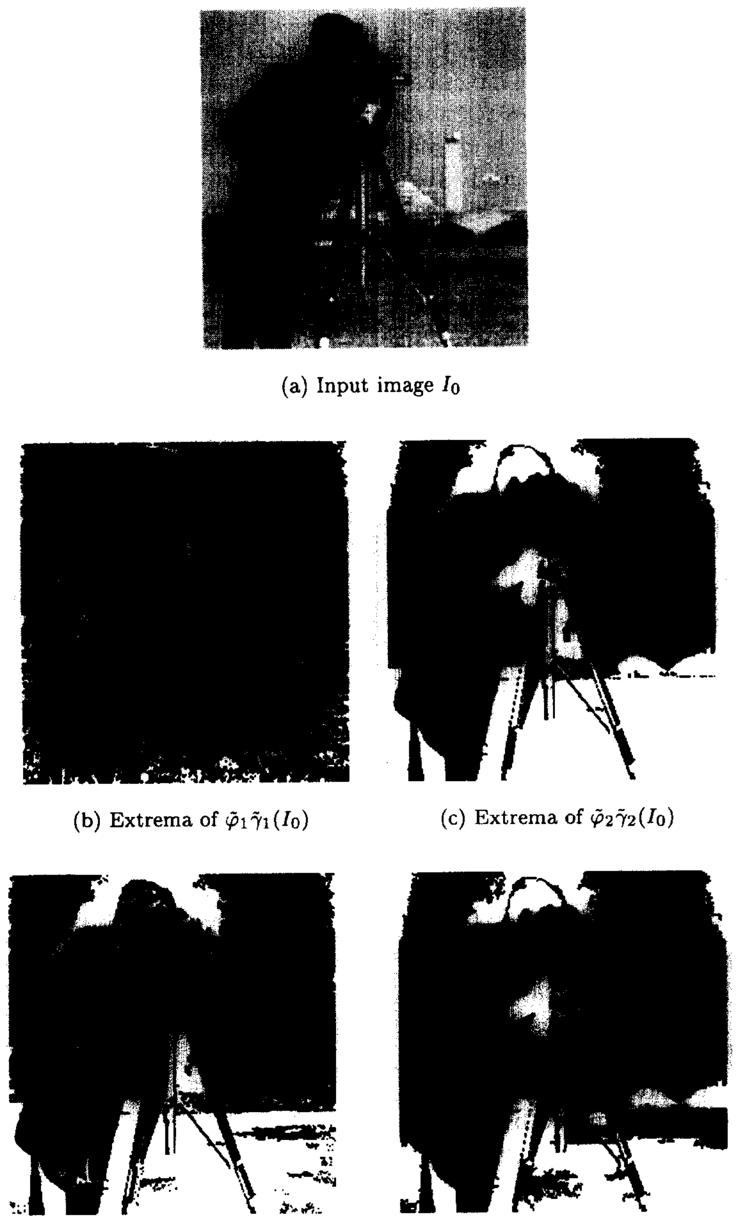
(c) $\tilde{\varphi}_2\tilde{\gamma}_2(I_0)$

(d) $\Psi_2 = \bigwedge_{i=1}^{2} \tilde{\varphi}_i\tilde{\gamma}_i(I_0)$

(e) $\mathrm{ASF}_2 = \tilde{\varphi}_2\tilde{\gamma}_2\tilde{\varphi}_1\tilde{\gamma}_1(I_0)$

Fig. 14. The filter $\Psi_n = \bigwedge_{i=1}^{n} \tilde{\varphi}_i\tilde{\gamma}_i$: gray-level example. In this figure, images (b) and (c) show two levels of alternating filters used to compute Ψ_2 (image (d)) and ASF_2 (image (e)). The criterion used for $\tilde{\gamma}_1$ and $\tilde{\gamma}_2$ is ε_B, where B is a 3×3 square for $\tilde{\gamma}_1$ and a 17×17 square for γ_2. $\tilde{\varphi}_1$ and $\tilde{\varphi}_2$ are the dual closings of $\tilde{\gamma}_1$ and $\tilde{\gamma}_2$, respectively. Notice that $\Psi_1 = \mathrm{ASF}_1 = \tilde{\varphi}_1\tilde{\gamma}_1$ (image (b)).

In order for the family $\{\Psi_n = \bigwedge_{i=1}^{n} \tilde{\varphi}_i\tilde{\gamma}_i\}$ to form a pyramid (see Definition 10), the necessary composition expression should have been $\Psi_p\Psi_n = \Psi_p$, $p \geqslant n$, which is not true. The next theorem will enable the relationship between the filters Ψ_n and Ψ_n^* and a pyramid to be established. Each of the families $\{\Psi_n\}$ and $\{\Psi_n^*\}$ does not form a pyramid when taken separately, but both of them do when they are composed in a sequentially alternated way as will be shown in Corollary 4.



(a) Input image I_0

(b) Extrema of $\tilde{\varphi}_1\tilde{\gamma}_1(I_0)$ (c) Extrema of $\tilde{\varphi}_2\tilde{\gamma}_2(I_0)$

(d) Extrema of $\Psi_2 = \bigwedge_{i=1}^{2} \tilde{\varphi}_1\tilde{\gamma}_1(I_0)$ (e) Extrema of $\mathrm{ASF}_2 = \tilde{\varphi}_2\tilde{\gamma}_2\tilde{\varphi}_1\tilde{\gamma}_1(I_0)$

Fig. 15. Comparison of extrema. This figure displays, in white, the extrema (i.e., both the regions that are maxima and the regions that are minimal of the primitives that were shown in the previous figure. Notice that the filter ψ_2 contains extrema regions of the levels $\tilde{\varphi}_1\tilde{\gamma}_1(I_0)$ and $\tilde{\varphi}_2\tilde{\gamma}_2(I_0)$ employed for its computation. This can be beneficial when extrema regions are to be preserved.

Theorem 5. *Let $\{\tilde{\gamma}_i\}$ and $\{\tilde{\varphi}_i\}$, where $i \in S = \{1, \ldots, n\}$ be, respectively, a granulometry and an antigranulometry by reconstruction. Let Ψ_n and Ψ_n^* be, respectively, $\bigwedge_{i=1}^n \tilde{\varphi}_i\tilde{\gamma}_i$ and $\bigvee_{i=1}^n \tilde{\gamma}_i\tilde{\varphi}_i$. Then $\Psi_n^*\Psi_n$ is equal to the (strong) filter $\tilde{\varphi}_n\tilde{\gamma}_n$.*

Proof. (i) $\Psi_n^*\Psi_n = (\bigvee_{i=1}^n \tilde{\gamma}_i\tilde{\varphi}_i)(\bigwedge_{j=1}^n \tilde{\varphi}_j\tilde{\gamma}_j) \geqslant \tilde{\gamma}_n\tilde{\varphi}_n(\bigwedge_{j=1}^n \tilde{\varphi}_j\tilde{\gamma}_j) \geqslant \tilde{\gamma}_n\tilde{\varphi}_n\tilde{\gamma}_n = \tilde{\varphi}_n\tilde{\gamma}_n$.
(ii) Using Proposition 7, $\Psi_n^*\Psi_n = (\bigvee_{i=1}^n \tilde{\gamma}_i\tilde{\varphi}_i)(\bigwedge_{j=1}^n \tilde{\varphi}_j\tilde{\gamma}_j) \leqslant \tilde{\varphi}_n(\bigwedge_{j=1}^n \tilde{\varphi}_j\tilde{\gamma}_j) = \bigwedge_{j=1}^n \tilde{\varphi}_n\tilde{\varphi}_j\tilde{\gamma}_j = \bigwedge_{j=1}^n \tilde{\varphi}_n\tilde{\gamma}_j = \tilde{\varphi}_n\tilde{\gamma}_n$. □

Under a different form, Theorem 5 was stated but not proved in [6].

As a particular case of Theorem 5, we have that $(I \vee \Psi_n^*)(I \wedge \Psi_n) = \tilde{\varphi}_n\tilde{\gamma}_n$. Notice that $I \wedge \Psi_n$ is an opening (unlike Ψ_n) and that $I \vee \Psi_n^*$ is a closing (unlike Ψ_n^*). It can also be deduced that $(I \wedge \Psi_n^*)(I \wedge \Psi_n) = (I \wedge \Psi_n)(I \wedge \Psi_n^*) = I \wedge \Psi_n$, from the fact that $\{I \wedge \Psi_n,\ I \wedge \Psi_n^*\}$ is a granulometry. Similarly, $(I \vee \Psi_n^*)(I \vee \Psi_n) = (I \vee \Psi_n)(I \vee \Psi_n^*) = I \vee \Psi_n^*$.

Corollary 4. *Let Δ_n be the alternating sequential composition $\Psi_n^*\Psi_n \cdots \Psi_i^*\Psi_i \cdots \Psi_1^*\Psi_1$, $n \geqslant i \geqslant 1$. Then, the family of filters $\{\Delta_n\}$ forms a pyramid.*

Corollary 4 follows directly from Theorem 5 since $\Delta_n = \Psi_n^*\Psi_n \cdots \Psi_i^*\Psi_i \cdots \Psi_1^*\Psi_1 = \tilde{\varphi}_n\tilde{\gamma}_n \cdots \tilde{\varphi}_i\tilde{\gamma}_i \cdots \tilde{\varphi}_1\tilde{\gamma}_1$, which is a member of an alternating sequential filter (ASF) pyramid by reconstruction. The interest of Corollary 4 is only theoretical. The pyramid $\{\Delta_n\}$ is identical to the ASF pyramid by reconstruction, which is simpler to compute.

5.5. Simplification effects of the filter $\Psi_n = \bigwedge_{i=1}^n \tilde{\varphi}_i\tilde{\gamma}_i$

The issue of determining which filter is best suited for image analysis is not simple because, in general, simplification effects vary greatly depending on the types of input images. Besides, effects that are often regarded as desirable in some applications can be unacceptable in others. In this section we discuss the effects achieved by the new filter $\Psi_n = \bigwedge_{i=1}^n \tilde{\varphi}_i\tilde{\gamma}_i$ and to compare them with respect to those obtained by using the 'traditional' alternating sequential filter by reconstruction $\mathrm{ASF}_n = \tilde{\varphi}_n\tilde{\gamma}_n \cdots \tilde{\varphi}_1\tilde{\gamma}_1$.

An example of the application of the filter Ψ_n for image simplification can be seen in Fig. 14. In general, Ψ_n preserves the *extrema* (maxima and minima) regions in an image better than ASF_n (and than $\tilde{\varphi}_n\tilde{\gamma}_n$). The reason is that whereas in the ASF_n case the effect that predominates is the simplification performed by level n, i.e., by the alternating filter $\tilde{\varphi}_n\tilde{\gamma}_n$ (the most severe – active – one) that is applied last in the sequential composition. On the other hand, in the Ψ_n case, all levels are treated in an equal manner by the inf operator; this allows finer (smaller index) levels to add some extrema regions. This can be beneficial or not depending on whether one desires a severe simplification or rather that extrema regions from several levels be preserved for later purposes. Fig. 15 depicts the extrema regions of the images in Fig. 14. The filter Ψ_n has been used, for example, in the *flat zone approach* to image segmentation presented in [4–6, 8], in which the extrema regions of the output of the filtering stage play an important role. The reason is that extrema regions are usually preceptually significant and therefore important to be taken into account in, for example, region-based coding applications (as was our goal). In Fig. 14 the eyes (minima – darker than their neighbors – regions) have been preserved by Ψ_2 but not by ASF_2. In the particular case of our region-based coding application, it was desirable that regions such as the eyes be preserved by the filtering (preprocessing) stage. Concerning implementation details, the results in Fig. 14 were computed using standard morphological erosions as increasing criteria for the openings by reconstruction $\tilde{\gamma}_1$ and

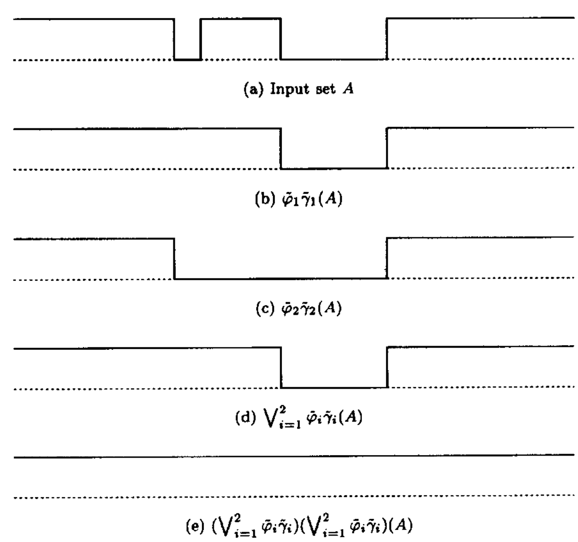


Fig. 16. The operator $\bigvee_{i=1}^n \tilde{\varphi}_i\tilde{\gamma}_i$: one-dimensional example. Notice the non-idempotence of $\bigvee_{i=1}^2 \tilde{\varphi}_i\tilde{\gamma}_i$: $\tilde{\varphi}_2$ can fill the pore in (d), which is smaller than the corresponding pore in (c), and therefore (d) and (e) can be different.

Table 3
Compendium of filter properties

	Idem-potent	$\wedge$-over-filter	$\vee$-under-filter	Strong filter
γ	Yes	Yes	Yes	Yes
φ	Yes	Yes	Yes	Yes
$\varphi\gamma$	Yes	Yes	No	No
$\gamma\varphi$	Yes	No	Yes	No
$\bigwedge_{i=1}^{n}\varphi_i\gamma_i$	No	No	No	No
$\bigvee_{i=1}^{n}\varphi_i\gamma_i$	No	Yes	No	No
$\tilde{\varphi}\tilde{\gamma}$	Yes	Yes	Yes	Yes
$\tilde{\gamma}\tilde{\varphi}$	Yes	Yes	Yes	Yes
$\bigwedge_{i=1}^{n}\tilde{\varphi}_i\tilde{\gamma}_i$	Yes	Yes	Yes	Yes
$\bigvee_{i=1}^{n}\tilde{\varphi}_i\tilde{\gamma}_i$	No	Yes	No	No

$\tilde{\gamma}_2$. The structuring elements employed were squares of sizes 3 × 3 and 17 × 17, respectively. The closings $\tilde{\varphi}_1$ and $\tilde{\varphi}_2$ were dual of, respectively, $\tilde{\gamma}_1$ and $\tilde{\gamma}_2$.

Let us notice that when only one level is used (i.e., $n = 1$), then both families $\{\Psi_n = \bigwedge_{i=1}^{n}\tilde{\varphi}_i\tilde{\gamma}_i\}$ and $\{\mathrm{ASF}_n = \tilde{\varphi}_n\tilde{\gamma}_n \cdots \tilde{\varphi}_1\tilde{\gamma}_1\}$ are clearly identical because $\Psi_1 = \mathrm{ASF}_1 = \varphi_1\gamma_1$.

Regarding computational issues, the filter Ψ_n is better suited to parallel implementation than ASF_n. Whereas in Ψ_n each simplification level (each alternating filter $\tilde{\varphi}_i\tilde{\gamma}_i$) is processed independently (so that their processing can be performed in a parallel manner), only one level can be processed at any given time in the ASF_n case since simplification levels are applied sequentially.

5.6. *The operator* $\bigvee_{i=1}^{n}\tilde{\varphi}_i\tilde{\gamma}_i$

It was observed before that the sup and the inf of filters by reconstruction are not in general filters by reconstruction because the idempotence of the resulting operator is not ensured. The families $\{\Psi_n = \bigwedge_{i=1}^{n}\tilde{\varphi}_i\tilde{\gamma}_i\}$ and its dual $\{\Psi_n^* = \bigvee_{i=1}^{n}\tilde{\gamma}_i\tilde{\varphi}_i\}$ are filters (i.e., they are idempotent), as treated in the previous section. This section will show that the other two possibilities of combining alternating filters by reconstruction, $\bigvee_{i=1}^{n}\tilde{\gamma}_i\tilde{\varphi}_i$ and $\bigwedge_{i=1}^{n}\tilde{\varphi}_i\tilde{\gamma}_i$, are not filters.

Let us apply $\bigvee_{i=1}^{n}\tilde{\varphi}_i\tilde{\gamma}_i$ twice: $(\bigvee_{i=1}^{n}\tilde{\varphi}_i\tilde{\gamma}_i)(\bigvee_{j=1}^{n}\tilde{\varphi}_j\tilde{\gamma}_j) \geqslant \bigvee_i\bigvee_j\tilde{\varphi}_i\tilde{\gamma}_i\tilde{\varphi}_j\tilde{\gamma}_j = (\bigvee_i\tilde{\varphi}_i\tilde{\gamma}_i) \vee (\bigvee_{i\,(i>j)}\bigvee_j\tilde{\varphi}_i\tilde{\gamma}_i\tilde{\varphi}_j\tilde{\gamma}_j) \geqslant \bigvee_i\tilde{\varphi}_i\tilde{\gamma}_i$. The equality does not hold because $\tilde{\varphi}_i\tilde{\gamma}_i\tilde{\varphi}_j\tilde{\gamma}_j \geqslant \tilde{\varphi}_i\tilde{\gamma}_i$ and, in general, $\tilde{\varphi}_i\tilde{\gamma}_i\tilde{\varphi}_j\tilde{\gamma}_j \neq \tilde{\varphi}_i\tilde{\gamma}_i$, for $i > j$.

An example of the non-idempotenace of $\bigvee_{i=1}^{n}\tilde{\varphi}_i\tilde{\gamma}_i$ is given in Fig. 16.

6. Conclusion

In this paper theoretical aspects of the filter by reconstruction class have been studied. Filters by reconstruction constitute a type of morphological filter that possesses interesting properties, which can be studied within the framework offered by the theory of morphological filtering.

First of all, filters by reconstruction are connected filters and, therefore, consider the connectivity that exists in the input image in the sense that no discontinuities are introduced. Second, certain combinations of filters by reconstruction have the strong-property (a robustness property) while being able to treat both bright and dark features. Finally, filters by reconstruction are well suited for multi-scale schemes because of the existance of relationships between the simplifications computed at several levels.

The study of filters by reconstruction has been performed through their grain and pore properties, i.e., through the effect of these filters on the connected components of the input set and its complement. Our focus has been the investigation of the families formed by the sup and inf of alternating filters by reconstruction when the component openings and closings belong, respectively, to a granulometry and antigranulometry. Commutation properties have been shown to exist in some cases when filters by reconstruction are applied to members of these families. An additional important result has been the introduction of a new family of strong filters, which constitutes an alternative to the well-known alternating filter by reconstruction family as multi-scale image analysis tool. Although our study has restricted to the set (binary image) case, all results are extendable for functions (by means of the so-called flat operators).

List of symbols

E	a space of points
$\mathscr{P}(E)$	power set of E
A, B	sets
A^c	complement of A
x, y	points
$\cup, \cap, \setminus$	set union, set intersection and set difference, respectively
T	lattice
$\leqslant, \wedge, \vee$	order relation (less than or equal to), inf- and sup-operation, respectively
ψ	operator
I, I^c	identity operator and complementation operator, respectively
$\gamma, \tilde{\gamma}, \varphi, \tilde{\varphi}$	opening, opening by reconstruction, closing and closing by reconstruction, respectively
γ_x, φ_x	point opening and its dual closing, respectively
γ_0, φ_0	trivial opening and trivial closing, respectively
$\{\gamma_i\}, \{\varphi_i\}$	granulometry and antigranulometry, respectively
ASF_i	$\varphi_i\gamma_i \cdots \varphi_j\gamma_j \cdots \varphi_1\gamma_1$, where $i \geqslant j \geqslant 1$, $\gamma_i \in \{\gamma_i\}$ and $\varphi_i \in \{\varphi_i\}$
Ψ_n	$\bigwedge_{i=1}^{n} \tilde{\varphi}_i\tilde{\gamma}_i$
Ψ_n^*	$\bigvee_{i=1}^{n} \tilde{\gamma}_i\tilde{\varphi}_i$
Δ_n	$\Psi_n^*\Psi_n \cdots \Psi_i^*\Psi_i \cdots \Psi_1^*\Psi_1$, where $i \geqslant j \geqslant 1$
I_0	input gray-level image
$\mathbb{Z}, \mathbb{R}$	the set of integers and the set of real numbers, respectively

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