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A Note on the Muskingum Flood-Routing Method

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Abstract—An exact method of solution of the flood-routing equation, when the storage is a linear function of weighted inflow and outflow, is developed. This operation is shown to be equivalent to routing a multiple of the inflow through reservoir storage and subtracting the excess inflow. Modified coefficients for the Muskingum equation are developed which do not depend on the routing interval being small relative to K.

Introduction—The Muskingum method, which is a finite difference method of solution of the flood routing equation, under the assumption that storage is a linear function of a weighted mean of inflow and outflow, $S = K[xI + (1 - x)Q]$, is widely used, both in its original form [*McCarthy*, 1938] and as the basis of a number of graphical or semi-graphical methods. In the use of these methods, however, it is sometimes overlooked that an essential requirement, to insure accuracy in such finite difference calculations, is that the finite interval T must be small relative to the other time elements involved. This fact was emphasized by *Clark* [1945] in discussing the Muskingum flood-routing method. Nevertheless, it still happens that values of T sensibly equal to K are recommended for use in actual calculation.

Whereas it is possible that in practice the inaccuracies so introduced are generally not significant relative to the inaccuracies introduced by the basic storage assumption, and the usual inaccuracies of the data, it may happen, particularly in theoretical work, that a high relative accuracy is required. The failure of the Muskingum method when T/K is not small is demonstrated by the widely accepted belief that routing through linear storage with $x = 0.5$ operates as a pure delay. This conclusion is based on the fact that the substitution of $T = K$ and $x = 0.5$ in the Muskingum equation yields $Q_1 = I_0$, the other coefficients being zero. That this conclusion is erroneous is demonstrated in this note. An exact method of solution under the storage assumption is developed, and modified equations for the Muskingum coefficients are derived. These equations are true even when T is not small relative to K.

Notation—

$I(t)$ = inflow, ft³/sec
$Q(t)$ = outflow, ft³/sec
x = a numerical parameter
$S(t)$ = storage, ft³ hrs/sec
K = a time parameter, hours
k = $K(1 - x)$
D = the differential operator d/dt
C_1 C_2 C_3 the Muskingum coefficients
c = $\exp - T/K(1 - x)$
q = the outflow from routing I through $S = K(1 - x)q$
T = the routing interval, hours
m = slope of inflow curve

The exact solution—The fundamental equations are

$$I = Q + \frac{ds}{dt} \tag{1}$$

$$S = K(xI + (1 - x)Q) \tag{2}$$

from which

$$I - xK\frac{dI}{dt} = Q + (1 - x)K\frac{dQ}{dt}$$

$$Q(t) = \frac{1 - xKD}{1 + (1 - x)KD} I(t) \tag{3}$$

When x = zero we have the corresponding reservoir case

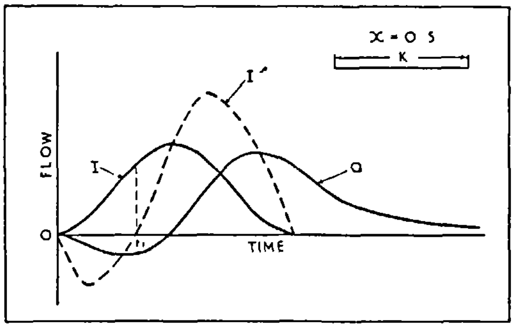

FIG. 1—Routing through storage with $x = 0.5$

$$Q(t) = \frac{1}{1 + KD} I(t) \qquad (4)$$

which has the solution

$$Q = \frac{1}{K} e^{-t/K} \int e^{t/K} I \, dt \qquad (5)$$

Now (3) may be looked upon as the result of operating on $I(t)$ successively with $1 - xKD$ and $1/[1 + (1 - x)KD]$. The operation $1 - xKD$ merely involves differentiation of the inflow and $1/[1 + (1 - x)KD]$ represents reservoir routing with $S = (1 - x)KQ$. Therefore (3) is equivalent to subtracting xK times the first derivative of I from I and routing the remainder through reservoir storage with $S = (1 - x)KQ$. These operations generally can be carried out graphically, and mathematically as well, when I is a simple function of time. We can learn more from (3). Let us define

$$I'(t) = (1 - xKD)I(t) \qquad (6)$$

This means that I' is the result of routing I backwards (that is, calculation of inflow from outflow) through linear reservoir storage $S = -xKI$. The effect of the negative xK is achieved by taking the routing procedure from right to left; that is, in the negative direction of time (Fig. 1).

When we come to t_1 at which I' becomes zero, I would fall off logarithmically and never actually reach zero unless I' took negative values. This means that when I starts from zero and rises at a finite rate, I' must always take negative values initially.

It is clear, too, that the interval between the centers of area of $I'(t)$ and $I(t)$ is xK. We must now route I' forwards through $S = (1 - x)KQ$ to obtain Q (Fig. 1). Clearly this involves a further shift of the center of area $(1 - x)K$ so that the total shift is K. However I and Q are not otherwise identical even when $x = 0.5$ as shown in Figure 1. It should be noted that the negative initial values of I' result in negative initial values of Q.

If we divide out the operator in (3) we obtain

$$Q = \left[-\frac{x}{1 - x} + \frac{1}{(1 - x)\{1 + (1 - x)KD\}} \right] I \qquad (7)$$

$$Q = \frac{1}{1 + (1 - x)KD} \frac{I}{(1 - x)} - \frac{xI}{1 - x} \qquad (8)$$

We see, therefore, that the outflow consists of the sum of two parts, the first of which we shall call q, being the result of routing $I/(1 - x)$ through $S = K(1 - x)q$, and the second part being simply the inflow multiplied by $-x/(1 - x)$.

There are various ways of routing through reservoir storage. Equation (5) may always be integrated graphically, or mathematically if I is in a suitable form. A simple graphical solution not involving integration has been demonstrated by *Nash* and *Farrell* [1955]. It frequently happens, however, that a coefficient solution is desired. Formulas for the coefficients are calculated in the next section.

Modification of the Muskingum coefficients—

$$Q_1 = C_1 I_0 + C_2 I_1 + C_3 Q_0 \qquad (9)$$

We shall use (8) to obtain the expression for the C's. In expressing Q as a function of $I_0 I_1$ and Q_0 only, we must neglect second and higher derivatives of I; that is, we must assume I to consist of straight-line segments. If the second or higher derivatives are required, we must use three or more values of I in (9). However, by choosing time intervals which are sufficiently short, the calculation using only I_0, I_1, and Q_0 can be made as precise as is desired. The only difference between the present calculation and the usual development of the Muskingum coefficient equation is that we are not limited to values of the time interval which are small compared with K.

The solution of (8) when I is a series of straight segments is obtained as follows. Let $m = (I_1 - I_0)/T$ be the slope of a segment.

Let

$$q(t) = \frac{1}{1 + (1 - x)KD} I(t)$$

then

$$Q = q/(1 - x) - xI/(1 - x) \qquad (10)$$

Let $k = (1 - x)K$ and $c = \exp[-T/K(1 - x)]$ to simplify the notation. From (5)

$$q = 1/ke^{-t/k} \int (I_0 + mt)e^{t/k}\, dt$$

$$q = 1/ke^{-t/k}[kI_0e^{t/k} + mk^2e^{t/k}(t/k - 1) + A]$$

$$q = I_0 + mk(t/k - 1) + A/ke^{-t/k} \qquad (11)$$

We solve for the arbitrary constant A by letting $q = q_0$ at $t = 0$ and obtain $q_0 = I_0 - mk + A/K$. Substituting in (11) we obtain

$$q = I_0 + mk(t/k - 1) + (q_0 - I_0 + mk)e^{-t/k}$$

Substituting $(I_1 - I_0)/T$ for m and letting $t = T$, we obtain

$$q_1 = I_0 + k/T(T/k - 1)(I_1 - I_0) + [q_0 - I_0 + k/T(I_1 - I_0)]c$$

$$q_1 = I_0[k/T(1 - c) - c] + I_1[-k/T(1 - c) + 1] + q_0c \qquad (12)$$

whence by (10)

$$Q_1 = I_0\left[\frac{k}{T}\frac{1 - c}{1 - x} - \frac{c}{1 - x}\right] + I_1\left[-\frac{k}{T}\frac{1 - c}{1 - x} + \frac{1}{1 - x} - \frac{x}{1 - x}\right] + q_0\frac{c}{1 - x}$$

But $q_0/(1 - x) = Q_0 - xI_0/(1 - x)$ which when substituted in (12), bearing in mind that $k = K(1 - x)$, gives

$$Q_1 = I_0[K/T(1 - c) - c] + I_1[-K/T(1 - c) + 1] + Q_0c \qquad (13)$$

This is the modified form of the Muskingum equation when T is not small relative to K. If T/K is taken very small the coefficients in (13) and in the Muskingum equation converge.

Conclusions—We have seen that the Muskingum method is equivalent to either routing the inflow backwards (that is, calculating I from Q) through storage $S = -xKI$ and subsequently forwards through $S = (1 - x)KQ$, or routing a multiple of the inflow $I/(1 - x)$ through $S = K(1 - x)q$ and subtracting $xI/(1 - x)$. We have also developed the values of the coefficients to be used when T/K is not small. The negative outflow, which may occur when the inflow rises steeply, has been explained as being essentially associated with the storage assumption, and not with any particular method of solution. This rather unrealistic consequence of the storage assumption suggests that some modification is desirable. As it is necessary in practice to determine x and K by experiment, it would seem more reasonable to abandon the storage assumption and consider the linear operation to consist of two parts, a pure delay and a single reservoir routing [*Hopkins*, 1956], the two parameters to be determined by experiment. The pure delay plus the storage factor K would be equal to the lag between the centers of area of inflow and outflow, and the ratio of the storage factor to the lag would form a dimensionless parameter which might be constant as a first approximation or reflect some characteristics, at present unknown, of the channel. It might be possible to determine this relation by means of a statistical correlation.

Acknowledgment—This note is published by permission of the Director, Hydraulics Research Station, Department of Scientific and Industrial Research, Howbery Park, Wallingford, Berkshire, England.

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